Improved Sobolev inequalities, relative entropy and fast diffusion equations

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The **fast diffusion equation**

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

1. Fast diffusion equations: entropy methods
   Barenblatt solutions, relative entropy functional, large time asymptotics, functional inequalities
2. Fast diffusion equations: linearization of the entropy
   weighted L^2 spaces, spectral gap, and asymptotic rates
3. Gagliardo-Nirenberg inequalities: improvements
   discarding asymptotic eigenmodes and getting improved functional inequalities
4. [Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows]
   extinction in the regime of separation of variables and concavity along the flow

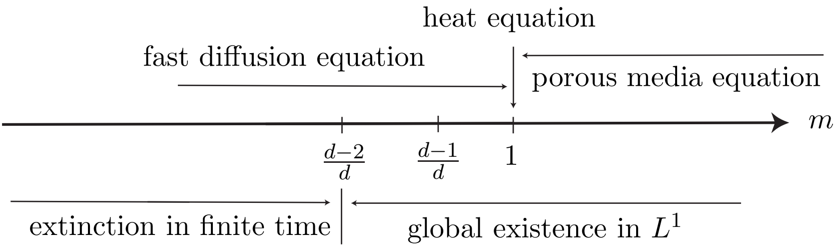
Fast diffusion equations: entropy methods

Existence, classical results

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$ as $t \to +\infty$

[Friedmann, Kamin, 1980] $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$



Existence theory, critical values of the parameter m

Intermediate asymptotics for fast diffusion & porous media

Some references

Generalized entropies and nonlinear diffusions (EDP, uncomplete): [Del Pino, J.D.], [Carrillo, Toscani], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vázquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub],... [del Pino, Sáez], [Daskalopulos, Sesum]...

Some methods

1) [J.D., del Pino] relate entropy and **Gagliardo-Nirenberg** inequalities
2) *entropy – entropy-production method* the **Bakry-Emery** point of view
3) mass transport techniques
4) hypercontractivity for appropriate semi-groups
5) the approach by **linearization** of the entropy

Focus: Fast diffusion equations and Gagliardo-Nirenberg inequalities

We follow the same scheme as for the heat equation

Time-dependent rescaling, Free energy

- **Time-dependent rescaling**: Take $u(\tau, y) = R^{-d}(t)\, v\,(t, y/R(\tau))$ where

$$\frac{\partial R}{\partial \tau} = R^{d(1-m)-1}\;, \quad R(0) = 1\;, \quad t = \log R$$

- The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x\, v)\;, \quad v_{|\tau=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional: **Generalized entropy** or **Free energy**

$$\Sigma[v] := \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2} |x|^2 v \right) dx - \Sigma_0$$

Entropy production is measured by the **Generalized Fisher information**

$$\frac{d}{dt}\Sigma[v] = -I[v]\;, \quad I[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$

Relative entropy and entropy production

- **Stationary solution:** choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) := \left(C + \tfrac{1-m}{2\,m}\,|x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix Σ_0 so that $\Sigma[v_\infty] = 0$. The entropy can be put in an m-homogeneous form: for $m \neq 1$,

$$\Sigma[v] = \int_{\mathbb{R}^d} \psi\left(\tfrac{v}{v_\infty}\right)\, v_\infty^m\, dx \quad \textit{with } \psi(t) = \frac{t^m - 1 - m\,(t-1)}{m-1}$$

- **Entropy – entropy production inequality**

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2\,\Sigma[v]$$

Corollary

A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\, u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\Sigma[v(t,\cdot)] \leq \Sigma[u_0]\, e^{-2\,t}$

An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\Sigma[v] = \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0 \le \frac{1}{2} \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2}\, I[v]$$

Rewrite it with $p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx + K \ge 0$$

- $1 < p = \frac{1}{2m-1} \le \frac{d}{d-2} \iff$ Fast diffusion case: $\frac{d-1}{d} \le m < 1$; $K < 0$
- $0 < p < 1 \iff$ Porous medium case: $m > 1$, $K > 0$
- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} v\, dx = \int_{\mathbb{R}^d} w^{2p}\, dx \right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal
- $m = m_1 := \frac{d-1}{d}$: Sobolev, $m \to 1$: logarithmic Sobolev

Theorem

[Del Pino, J.D.] *Assume that $1 < p \le \frac{d}{d-2}$ (fast diffusion case) and* $d \ge 3$

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le A \|\nabla w\|^\theta_{L^2(\mathbb{R}^d)} \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}, \quad \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \quad y = \frac{p+1}{p-1}$$

Intermediate asymptotics

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau} +$ Csiszár-Kullback inequalities
Undo the change of variables, with

$$u_\infty(t,x) = R^{-d}(t)\, v_\infty\,(x/R(t))$$

Theorem

[Del Pino, J.D.] *Consider a solution of* $u_t = \Delta u^m$ *with initial data* $u_0 \in L^1_+(\mathbb{R}^d)$ *such that* $|x|^2\, u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$

- **Fast diffusion case:** $\frac{d-1}{d} < m < 1$ *if* $d \geq 3$

$$\limsup_{t \to +\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}} \, \| u^m - u_\infty^m \|_{L^1} < +\infty$$

- **Porous medium case:** $1 < m < 2$

$$\limsup_{t \to +\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}} \, \| \,[u - u_\infty]\, u_\infty^{m-1} \,\|_{L^1} < +\infty$$

Fast diffusion equations: the finite mass regime

Can we consider $m < m_1$ *?*

- If $m \ge 1$: porous medium regime or $m_1 := \frac{d-1}{d} \le m < 1$, the decay of the entropy is governed by Gagliardo-Nirenberg inequalities, and to the limiting case $m = 1$ corresponds the logarithmic Sobolev inequality
- Displacement convexity holds in the same range of exponents, $m \in (m_1, 1)$, as for the Gagliardo-Nirenberg inequalities
- The fast diffusion equation can be seen as the gradient flow of the generalized entropy with respect to the Wasserstein distance if $m > \widetilde{m}_1 := \frac{d}{d+2}$
- If $m_c := \frac{d-2}{d} \le m < m_1$, solutions globally exist in L^1 and the Barenblatt self-similar solution has finite mass

