Uniqueness and symmetry based on nonlinear flows

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A result of uniqueness on a classical example

On the sphere $\mathbb{S}^d$, let us consider the positive solutions of

$$-\Delta u + \lambda u = u^{p-1}$$

$p \in [1, 2) \cup (2, 2^*] \text{ if } d \geq 3, \quad 2^* = \frac{2d}{d-2}$

$p \in [1, 2) \cup (2, +\infty) \text{ if } d = 1, 2$

**Theorem**

If $\lambda \leq d$, $u \equiv \lambda^{1/(p-2)}$ is the unique solution

Bifurcation point of view

Figure: \((p - 2) \lambda \mapsto (p - 2) \mu(\lambda)\) with \(d = 3\)

\[
\|\nabla u\|_{L^2(S^d)}^2 + \lambda \|u\|_{L^2(S^d)}^2 \geq \mu(\lambda) \|u\|_{L^p(S^d)}^2
\]

Taylor expansion of \(u = 1 + \varepsilon \varphi_1\) as \(\varepsilon \to 0\) with \(-\Delta \varphi_1 = d \varphi_1\)

\[
\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > \frac{d}{p - 2}
\]

The inequality holds with \(\mu(\lambda) = \lambda = \frac{d}{p - 2}\) [Bakry & Emery, 1985] [Beckner, 1993], [Bidaut-Véron & Véron, 1991, Corollary 6.1]
Inequalities without weights and fast diffusion equations: optimality and uniqueness of the critical points

- The Bakry-Emery method (compact manifolds)
  - The Fokker-Planck equation
  - The Bakry-Emery method on the sphere: a parabolic method
  - The Moser-Trudinger-Onofri inequality (on a compact manifold)

- Fast diffusion equations on the Euclidean space (without weights)
  - Euclidean space: Rényi entropy powers
  - Euclidean space: self-similar variables and relative entropies
  - The role of the spectral gap
The Fokker-Planck equation

The linear Fokker-Planck (FP) equation

\[
\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla \phi)
\]

on a domain \( \Omega \subset \mathbb{R}^d \), with no-flux boundary conditions

\[(\nabla u + u \nabla \phi) \cdot \nu = 0 \quad \text{on} \quad \partial \Omega\]

is equivalent to the Ornstein-Uhlenbeck (OU) equation

\[
\frac{\partial v}{\partial t} = \Delta v - \nabla \phi \cdot \nabla v =: \mathcal{L} v
\]

(Bakry, Emery, 1985), (Arnold, Markowich, Toscani, Unterreiter, 2001)

With mass normalized to 1, the unique stationary solution of (FP) is

\[u_s = \frac{e^{-\phi}}{\int_{\Omega} e^{-\phi} \, dx} \quad \iff \quad v_s = 1\]
The Bakry-Emery method

With \( d\gamma = u_s \, dx \) and \( \nu \) such that \( \int_{\Omega} \nu \, d\gamma = 1 \), \( q \in (1, 2] \), the \( q \)-entropy is defined by

\[
\mathcal{E}_q[\nu] := \frac{1}{q - 1} \int_{\Omega} (\nu^q - 1 - q(\nu - 1)) \, d\gamma
\]

Under the action of (OU), with \( w = \nu^{q/2} \), \( \mathcal{I}_q[\nu] := \frac{4}{q} \int_{\Omega} |\nabla w|^2 \, d\gamma \),

\[
\frac{d}{dt} \mathcal{E}_q[\nu(t, \cdot)] = -\mathcal{I}_q[\nu(t, \cdot)] \quad \text{and} \quad \frac{d}{dt} \left( \mathcal{I}_q[\nu] - 2\lambda \mathcal{E}_q[\nu] \right) \leq 0
\]

with \( \lambda := \inf_{w \in H^1(\Omega, d\gamma) \setminus \{0\}} \frac{\int_{\Omega} \left( 2 \frac{q-1}{q} \|\text{Hess } w\|^2 + \text{Hess } \phi : \nabla w \otimes \nabla w \right) \, d\gamma}{\int_{\Omega} |\nabla w|^2 \, d\gamma} \).