...the Bakry-Emery method

We follow the same scheme as for the heat equation

Consider the generalized Fisher information

$$I[v] := \int_{\mathbb{R}^d} v\,|Z|^2\,dx \quad \text{with} \quad Z := \frac{\nabla v^m}{v} + x$$

and compute

$$\frac{d}{dt}\,I[v(t,\cdot)] + 2\,I[v(t,\cdot)] = -2\,(m-1)\int_{\mathbb{R}^d} u^m\,(\operatorname{div} Z)^2\,dx - 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} u^m\,(\partial_i Z^j)^2\,dx$$

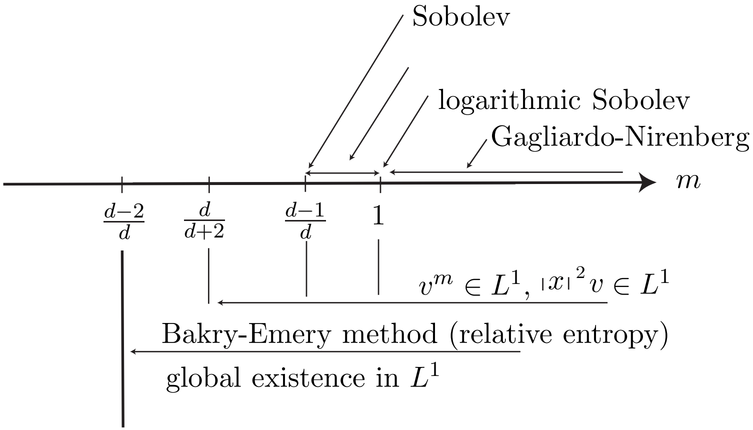
- the Fisher information decays exponentially: $I[v(t,\cdot)] \le I[u_0]\,e^{-\,2\,t}$
- $\lim_{t\to\infty} I[v(t,\cdot)] = 0$ and $\lim_{t\to\infty} \Sigma[v(t,\cdot)] = 0$
- $\frac{d}{dt}\Big(I[v(t,\cdot)] - 2\,\Sigma[v(t,\cdot)]\Big) \le 0$ means $I[v] \ge 2\,\Sigma[v]$

[Carrillo, Toscani], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]

$I[v] \ge 2\,\Sigma[v]$ holds for any $m > m_c$, at least for radial solutions

Fast diffusion: finite mass regime

Inequalities...



... existence of solutions of $u_t = \Delta u^m$

More references: Extensions and related results

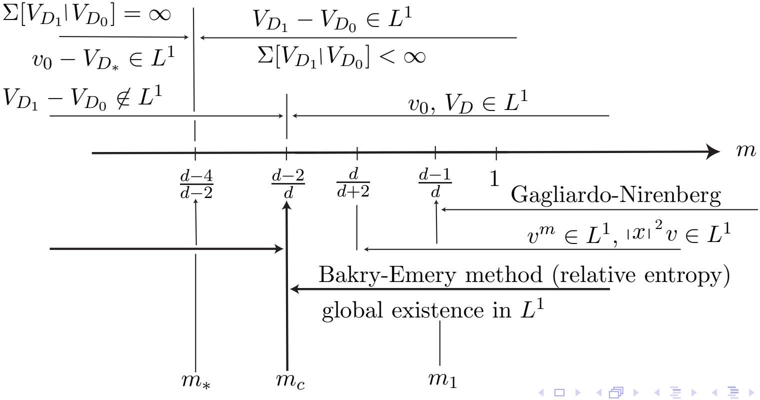
- Mass transport methods: inequalities / rates [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]
- General nonlinearities [Biler, J.D., Esteban], [Carrillo-DiFrancesco], [Carrillo-Juengel-Markowich-Toscani-Unterreiter] and gradient flows [Jordan-Kinderlehrer-Otto], [Ambrosio-Savaré-Gigli], [Otto-Westdickenberg] [J.D.-Nazaret-Savaré], etc
- Non-homogeneous nonlinear diffusion equations [Biler, J.D., Esteban], [Carrillo, DiFrancesco]
- Extension to systems and connection with Lieb-Thirring inequalities [J.D.-Felmer-Loss-Paturel, 2006], [J.D.-Felmer-Mayorga]
- Drift-diffusion problems with mean-field terms. An example: the Keller-Segel model [J.D-Perthame, 2004], [Blanchet-J.D-Perthame, 2006], [Biler-Karch-Laurençot-Nadzieja, 2006], [Blanchet-Carrillo-Masmoudi, 2007], etc
- ... connection with linearized problems [Markowich-Lederman], [Carrillo-Vázquez], [Denzler-McCann], [McCann, Slepčev], [Kim, McCann], [Koch, McCann, Slepčev]

Fast diffusion equations: the infinite mass regime
Linearization of the entropy

Extension to the infinite mass regime, finite time vanishing

- If $m > m_c := \frac{d-2}{d} \le m < m_1$, solutions globally exist in $L^1(\mathbb{R}^d)$ and the Barenblatt self-similar solution has finite mass.
- For $m \le m_c$, the Barenblatt self-similar solution has infinite mass

Extension to $m \le m_c$? Work in relative variables !