Proposition

*(Bakry, Emery, 1984) (JD, Nazaret, Savaré, 2008)* Let \( \Omega \) be convex. If \( \lambda > 0 \) and \( \nu \) is a solution of (OU), then \( \mathcal{I}_q[\nu(t, \cdot)] \leq \mathcal{I}_q[\nu(0, \cdot)] e^{-2\lambda t} \) and \( \mathcal{E}_q[\nu(t, \cdot)] \leq \mathcal{E}_q[\nu(0, \cdot)] e^{-2\lambda t} \) for any \( t \geq 0 \) and, as a consequence,

\[
\mathcal{I}_q[\nu] \geq 2\lambda \mathcal{E}_q[\nu] \quad \forall \nu \in H^1(\Omega, d\gamma)
\]
A proof of the interpolation inequality by the \textit{carré du champ} method

\[ \|\nabla u\|^2_{L^2(S^d)} \geq \frac{d}{p-2} \left( \|u\|^2_{L^p(S^d)} - \|u\|^2_{L^2(S^d)} \right) \quad \forall u \in H^1(S^d) \]

\[ p \in [1, 2) \cup (2, 2^*) \text{ if } d \geq 3, \quad 2^* = \frac{2d}{d-2} \]

\[ p \in [1, 2) \cup (2, +\infty) \text{ if } d = 1, 2 \]
The Bakry-Emery method on the sphere

Entropy functional

\[
E_p[\rho] := \frac{1}{p-2} \left[ \int_{\mathbb{S}^d} \rho^2 d\mu - \left( \int_{\mathbb{S}^d} \rho \, d\mu \right)^{\frac{2}{p}} \right] \quad \text{if} \quad p \neq 2
\]
\[
E_2[\rho] := \int_{\mathbb{S}^d} \rho \log \left( \frac{\rho}{\|\rho\|_{L^1(\mathbb{S}^d)}} \right) \, d\mu
\]

Fisher information functional

\[
I_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho_\frac{1}{p}|^2 \, d\mu
\]

[Bakry & Emery, 1985] carré du champ method: use the heat flow

\[
\frac{\partial \rho}{\partial t} = \Delta \rho
\]

and observe that \( \frac{d}{dt} E_p[\rho] = - I_p[\rho] \),

\[
\frac{d}{dt} \left( I_p[\rho] - d \, E_p[\rho] \right) \leq 0 \quad \Rightarrow \quad I_p[\rho] \geq d \, E_p[\rho]
\]

with \( \rho = |u|^p \), if \( p \leq 2^\# := \frac{2d^2+1}{(d-1)^2} \)
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

(Demange), (JD, Esteban, Kowalczyk, Loss): for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt} \left( I_p[\rho] - d \mathcal{E}_p[\rho] \right) \leq 0$$

$(p, m)$ admissible region, $d = 5$
Cylindrical coordinates, Schwarz symmetrization, stereographic projection...
... and the ultra-spherical operator

Change of variables $z = \cos \theta$, $\nu(\theta) = f(z)$, $d\nu_d := \nu^{d-1} d z / Z_d$, $\nu(z) := 1 - z^2$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L} f := (1 - z^2) f'' - d z f' = \nu f'' + \frac{d}{2} \nu' f'$$

which satisfies $\langle f_1, \mathcal{L} f_2 \rangle = -\int_{-1}^{1} f'_1 f'_2 \nu \, d\nu_d$

**Proposition**

*Let $p \in [1, 2) \cup (2, 2^*], d \geq 1$. For any $f \in H^1([-1, 1], d\nu_d)$,*

$$-\langle f, \mathcal{L} f \rangle = \int_{-1}^{1} |f'|^2 \nu \, d\nu_d \geq d \frac{\|f\|_{L^p(S^d)}^2 - \|f\|_{L^2(S^d)}^2}{p - 2}$$
The heat equation \( \frac{\partial g}{\partial t} = \mathcal{L} g \) for \( g = f^p \) can be rewritten in terms of \( f \) as

\[
\frac{\partial f}{\partial t} = \mathcal{L} f + (p - 1) \frac{|f'|^2}{f} \nu
\]