Entropy methods and linearization: intermediate asymptotics, vanishing

[A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez], [J.D., Toscani]

- work in relative variables
- use the properties of the flow
- write everything as relative quantities (to the Barenblatt profile)
- compare the functionals (entropy, Fisher information) to their linearized counterparts

$\Longrightarrow$ *Extend the domain of validity of the method to the price of a restriction of the set of admissible solutions*

Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where $m_c := \frac{d-2}{d}$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$
- $0 < m < m_c$, $T < +\infty$: vanishing in finite time $\lim_{\tau \nearrow T} u(\tau, y) = 0$

Alternative approach by comparison techniques: [Daskalopoulos, Sesum] (without rates)

Fast diffusion equation and Barenblatt solutions

$$\frac{\partial u}{\partial \tau} = -\nabla \cdot (u\,\nabla u^{m-1}) = \frac{1-m}{m}\,\Delta u^m \tag{1}$$

with $m<1$. We look for positive solutions $u(\tau,y)$ for $\tau \ge 0$ and $y \in \mathbb{R}^d$, $d \ge 1$, corresponding to nonnegative initial-value data $u_0 \in L^1_{\rm loc}(dx)$

In the limit case $m=0$, u^m/m has to be replaced by $\log u$

Barenblatt type solutions are given by

$$U_{D,T}(\tau,y) := \frac{1}{R(\tau)^d}\left(D + \frac{1-m}{2\,d\,|m-m_c|}\left|\frac{y}{R(\tau)}\right|^2\right)_+^{-\frac{1}{1-m}}$$

- If $m > m_c := (d-2)/d$, $U_{D,T}$ with $R(\tau) := (T+\tau)^{\frac{1}{d\,(m-m_c)}}$ describes the large time asymptotics of the solutions of equation (1) as $\tau \to \infty$ (mass is conserved)
- If $m < m_c$ the parameter T now denotes the *extinction time* and $R(\tau) := (T-\tau)^{-\frac{1}{d\,(m_c-m)}}$
- If $m = m_c$ take $R(\tau) = e^\tau$, $U_{D,T}(\tau,y) = e^{-d\,\tau}\left(D + e^{-2\tau}\,|y|^2/2\right)^{-d/2}$

Two crucial values of m: $m_* := \frac{d-4}{d-2} < m_c := \frac{d-2}{d} < 1$

Rescaling

A time-dependent change of variables

$$t := \frac{1-m}{2} \log\left(\frac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\frac{1}{2\,d\,|m-m_c|}}\,\frac{y}{R(\tau)}$$

If $m = m_c$, we take $t = \tau/d$ and $x = e^{-\tau}\,y/\sqrt{2}$

The generalized Barenblatt functions $U_{D,T}(\tau, y)$ are transformed into stationary *generalized Barenblatt profiles* $V_D(x)$

$$V_D(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}} \quad x \in \mathbb{R}^d$$

If u is a solution to (1), the function $v(t, x) := R(\tau)^d\, u(\tau, y)$ solves

$$\frac{\partial v}{\partial t} = -\nabla \cdot \left[v\,\nabla\left(v^{m-1} - V_D^{m-1}\right)\right] \quad t > 0\,, \quad x \in \mathbb{R}^d \qquad (2)$$

with initial condition $v(t = 0, x) = v_0(x) := R(0)^{-d}\, u_0(y)$

Goal

We are concerned with the *sharp rate* of convergence of a solution v of the rescaled equation to the *generalized Barenblatt profile* V_D in the whole range $m<1$

Convergence is measured in terms of the **relative entropy**

$$\mathcal{E}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[v^m-V_D^m-m\,V_D^{m-1}\,(v-V_D)\right]\,dx$$

for all $m\neq 0$, $m<1$

Assumptions on the initial datum v_0

(H1) *$V_{D_0}\le v_0\le V_{D_1}$ for some $D_0>D_1>0$*

(H2) *if $d\ge 3$ and $m\le m_*$, (v_0-V_D) is integrable for a suitable $D\in[D_1,D_0]$*

- The case $m=m_*=\frac{d-4}{d-2}$ will be discussed later
- If $m>m_*$, we define D as the unique value in $[D_1,D_0]$ such that $\int_{\mathbb{R}^d}(v_0-V_D)\,dx=0$

Our goal is to find the best possible rate of decay of $\mathcal{E}[v]$ if v solves (2)

Sharp rates of convergence

Theorem

[Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if $m<1$ and $m\neq m_*$, the entropy decays according to*

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,(1-m)\,\Lambda\,t} \quad \forall\; t\geq 0$$

The sharp decay rate Λ is equal to the best constant $\Lambda_{\alpha,d}>0$ in the Hardy–Poincaré inequality of Theorem 7 with $\alpha:=1/(m-1)<0$
The constant $C>0$ depends only on m, d, D_0, D_1, D and $\mathcal{E}[v_0]$

- Notion of *sharp rate* has to be discussed
- Rates of convergence in more standard norms: $L^q(dx)$ for $q\geq\max\{1, d\,(1-m)/\,[2\,(2-m)+d\,(1-m)]\}$, or C^k by interpolation
- By undoing the time-dependent change of variables, we deduce results on the *intermediate asymptotics* of (1), i.e. rates of decay of $u(\tau,y)-U_{D,T}(\tau,y)$ as $\tau\to+\infty$ if $m\in[m_c,1)$, or as $\tau\to T$ if $m\in(-\infty,m_c)$

Strategy of proof

Assume that $D=1$ and consider $d\mu_\alpha := h_\alpha\,dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$, with $\alpha = 1/(m-1) < 0$, and $\mathcal{L}_{\alpha,d} := -h_{1-\alpha}\,\mathrm{div}\,[\,h_\alpha\,\nabla\cdot\,]$ on $L^2(d\mu_\alpha)$:

$$\int_{\mathbb{R}^d} f\,(\mathcal{L}_{\alpha,d}\,f)\,d\mu_{\alpha-1} = \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu_\alpha$$