\[
- \frac{1}{2} \frac{d}{dt} \int_{-1}^{1} |f'|^2 \nu \, d\nu_d = \frac{1}{2} \frac{d}{dt} \langle f, \mathcal{L} f \rangle = \langle \mathcal{L} f, \mathcal{L} f \rangle + (p - 1) \left\langle \frac{|f'|^2}{f} \nu, \mathcal{L} f \right\rangle
\]

\[
\frac{d}{dt} I[g(t, \cdot)] + 2 d I[g(t, \cdot)] = \frac{d}{dt} \int_{-1}^{1} |f'|^2 \nu \, d\nu_d + 2 d \int_{-1}^{1} |f'|^2 \nu \, d\nu_d
\]

\[
= -2 \int_{-1}^{1} \left( |f''|^2 + (p - 1) \frac{d}{d + 2} \frac{|f'|^4}{f^2} - 2 (p - 1) \frac{d - 1}{d + 2} \frac{|f'|^2 f''}{f} \right) \nu^2 \, d\nu_d
\]

is nonpositive if

\[
|f''|^2 + (p - 1) \frac{d}{d + 2} \frac{|f'|^4}{f^2} - 2 (p - 1) \frac{d - 1}{d + 2} \frac{|f'|^2 f''}{f}
\]

is pointwise nonnegative, which is granted if

\[
\left[ (p - 1) \frac{d - 1}{d + 2} \right]^2 \leq (p - 1) \frac{d}{d + 2} \iff p \leq \frac{2 d^2 + 1}{(d - 1)^2} = 2^\# < \frac{2 d}{d - 2} = 2^*
\]
The elliptic point of view (nonlinear flow)

\[
\frac{\partial u}{\partial t} = u^{2-2\beta} \left( \mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right), \quad \kappa = \beta (p - 2) + 1
\]

\[
- \mathcal{L} u - (\beta - 1) \frac{|u'|^2}{u} \nu + \frac{\lambda}{p - 2} u = \frac{\lambda}{p - 2} u^\kappa
\]

Multiply by \( \mathcal{L} u \) and integrate

\[
\ldots \int_{-1}^{1} \mathcal{L} u \, u^\kappa \, d\nu_d = - \kappa \int_{-1}^{1} u^\kappa \frac{|u'|^2}{u} \, d\nu_d
\]

Multiply by \( \kappa \frac{|u'|^2}{u} \) and integrate

\[
\ldots = + \kappa \int_{-1}^{1} u^\kappa \frac{|u'|^2}{u} \, d\nu_d
\]

The two terms cancel and we are left only with

\[
\int_{-1}^{1} \left| \frac{u''}{6-p} - \frac{p+2}{6-p} \frac{|u'|^2}{u} \right|^2 \nu^2 \, d\nu_d = 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}
\]
The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban

Extension to compact Riemannian manifolds of dimension 2...
We shall also denote by $\mathcal{R}$ the Ricci tensor, by $H_g u$ the Hessian of $u$ and by

$$L_g u := H_g u - \frac{g}{d} \Delta_g u$$

the trace free Hessian. Let us denote by $M_g u$ the trace free tensor

$$M_g u := \nabla u \otimes \nabla u - \frac{g}{d} |\nabla u|^2$$

We define

$$\lambda_* := \inf_{u \in \mathcal{H}^2(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \left[ \| L_g u - \frac{1}{2} M_g u \|^2 + \mathcal{R}(\nabla u, \nabla u) \right] e^{-u/2} d\nu_g}{\int_{\Omega} |\nabla u|^2 e^{-u/2} d\nu_g}$$
**Theorem**

Assume that $d = 2$ and $\lambda_* > 0$. If $u$ is a smooth solution to

$$-rac{1}{2} \Delta_g u + \lambda = e^u$$

then $u$ is a constant function if $\lambda \in (0, \lambda_*)$.

The Moser-Trudinger-Onofri inequality on $\mathcal{M}$

$$\frac{1}{4} \|\nabla u\|^2_{L^2(\mathcal{M})} + \lambda \int_{\mathcal{M}} u \, d\nu_g \geq \lambda \log \left( \int_{\mathcal{M}} e^u \, d\nu_g \right) \quad \forall u \in H^1(\mathcal{M})$$

for some constant $\lambda > 0$. Let us denote by $\lambda_1$ the first positive eigenvalue of $-\Delta_g$.

**Corollary**

If $d = 2$, then the MTO inequality holds with $\lambda = \Lambda := \min\{4 \pi, \lambda_*\}$. Moreover, if $\Lambda$ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than $\Lambda$. 
The flow

\[ \frac{\partial f}{\partial t} = \Delta_g (e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2} \]

\[
G_\lambda[f] := \int_M \| L_g f - \frac{1}{2} M_g f \|^2 e^{-f/2} \, d\nu_g + \int_M \mathcal{R}(\nabla f, \nabla f) e^{-f/2} \, d\nu_g \\
- \lambda \int_M |\nabla f|^2 e^{-f/2} \, d\nu_g
\]

Then for any \( \lambda \leq \lambda_* \) we have

\[
\frac{d}{dt} \mathcal{F}_\lambda[f(t, \cdot)] = \int_M (-\frac{1}{2} \Delta_g f + \lambda) \left( \Delta_g (e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2} \right) \, d\nu_g \\
= -G_\lambda[f(t, \cdot)]
\]

Since \( \mathcal{F}_\lambda \) is nonnegative and \( \lim_{t \to \infty} \mathcal{F}_\lambda[f(t, \cdot)] = 0 \), we obtain that

\[
\mathcal{F}_\lambda[u] \geq \int_0^\infty G_\lambda[f(t, \cdot)] \, dt
\]
Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

On the Euclidean space $\mathbb{R}^2$, given a general probability measure $\mu$ does the inequality

$$
\frac{1}{16}\pi \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq \lambda \left[ \log \left( \int_{\mathbb{R}^d} e^u \, d\mu \right) - \int_{\mathbb{R}^d} u \, d\mu \right]
$$

hold for some $\lambda > 0$? Let

$$
\Lambda_* := \inf_{x \in \mathbb{R}^2} \frac{-\Delta \log \mu}{8 \pi \mu}
$$

**Theorem**

Assume that $\mu$ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda < \Lambda_*$ and the inequality holds with $\lambda = \Lambda_*$ if equality is achieved among radial functions.
Euclidean space: Rényi entropy powers and fast diffusion

- The Euclidean space without weights

- Rényi entropy powers, the entropy approach without rescaling: (Savaré, Toscani): scalings, nonlinearity and a concavity property inspired by information theory
The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial v}{\partial t} = \Delta v^m$$

with initial datum $v(x, t = 0) = v_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} v_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 v_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$U^*(t, x) := \frac{1}{(\kappa t^{1/\mu})^d} B^*\left(\frac{x}{\kappa t^{1/\mu}}\right)$$

where

$$\mu := 2 + d(m - 1), \quad \kappa := \left|\frac{2 \mu m}{m - 1}\right|^{1/\mu}$$

and $B^*$ is the Barenblatt profile

$$B^*_*(x) := \begin{cases} 
(C^*_* - |x|^2)^{1/(m-1)} & \text{if } m > 1 \\
(C^*_* + |x|^2)^{1/(m-1)} & \text{if } m < 1
\end{cases}$$

J. Dolbeault

Uniqueness and symmetry
The Rényi entropy power $F$

The *entropy* is defined by

$$E := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$I := \int_{\mathbb{R}^d} v \, |\nabla p|^2 \, dx \quad \text{with} \quad p = \frac{m}{m - 1} v^{m - 1}$$

If $v$ solves the fast diffusion equation, then

$$E' = (1 - m) I$$

To compute $I'$, we will use the fact that

$$\frac{\partial p}{\partial t} = (m - 1) p \Delta p + |\nabla p|^2$$

$$F := E^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d (1 - m)} = 1 + \frac{2}{1 - m} \left( \frac{1}{d} + m - 1 \right) = \frac{2}{d} \frac{1}{1 - m} - 1$$

has a linear growth asymptotically as $t \to +\infty$
The variation of the Fisher information