A first order expansion of $v(t,x) = h_\alpha(x)\left[1+\varepsilon\,f(t,x)\,h_\alpha^{1-m}(x)\right]$ solves

$$\frac{\partial f}{\partial t} + \mathcal{L}_{\alpha,d}\,f = 0$$

Theorem

[A. Blanchet, M. Bonforte, J.D., G. Grillo, J.-L. Vázquez] Let $d\geq 3$. For any $\alpha\in(-\infty,0)\setminus\{\alpha_*\}$, there is a positive constant $\Lambda_{\alpha,d}$ such that

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d}|f|^2\,d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d}|\nabla f|^2\,d\mu_\alpha \quad \forall\, f\in H^1(d\mu_\alpha)$$

under the additional condition $\int_{\mathbb{R}^d} f\,d\mu_{\alpha-1}=0$ if $\alpha<\alpha_*$

$$\Lambda_{\alpha,d} = \begin{cases} \frac{1}{4}\,(d-2+2\,\alpha)^2 & \text{if } \alpha\in\left[-\frac{d+2}{2},\alpha_*\right)\cup(\alpha_*,0) \\ -4\,\alpha-2\,d & \text{if } \alpha\in\left[-d,-\frac{d+2}{2}\right) \\ -2\,\alpha & \text{if } \alpha\in(-\infty,-d) \end{cases}$$

Relative entropy and Fisher information, interpolation

For $m \neq 0, 1$, the *relative entropy* of J. Ralston and W.I. Newmann and the *generalized relative Fisher information* are given by

$$\mathcal{F}[w] := \frac{m}{1-m} \int_{\mathbb{R}^d} \left[w - 1 - \frac{1}{m}\left(w^m - 1\right)\right] \, V_D^m \, dx$$

$$\mathcal{I}[w] := \int_{\mathbb{R}^d} \left| \frac{1}{m-1} \nabla \left[\left(w^{m-1} - 1\right) V_D^{m-1} \right] \right|^2 v \, dx$$

where $w = \frac{v}{V_D}$. If v is a solution of (2): $\frac{d}{dt}\mathcal{F}[w(t,\cdot)] = -\mathcal{I}[w(t,\cdot)]$

- **Linearization:** $f := (w-1)\, V_D^{m-1}$, $h := \max\{h_2, 1/h_1\}$, $h_1(t) := \inf w(t,\cdot)$, $h_2(t) := \sup w(t,\cdot)$. With $|im_{t\to\infty} h(t) = 1$

$$h^{m-2} \int_{\mathbb{R}^d} |f|^2 \, V_D^{2-m} \, dx \le \frac{2}{m} \mathcal{F}[w] \le h^{2-m} \int_{\mathbb{R}^d} |f|^2 \, V_D^{2-m} \, dx$$

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, V_D \, dx \le [1 + X(h)]\, \mathcal{I}[w] + Y(h) \int_{\mathbb{R}^d} |f|^2 \, V_D^{2-m} \, dx$$

where $h^{5-2m} =: 1 + X(h)$, $d\,(1-m)\left[h^{4(2-m)} - 1\right] =: Y(h)$ and $\lim_{h\to 1} X(h) = \lim_{h\to 1} Y(h) = 0$

Proof (continued)

- A new **interpolation** inequality: for $h > 0$ small enough

$$\mathcal{F}[w] \le \frac{h^{2-m}\,[1+X(h)]}{2\left[\Lambda_{\alpha,d} - m\,Y(h)\right]}\, m\,\mathcal{I}[w]$$

- Another **interpolation** allows to close the system of estimates: for some C, t large enough,

$$0 \le h-1 \le \mathsf{C}\,\mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$$

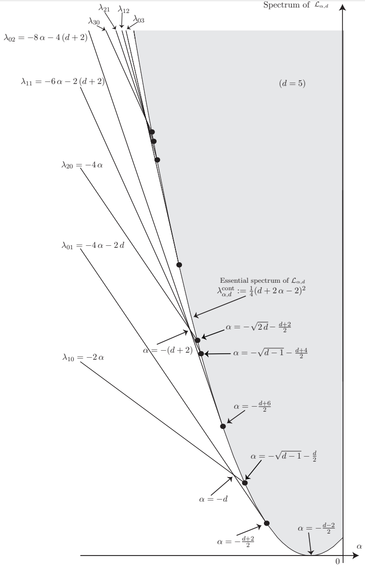
Hence we have a nonlinear differential inequality

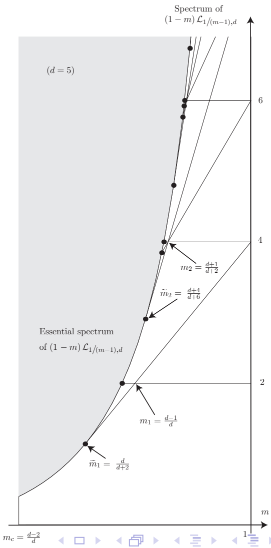
$$\frac{d}{dt}\mathcal{F}[w(t,\cdot)] \le -2\,\frac{\Lambda_{\alpha,d} - m\,Y(h)}{\left[1+X(h)\right]\,h^{2-m}}\,\mathcal{F}[w(t,\cdot)]$$

- A **Gronwall** lemma (take $h = 1 + \mathsf{C}\,\mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$) then shows that

$$\limsup_{t\to\infty} \mathrm{e}^{2\,\Lambda_{\alpha,d}\,t}\,\mathcal{F}[w(t,\cdot)] < +\infty$$

Plots ($d = 5$)


Spectrum of $\mathcal{L}_{\alpha,d}$
λ_{21}
λ_{12}
λ_{30}
λ_{03}
$\lambda_{02} = -8\,\alpha - 4\,(d+2)$
$\lambda_{11} = -6\,\alpha - 2\,(d+2)$
$(d = 5)$
$\lambda_{20} = -4\,\alpha$
$\lambda_{01} = -4\,\alpha - 2\,d$
Essential spectrum of $\mathcal{L}_{\alpha,d}$
$\lambda^{\rm cont}_{\alpha,d} := \frac{1}{4}(d + 2\,\alpha - 2)^2$
$\alpha = -\sqrt{2\,d} - \frac{d+2}{2}$
$\alpha = -(d+2)$
$\alpha = -\sqrt{d-1} - \frac{d+4}{2}$
$\lambda_{10} = -2\,\alpha$
$\alpha = -\frac{d+6}{2}$
$\alpha = -\sqrt{d-1} - \frac{d}{2}$
$\alpha = -d$
$\alpha = -\frac{d-2}{2}$
$\alpha = -\frac{d+2}{2}$
α
0