Lemma

If $v$ solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $1 - \frac{1}{d} \leq m < 1$, then

$$I' = \frac{d}{dt} \int_{\mathbb{R}^d} v |\nabla p|^2 \, dx = -2 \int_{\mathbb{R}^d} v^m \left( \|D^2 p\|^2 + (m - 1) (\Delta p)^2 \right) \, dx$$

Explicit arithmetic geometric inequality

$$\|D^2 p\|^2 - \frac{1}{d} (\Delta p)^2 = \left\| D^2 p - \frac{1}{d} \Delta p \text{Id} \right\|^2$$

.... there are no boundary terms in the integrations by parts?
The concavity property

**Theorem**

[Toscani-Savaré] Assume that \( m \geq 1 - \frac{1}{d} \) if \( d > 1 \) and \( m > 0 \) if \( d = 1 \). Then \( F(t) \) is increasing, \((1 - m) F''(t) \leq 0\) and

\[
\lim_{t \to +\infty} \frac{1}{t} F(t) = (1 - m) \sigma \lim_{t \to +\infty} E^{\sigma - 1} I = (1 - m) \sigma E_*^{\sigma - 1} I_\star
\]

[Dolbeault-Toscani] The inequality

\[
E^{\sigma - 1} I \geq E_*^{\sigma - 1} I_\star
\]

is equivalent to the Gagliardo-Nirenberg inequality

\[
\| \nabla w \|^{\theta}_{L^2(\mathbb{R}^d)} \| w \|^{1 - \theta}_{L^{q + 1}(\mathbb{R}^d)} \geq C_{GN} \| w \|_{L^2(\mathbb{R}^d)}
\]

if \( 1 - \frac{1}{d} \leq m < 1 \). Hint: \( \nu^{m-1/2} = \frac{w}{\| w \|_{L^2(\mathbb{R}^d)}}, q = \frac{1}{2m-1} \)
Euclidean space: self-similar variables and relative entropies

In the Euclidean space, it is possible to characterize the optimal constants using a spectral gap property
Self-similar variables and relative entropies

The large time behavior of the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ is governed by the source-type Barenblatt solutions

$$v_*(t, x) := \frac{1}{\kappa^d(\mu t)^d/\mu} B_*(\frac{x}{\kappa(\mu t)^{1/\mu}}) \quad \text{where} \quad \mu := 2 + d(m - 1)$$

where $B_*$ is the Barenblatt profile (with appropriate mass)

$$B_*(x) := (1 + |x|^2)^{1/(m-1)}$$

A time-dependent rescaling: self-similar variables

$$v(t, x) = \frac{1}{\kappa^d R^d} u(\tau, \frac{x}{\kappa R}) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \frac{1}{2} \log \left( \frac{R(t)}{R_0} \right)$$

Then the function $u$ solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[ u \left( \nabla u^{m-1} - 2x \right) \right] = 0$$
Free energy and Fisher information

The function $u$ solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[ u \left( \nabla u^{m-1} - 2x \right) \right] = 0$$

(Ralston, Newman, 1984) Lyapunov functional:

*Generalized entropy or Free energy*

$$\mathcal{E}[u] := \int_{\mathbb{R}^d} \left( - \frac{u^m}{m} + |x|^2 u \right) \, dx - \mathcal{E}_0$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt} \mathcal{E}[u] = - \mathcal{I}[u] , \quad \mathcal{I}[u] := \int_{\mathbb{R}^d} u \left| \nabla u^{m-1} + 2x \right|^2 \, dx$$
Without weights: relative entropy, entropy production

**Stationary solution:** choose $C$ such that $\|u_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$u_\infty(x) := (C + |x|^2)^{-1/(1-m)}$$

**Entropy – entropy production inequality** (del Pino, JD)

**Theorem**

$$d \geq 3, \ m \in \left[\frac{d-1}{d}, +\infty\right), \ m > \frac{1}{2}, \ m \neq 1$$

$$\mathcal{I}[u] \geq 4 \mathcal{E}[u]$$

$$p = \frac{1}{2m-1}, \ u = w^{2p}: \ (GN) \ \|\nabla w\|_{L^2(\mathbb{R}^d)}^{\theta} \|w\|_{L^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq C_{GN} \|w\|_{L^2(\mathbb{R}^d)}$$

**Corollary**

(del Pino, JD) A solution $u$ with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2 u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\mathcal{E}[u(t, \cdot)] \leq \mathcal{E}[u_0] e^{-4t}$
A computation on a large ball, with boundary terms

\[ \frac{\partial u}{\partial \tau} + \nabla \cdot \left[ u \left( \nabla u^{m-1} - 2x \right) \right] = 0 \quad \tau > 0, \quad x \in B_R \]

where \( B_R \) is a centered ball in \( \mathbb{R}^d \) with radius \( R > 0 \), and assume that \( u \) satisfies zero-flux boundary conditions

\[ \left( \nabla u^{m-1} - 2x \right) \cdot \frac{x}{|x|} = 0 \quad \tau > 0, \quad x \in \partial B_R. \]

With \( z(\tau, x) := \nabla Q(\tau, x) := \nabla u^{m-1} - 2x \), the relative Fisher information is such that

\[ \frac{d}{d\tau} \int_{B_R} u |z|^2 \, dx + 4 \int_{B_R} u |z|^2 \, dx + 2 \frac{1-m}{m} \int_{B_R} u^m \left( \|D^2Q\|^2 - (1 - m)(\Delta Q)^2 \right) \, dx \]

\[ = \int_{\partial B_R} u^m (\omega \cdot \nabla |z|^2) \, d\sigma \leq 0 \quad (\text{by Grisvard’s lemma}) \]
Spectral gap: sharp asymptotic rates of convergence

Assumptions on the initial datum \( v_0 \)

\[(H1) \quad V_{D_0} \leq v_0 \leq V_{D_1} \text{ for some } D_0 > D_1 > 0 \]

\[(H2) \quad \text{if } d \geq 3 \text{ and } m \leq m_\ast, (v_0 - V_D) \text{ is integrable for a suitable } D \in [D_1, D_0] \]

**Theorem**

(Blanchet, Bonforte, JD, Grillo, Vázquez) Under Assumptions \((H1)-(H2)\), if \( m < 1 \) and \( m \neq m_\ast := \frac{d-4}{d-2} \), the entropy decays according to

\[ \mathcal{E}[v(t, \cdot)] \leq C e^{-2(1-m)} \Lambda_{\alpha,d} t \quad \forall \ t \geq 0 \]

where \( \Lambda_{\alpha,d} > 0 \) is the best constant in the Hardy–Poincaré inequality

\[ \Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 d\mu_\alpha \quad \forall \ f \in H^1(d\mu_\alpha), \int_{\mathbb{R}^d} f d\mu_{\alpha-1} = 0 \]

with \( \alpha := 1/(m - 1) < 0, \ d\mu_\alpha := h_\alpha \, dx, \ h_\alpha(x) := (1 + |x|^2)^\alpha \)
Spectral gap and best constants

\[ m_c = \frac{d-2}{d} \]

\[ m_1 = \frac{d-1}{d} \]

\[ \tilde{m}_1 := \frac{d}{d+2} \]

\[ m_2 = \frac{d+1}{d+2} \]

\[ \tilde{m}_2 := \frac{d+4}{d+6} \]

\[ \gamma(m) \]

Case 1
Case 2
Case 3
Caffarelli-Kohn-Nirenberg, symmetry and symmetry breaking results, and weighted nonlinear flows

- The critical Caffarelli-Kohn-Nirenberg inequality
  [JD, Esteban, Loss]

- A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities
  [JD, Esteban, Loss, Muratori]

- Large time asymptotics and spectral gaps

- Optimality cases
Linear and nonlinear flows: entropy methods
CKN inequalities, symmetry breaking and weighted nonlinear flows

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in L^p (\mathbb{R}^d, |x|^{-b} \, dx) : |x|^{-a} |\nabla v| \in L^2 (\mathbb{R}^d, dx) \right\}$

$$\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} \, dx \quad \forall v \in \mathcal{D}_{a,b}$$

holds under conditions on $a$ and $b$

$$p = \frac{2 \, d}{d - 2 + 2(b-a)} \quad \text{(critical case)}$$

▷ An optimal function among radial functions:

$$v_\star (x) = \left( 1 + |x|^{(p-2)(a_c - a)} \right)^{-\frac{2}{p-2}} \quad \text{and} \quad C_{a,b}^* = \frac{\| |x|^{-b} v_\star \|^2_p}{\| |x|^{-a} \nabla v_\star \|^2_2}$$