Spectrum of
$(1-m)\,\mathcal{L}_{1/(m-1),d}$
$(d = 5)$
6
4
$m_2 = \frac{d+1}{d+2}$
$\widetilde{m}_2 = \frac{d+4}{d+6}$
Essential spectrum
of $(1-m)\,\mathcal{L}_{1/(m-1),d}$
2
$m_1 = \frac{d-1}{d}$
$\widetilde{m}_1 = \frac{d}{d+2}$
m
1
$m_c = \frac{d-2}{d}$

Remarks, improvements

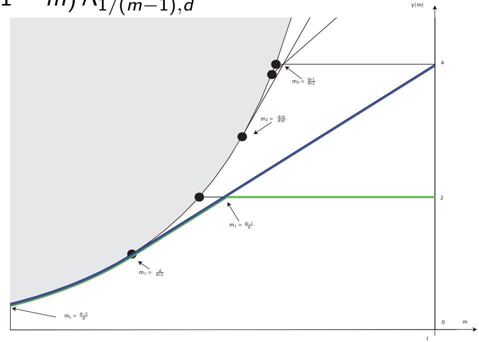
- Optimal constants in interpolation inequalities does not mean optimal *asymptotic* rates
- The critical case ($m = m_*$, $d \geq 3$): **Slow asymptotics** [Bonforte, Grillo, Vázquez] If $|v_0 - V_D|$ is bounded a.e. by a radial $L^1(dx)$ function, then there exists a positive constant C^* such that $\mathcal{E}[v(t,\cdot)] \leq C^*\, t^{-1/2}$ for any $t \geq 0$
- Can we improve the rates of convergence by imposing restrictions on the initial data ?
  - [Carrillo, Lederman, Markowich, Toscani (2002)] Poincaré inequalities for linearizations of very fast diffusion equations (radially symmetric solutions)
  - Formal or partial results: [Denzler, McCann (2005)], [McCann, Slepčev (2006)], [Denzler, Koch, McCann (announcement)],
- Faster convergence ?
  - Improved Hardy-Poincaré inequality: under the conditions $\int_{\mathbb{R}^d} f\, d\mu_{\alpha-1} = 0$ and $\int_{\mathbb{R}^d} x\, f\, d\mu_{\alpha-1} = 0$ (center of mas),
  $$\widetilde{\Lambda}_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha$$
  - Next ? Can we kill other linear modes ?

Improved asymptotic rates

[Bonforte, J.D., Grillo, Vázquez] Assume that $m \in (m_1, 1)$, $d \geq 3$. Under Assumption (H1), if v is a solution of (2) with initial datum v_0 such that $\int_{\mathbb{R}^d} x\, v_0\, dx = 0$ and if D is chosen so that $\int_{\mathbb{R}^d} (v_0 - V_D)\, dx = 0$, then

$$\mathcal{E}[v(t,\cdot)] \leq \widetilde{C}\, \mathrm{e}^{-\gamma(m)\, t} \quad \forall\, t \geq 0$$

with $\gamma(m) = (1-m)\, \widetilde{\Lambda}_{1/(m-1),d}$



Higher order matching asymptotics

For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1} \right) = 0$$

The strategy is easy to understand using a time-dependent rescaling and the relative entropy formalism. Define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t, x) \,, \quad R = R(\tau) \,, \quad t = \tfrac{1}{2} \log R \,, \quad x = \frac{y}{R}$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \, \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2\, x \right) \right] = 0 \quad t > 0 \,, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on R)

$$2\, \sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \frac{1}{\sigma} |x|^2 \right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d \tag{3}$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1} (v - B_\sigma) \right] dx$$

Let us briefly sketch the strategy of our method before giving all details

The time derivative of this relative entropy is

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt} \left(\frac{d}{d\sigma} \mathcal{F}_\sigma[v] \right)_{|\sigma=\sigma(t)}}_{\text{choose it } = 0} + \frac{m}{m-1} \int_{\mathbb{R}^d} \left(v^{m-1} - B_{\sigma(t)}^{m-1} \right) \frac{\partial v}{\partial t}\, dx$$

$$\Longleftrightarrow$$

Minimize $\mathcal{F}_\sigma[v]$ w.r.t. σ $\Longleftrightarrow$ $\int_{\mathbb{R}^d} |x|^2 B_\sigma \, dx = \int_{\mathbb{R}^d} |x|^2 v \, dx$

The entropy / entropy production estimate

According to the definition of B_σ, we know that
$2\,x = \sigma^{\frac{d}{2}(m-m_c)}\,\nabla B_\sigma^{m-1}$
Using the new change of variables, we know that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} v\left|\nabla\left[v^{m-1}-B_{\sigma(t)}^{m-1}\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[v] = \frac{m}{1-m}\int_{\mathbb{R}^d}\left[w-1-\frac{1}{m}\left(w^m-1\right)\right] B_\sigma^m\,dx$$

(Repeating) define the *relative Fisher information* by

$$\mathcal{I}_\sigma[v] := \int_{\mathbb{R}^d}\left|\frac{1}{m-1}\,\nabla\left[\left(w^{m-1}-1\right)B_\sigma^{m-1}\right]\right|^2 B_\sigma\,w\,dx$$

so that $\quad \frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\,m\,(1-m)\,\sigma(t)\,\mathcal{I}_{\sigma(t)}[v(t,\cdot)] \quad \forall\, t>0$