Question: $C_{a,b} = C_{a,b}^*$ (symmetry) or $C_{a,b} > C_{a,b}^*$ (symmetry breaking)?
Critical CKN: range of the parameters

Figure: $d = 3$

$$\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|
abla v|^2}{|x|^{2a}} \, dx$$

$a \leq b \leq a + 1$ if $d \geq 3$

$a < b \leq a + 1$ if $d = 2$, $a + 1/2 < b \leq a + 1$ if $d = 1$

and $a < a_c := (d - 2)/2$

$$p = \frac{2d}{d - 2 + 2(b - a)}$$

(Glaser, Martin, Grosse, Thirring (1976))
(Caffarelli, Kohn, Nirenberg (1984))
[F. Catrina, Z.-Q. Wang (2001)]
Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{FS}(a) := \frac{d(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$

[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]

The functional

$$C^\star_{a,b} \int_{\mathbb{R}^d} \frac{\left|\nabla v\right|^2}{|x|^{2a}} \ dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} \ dx\right)^{2/p}$$

is linearly instable at $$v = v^\star$$.
Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (2016)]

**Theorem**

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{FS}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric.
The Emden-Fowler transformation and the cylinder

With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder

\[ v(r, \omega) = r^{a-c} \varphi(s, \omega) \quad \text{with} \quad r = |x|, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r} \]

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the subcritical interpolation inequality

\[ \| \partial_s \varphi \|^2_{L^2(C)} + \| \nabla \omega \varphi \|^2_{L^2(C)} + \Lambda \| \varphi \|^2_{L^2(C)} \geq \mu(\Lambda) \| \varphi \|^2_{L^p(C)} \quad \forall \varphi \in H^1(C) \]

where \( \Lambda := (a_c - a)^2 \), \( C = \mathbb{R} \times S^{d-1} \) and the optimal constant \( \mu(\Lambda) \) is

\[ \mu(\Lambda) = \frac{1}{C_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda} \]
Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_*(s, \omega) = (\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$H^1(C) \ni \varphi \mapsto \|\partial_s \varphi\|_{L^2(C)}^2 + \|\nabla \omega \varphi\|_{L^2(C)}^2 + \Lambda \|\varphi\|_{L^2(C)}^2$$

under a constraint on $\|\varphi\|_{L^p(C)}^2$

$\varphi_*$ is not optimal for (CKN) if the Pöschl-Teller operator

$$-\partial_s^2 - \Delta \omega + \Lambda - \varphi_*^{p-2} = -\partial_s^2 - \Delta \omega + \Lambda - \frac{1}{(\cosh s)^2}$$

has a negative eigenvalue, i.e., for $\Lambda > \Lambda_1$ (explicit)
The variational problem on the cylinder

\[ \Lambda \mapsto \mu(\Lambda) := \min_{\varphi \in H^1(C)} \frac{\| \partial_s \varphi \|_{L^2(C)}^2 + \| \nabla \omega \varphi \|_{L^2(C)}^2 + \Lambda \| \varphi \|_{L^2(C)}^2}{\| \varphi \|_{L^p(C)}^2} \]

is a concave increasing function.

Restricted to symmetric functions, the variational problem becomes

\[ \mu_*(\Lambda) := \min_{\varphi \in H^1(\mathbb{R})} \frac{\| \partial_s \varphi \|_{L^2(\mathbb{R}^d)}^2 + \Lambda \| \varphi \|_{L^2(\mathbb{R}^d)}^2}{\| \varphi \|_{L^p(\mathbb{R}^d)}^2} = \mu_*(1) \Lambda^\alpha \]

Symmetry means \( \mu(\Lambda) = \mu_*(\Lambda) \)
Symmetry breaking means \( \mu(\Lambda) < \mu_*(\Lambda) \)
Parametric plot of the branch of optimal functions for $p = 2.8, \, d = 5$. Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point $\Lambda_1$ computed by V. Felli and M. Schneider. The branch behaves for large values of $\Lambda$ as predicted by F. Catrina and Z.-Q. Wang.
When the local criterion (linear stability) differs from global results in a larger family of inequalities (center, right)...
The uniqueness result and the strategy of the proof
The elliptic problem: rigidity