When linearizing, one more mode is killed and $\sigma(t)$ scales out

Improved rates of convergence

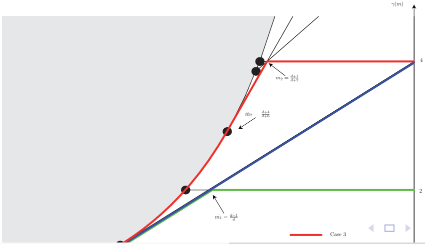
Theorem

Let $m \in (\widetilde{m}_1, 1)$, $d \geq 2$, $v_0 \in L^1_+(\mathbb{R}^d)$ *such that* v_0^m, $|y|^2\, v_0 \in L^1(\mathbb{R}^d)$

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,\gamma(m)\,t} \quad \forall\; t \geq 0$$

where

$$\gamma(m) = \begin{cases} \frac{((d-2)\,m-(d-4))^2}{4\,(1-m)} & \text{if } m \in (\widetilde{m}_1, \widetilde{m}_2] \\ 4\,(d+2)\,m - 4\,d & \text{if } m \in [\widetilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$



Gagliardo-Nirenberg inequalities: improvements

Gagliardo-Nirenberg inequalities: further improvements

A brief summary of the strategy for further improvements

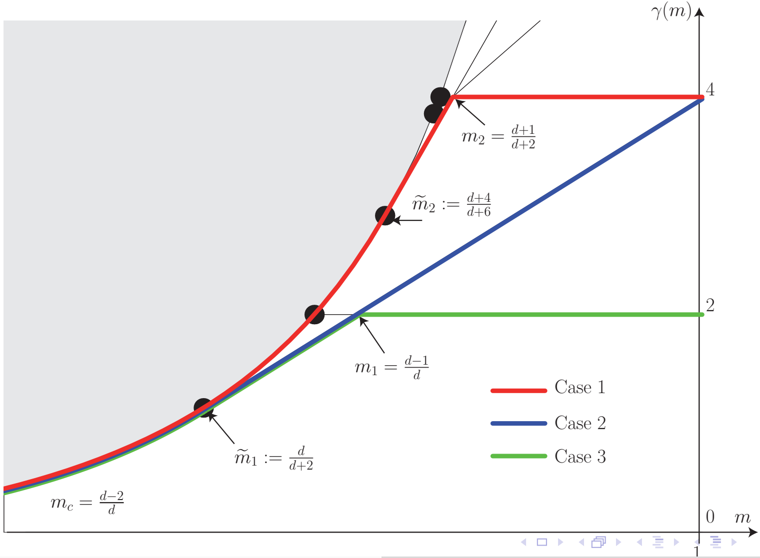
- In the basin of attraction of Barenblatt functions: improving the *asymptotic rates of convergence* for any m

$$\frac{\partial v}{\partial t}+\nabla\cdot\left(v\,\nabla v^{m-1}\right)=0\quad t>0\,,\quad x\in\mathbb{R}^d$$

  with $m\in(\frac{d-1}{d},1)$, $d\geq 3$

- The $\frac{1}{2}$ factor in the entropy - entropy production inequality can be explained by *spectral gap* considerations
- This factor can be improved for well prepared initial data, if $m>\frac{d-1}{d}$
- **Global improvements** can be obtained using rescalings which depend on the second moment, even for $m=\frac{d-1}{d}$

Spectral gaps and best constants



Best matching Barenblatt profiles

Consider the *fast diffusion equation*

$$\frac{\partial u}{\partial t}+\nabla\cdot\left[u\left(\sigma^{\frac{d}{2}(m-m_c)}\,\nabla u^{m-1}-2\,x\right)\right]=0\quad t>0\,,\quad x\in\mathbb{R}^d$$

with a nonlocal, time-dependent diffusion coefficient

$$\sigma(t)=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u(x,t)\,dx\,,\quad K_M:=\int_{\mathbb{R}^d}|x|^2\,B_1(x)\,dx$$

where

$$B_\lambda(x):=\lambda^{-\frac{d}{2}}\left(C_M+\tfrac{1}{\lambda}\,|x|^2\right)^{\frac{1}{m-1}}\quad\forall\,x\in\mathbb{R}^d$$

and define the relative entropy

$$\mathcal{F}_\lambda[u]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m-B_\lambda^m-m\,B_\lambda^{m-1}\,(u-B_\lambda)\right]\,dx$$

Three ingredients for *global improvements*

1. $\inf_{\lambda>0} \mathcal{F}_\lambda[u(x,t)] = \mathcal{F}_{\sigma(t)}[u(x,t)]$ so that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[u(x,t)] = -\mathcal{J}_{\sigma(t)}[u(\cdot,t)]$$

   where the relative Fisher information is

$$\mathcal{J}_\lambda[u] := \lambda^{\frac{d}{2}(m-m_c)} \frac{m}{1-m} \int_{\mathbb{R}^d} u \left|\nabla u^{m-1} - \nabla B_\lambda^{m-1}\right|^2 dx$$

2. In the *Bakry-Emery method*, there is an additional (good) term

$$4\left[1 + 2\, C_{m,d} \frac{\mathcal{F}_{\sigma(t)}[u(\cdot,t)]}{M^\gamma\, \sigma_0^{\frac{d}{2}(1-m)}}\right] \frac{d}{dt}\left(\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\right) \ge \frac{d}{dt}\left(\mathcal{J}_{\sigma(t)}[u(\cdot,t)]\right)$$

3. The *Csiszár-Kullback inequality* is also improved

$$\mathcal{F}_\sigma[u] \ge \frac{m}{8\int_{\mathbb{R}^d} B_1^m\, dx}\, C_M^2 \|u - B_\sigma\|^2_{\mathrm{L}^1(\mathbb{R}^d)}$$

An improved Gagliardo-Nirenberg inequality (1/2)

Relative entropy functional

$$\mathcal{R}^{(p)}[f] := \inf_{g \in \mathfrak{M}_d^{(p)}} \int_{\mathbb{R}^d} \left[g^{1-p} \left(|f|^{2\,p} - g^{2\,p} \right) - \tfrac{2\,p}{p+1} \left(|f|^{p+1} - g^{p+1} \right) \right] dx$$

Theorem

Let $d \geq 2$, $p > 1$ *and assume that* $p < d/(d-2)$ *if* $d \geq 3$. *If*

$$\frac{\int_{\mathbb{R}^d} |x|^2\,|f|^{2\,p}\,dx}{\left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx\right)^\gamma} = \frac{d\,(p-1)\,\sigma_*\,M_*^{\gamma-1}}{d+2-p\,(d-2)}\,,\quad \sigma_*(p) := \left(4\,\frac{d+2-p\,(d-2)}{(p-1)^2\,(p+1)}\right)^{\frac{4\,p}{d-p\,(d-4)}}$$

for any $f \in \mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$, *then we have*

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx - \mathsf{K}_{p,d} \left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx \right)^\gamma \geq \mathsf{C}_{p,d}\, \frac{\left(\mathcal{R}^{(p)}[f]\right)^2}{\left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx\right)^\gamma}$$

An improved Gagliardo-Nirenberg inequality (2/2)

A Csiszár-Kullback inequality

$$\mathcal{R}^{(p)}[f] \geq \mathsf{C}_{\mathrm{CK}}\, \|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p(\gamma-2)} \inf_{g \in \mathfrak{M}_d^{(p)}} \| |f|^{2p} - g^{2p} \|_{\mathrm{L}^1(\mathbb{R}^d)}^2$$

with $\mathsf{C}_{\mathrm{CK}} = \frac{p-1}{p+1}\,\frac{d+2-p\,(d-2)}{32\,p}\,\sigma_*^{\,d\,\frac{p-1}{4p}}\,M_*^{1-\gamma}$. Let

$$\mathfrak{C}_{p,d} := \mathsf{C}_{d,p}\,{\mathsf{C}_{\mathrm{CK}}}^2$$

Corollary

Under previous assumptions, we have

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx - \mathsf{K}_{p,d} \left(\int_{\mathbb{R}^d} |f|^{2p}\,dx \right)^{\gamma} \geq \mathfrak{C}_{p,d}\, \|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p(\gamma-4)} \inf_{g \in \mathfrak{M}_d(p)} \| |f|^{2p} - g^{2p} \|_{\mathrm{L}^1(\mathbb{R}^d)}^4$$

... but this is not all !

Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Outline

- Legendre duality
- Sobolev and HLS inequalities can be related using a nonlinear flow *compatible with Legendre's duality*
- The asymptotic behaviour close to the *vanishing time* is determined by a solution with *separation of variables* based on the Aubin-Talenti solution
- The vanishing time T can be estimated using a priori estimates
- The entropy H is negative, concave, and we can relate $\mathsf{H}(0)$ with $\mathsf{H}'(0)$ by integrating estimates on $(0, T)$, which provides *an improvement of Sobolev's inequality* if $d \geq 5$

Legendre duality

To a convex functional F, we may associate the functional F^* defined by Legendre's duality as

$$F^*[v] := \sup \left(\int_{\mathbb{R}^d} u\, v\, dx - F[u] \right)$$

- To $F_1[u] = \frac{1}{2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}$, we associate $F_1^*[v] = \frac{1}{2}\,\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}$ where p and q are Hölder conjugate exponents: $1/p + 1/q = 1$
- To $F_2[u] = \frac{1}{2}\,\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, we associate

$$F_2^*[v] = \frac{1}{2}\,\mathsf{S}_d^{-1} \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\, dx$$

where $(-\Delta)^{-1} v = G_d * v$, $G_d(x) = \frac{1}{d-2}\,|\mathbb{S}^{d-1}|^{-1}\,|x|^{2-d}$ if $d \geq 3$
As a straightforward consequence of Legendre's duality, if we have a functional inequality of the form $F_1[u] \leq F_2[u]$, then we have the dual inequality $F_1^*[v] \geq F_2^*[v]$

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall \, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{5}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx \quad \forall \, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \tag{6}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{7}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\; dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\; dx + \mathsf{S}_d \left(\int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\; dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\; dx$$

where $v = v(t,\cdot)$ is a solution of (7). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2d}{d+2}$

A first statement

Proposition

[J.D.] *Assume that $d \geq 3$ and $m = \frac{d-2}{d+2}$. If v is a solution of (7) with nonnegative initial datum in $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, then*

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}\right] = \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (5) if v is optimal for (6)

Improved Sobolev inequality

By integrating along the flow defined by (7), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (5), but only when $d \geq 5$ for integrability reasons

Theorem

[J.D.] *Assume that* $d \geq 5$ *and let* $q = \frac{d+2}{d-2}$*. There exists a positive constant* $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ *such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Solutions with *separation of variables*

Consider the solution vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}} \quad \forall\,(t,x) \in (0,T)\times\mathbb{R}^d$$

where $\alpha = (d+2)/4$, $c^{1-m} = 4\,m\,d$, $m = \frac{d-2}{d+2}$, $p = d/(d-2)$ and F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[M. delPino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodríguez]
For any solution v of (7) with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot) - 1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

A first set of *a priori* integral estimates

Let $\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$. Let $d \ge 3$ and $m = (d-2)/(d+2)$

Lemma

[J.D.] *If v is a solution of (7) vanishing at time $T > 0$ with $v_0 \in \mathrm{L}^{2^*}_+(\mathbb{R}^d)$*

$$\left(\tfrac{4\,(T-t)}{(d+2)\,\mathsf{S}_d}\right)^{\frac{d}{2}} \le \mathsf{J}(t) \le \mathsf{J}(0)\,, \quad \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mathsf{S}_d^{-1}\left(\tfrac{4\,(T-t)}{d+2}\right)^{\frac{d}{2}-1}$$

$$T \le \tfrac{1}{4}\,(d+2)\,\mathsf{S}_d\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{\frac{2}{d}}$$

for any $t \in (0,T)$. Moreover, if $d \ge 5$, we also have

$$\mathsf{J}(t) = \int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx \ge \int_{\mathbb{R}^d} v_0^{m+1}\,dx - \tfrac{2\,d}{d+2}\,t\,\|\nabla v_0^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

$$\|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla v_0^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

$$T \ge \frac{d+2}{2\,d}\int_{\mathbb{R}^d} v_0^{m+1}\,dx\,\|\nabla v_0^m\|^{-2}_{\mathrm{L}^2(\mathbb{R}^d)}$$

Proofs (1/2)

$\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$ satisfies

$$\mathsf{J}' = -(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}$$

If $d \ge 5$, then we also have

$$\mathsf{J}'' = 2\,m\,(m+1)\int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \ge 0$$

Such an estimate makes sense if $v = \overline{v}_T$. This is also true for any solution v as can be seen by rewriting the problem on $\mathbb{S}^d$: integrability conditions for v are exactly the same as for $\overline{v}_T$ □

Notice that

$$\frac{\mathsf{J}'}{\mathsf{J}} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{-\frac{2}{d}} \le -\kappa \quad \text{with} \quad \kappa := \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-\frac{2}{d}} \le \frac{d}{2\,T}$$

Proofs (2/2)

By the Cauchy-Schwarz inequality, we have

$$\|\nabla v^m\|^4_{\mathrm{L}^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} v^{(m-1)/2}\,\Delta v^m \cdot v^{(m+1)/2}\,dx\right)^2 \le \int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \int_{\mathbb{R}^d} v^{m+1}\,dx$$

so that $\mathsf{Q}(t) := \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx\right)^{-(d-2)/d}$ is monotone decreasing, and

$$\mathsf{H}' = 2\,\mathsf{J}\,(\mathsf{S}_d\,\mathsf{Q} - 1)\,, \quad \mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' + 2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}' \le \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' \le 0$$

$$\mathsf{H}'' \le -\kappa\,\mathsf{H}' \quad \text{with} \quad \kappa = \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-2/d}$$

By writing that $-\mathsf{H}(0) = \mathsf{H}(T) - \mathsf{H}(0) \le \mathsf{H}'(0)\,(1 - e^{-\kappa\,T})/\kappa$ and using the estimate $\kappa\,T \le d/2$, the proof is completed □

The two-dimensional case: Legendre duality

Onofri's inequality amounts to $F_1[u] \le F_2[u]$ with

$$F_1[u] := \log\left(\int_{\mathbb{R}^2} e^u \, d\mu\right) \quad \text{and} \quad F_2[u] := \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} u\,\mu\,dx$$

Proposition

[E. Carlen, M. Loss], [V. Calvez, L. Corrias] *For any* $v \in \mathrm{L}^1_+(\mathbb{R}^2)$ *with* $\int_{\mathbb{R}^2} v\,dx = 1$, *such that* $v \log v$ *and* $(1+\log|x|^2)\,v \in \mathrm{L}^1(\mathbb{R}^2)$, *we have*

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v \log\left(\frac{v}{\mu}\right) dx - 4\,\pi \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx \ge 0$$

Notice that $-\Delta \log \mu = 8\,\pi\,\mu$ can be inverted as

$$(-\Delta)^{-1}\mu = \frac{1}{8\,\pi}\log(\pi\,\mu)$$

The two-dimensional case: log HLS and...

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx$$

Assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta\log\left(\frac{v}{\mu}\right) \quad t>0\,, \quad x\in\mathbb{R}^2$$

Proposition

[J.D.] *If v is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\,dx = 1$, $v_0\,\log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu \in \mathrm{L}^1(\mathbb{R}^2)$, then*

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu$$

with $\log(v/\mu) = u/2$

The two-dimensional case: ...Onofri's inequality

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu$$

The right hand side is nonnegative by Onofri's inequality:

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] \geq \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \int_{\mathbb{R}^2}u\,d\mu - \log\left(\int_{\mathbb{R}^2}e^u\,d\mu\right) \geq 0$$

- If $\int_{\mathbb{R}^2} u\,d\mu = 1$, then

$$-\int_{\mathbb{R}^2}e^{\frac{u}{2}}u\,d\mu \geq -\log\left(\int_{\mathbb{R}^2}e^u\,d\mu\right)$$

- Corollary: for any $u \in \mathcal{D}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^2} e^{\frac{u}{2}}\,d\mu = 1$, we have

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \geq \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu$$

The two-dimensional case: the sphere setting

The image w of v by the inverse stereographic projection on the sphere $\mathbb{S}^2$, up to a scaling, solves the equation

$$\frac{\partial w}{\partial t} = \Delta_{\mathbb{S}^2} \log w \quad t > 0 \,, \quad y \in \mathbb{S}^2$$

More precisely, if $x = (x_1, x_2) \in \mathbb{R}^2$, then u and w are related by

$$w(t,y) = \frac{u(t,x)}{4\,\pi\,\mu(x)} \,, \quad y = \left(\tfrac{2\,(x_1,x_2)}{1+|x|^2}, \tfrac{1-|x|^2}{1+|x|^2} \right) \in \mathbb{S}^2$$

The loss of mass of the solution of

$$\frac{\partial v}{\partial t} = \Delta \log v \quad t > 0 \,, \quad x \in \mathbb{R}^2$$

is compensated in case of

$$\frac{\partial v}{\partial t} = \Delta \log \left(\frac{v}{\mu} \right) \quad t > 0 \,, \quad x \in \mathbb{R}^2$$

by the source term $-\Delta \log \mu$

Thank you for your attention !