The symmetry issue can be reformulated as a uniqueness (rigidity) issue. An optimal function for the inequality

\[ \left( \int_\mathbb{R}^d \frac{|v|^p}{|x|^{bp}} \, dx \right)^{2/p} \leq C_{a,b} \int_\mathbb{R}^d \frac{|
abla v|^2}{|x|^{2a}} \, dx \]

solves the (elliptic) Euler-Lagrange equation

\[ -\nabla \cdot (|x|^{-2a} \nabla v) = |x|^{-bp} v^{p-1} \]

(up to a scaling and a multiplication by a constant). Is any nonnegative solution of such an equation equal to

\[ v_*(x) = (1 + |x|^{(p-2)(a_c-a)})^{-\frac{2}{p-2}} \]

(up to invariances)? On the cylinder

\[ -\partial_s^2 \varphi - \partial_\omega \varphi + \Lambda \varphi = \varphi^{p-1} \]

Up to a normalization and a scaling

\[ \varphi_*(s, \omega) = (\cosh s)^{-\frac{1}{p-2}} \]
Symmetry in one slide: 3 steps

- A change of variables: $v(|x|^{\alpha-1} x) = w(x)$, $D_\alpha v = (\alpha \frac{\partial v}{\partial s}, \frac{1}{s} \nabla_\omega v)$

\[
\|v\|_{L^{2p,d-n}(\mathbb{R}^d)} \leq K_{\alpha,n,p} \|D_\alpha v\|_{L^{2,d-n}(\mathbb{R}^d)}^{\frac{\gamma}{2}} \|v\|_{L^{p+1,d-n}(\mathbb{R}^d)}^{1-\frac{\gamma}{2}} \quad \forall \ v \in H^{p}_{d-n,d-n}(\mathbb{R}^d)
\]

- Concavity of the Rényi entropy power: with

\[
L_{\alpha} = -D_\alpha^* D_\alpha = \alpha^2 \left( u'' + \frac{n-1}{s} u' \right) + \frac{1}{s^2} \Delta_\omega u \quad \text{and} \quad \frac{\partial u}{\partial t} = L_{\alpha} u^m
\]

\[
- \frac{d}{dt} G[u(t, \cdot)] \left( \int_{\mathbb{R}^d} u^m \, d\mu \right)^{1-\sigma} \geq (1 - m) (\sigma - 1) \int_{\mathbb{R}^d} u^m \left| L_{\alpha} P - \frac{\int_{\mathbb{R}^d} u |D_\alpha P|^2 \, d\mu}{\int_{\mathbb{R}^d} u^m \, d\mu} \right|^2 \, d\mu
\]

\[
+ 2 \int_{\mathbb{R}^d} \left( \alpha^4 \left( 1 - \frac{1}{n} \right) \left| P'' - \frac{P'}{s} - \frac{\Delta_\omega P}{\alpha^2 (n-1) s^2} \right|^2 + \frac{2 \alpha^2}{s^2} \left| \nabla_\omega P' - \frac{\nabla_\omega P}{s} \right|^2 \right) u^m \, d\mu
\]

\[
+ 2 \int_{\mathbb{R}^d} \left( (n-2) \left( \alpha_{FS}^2 - \alpha^2 \right) \left| \nabla_\omega P \right|^2 + c(n, m, d) \frac{|\nabla_\omega P|^4}{P^2} \right) u^m \, d\mu
\]

- Elliptic regularity and the Emden-Fowler transformation: justifying the integrations by parts

J. Dolbeault

Uniqueness and symmetry
Proof of symmetry (1/3: changing the dimension)

We rephrase our problem in a space of higher, artificial dimension \( n > d \) (here \( n \) is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight \( |x|^{n-d} \) which is the same in all norms. With \( \beta = 2a \) and \( \gamma = b \),

\[
\nu(|x|^{\alpha-1} x) = w(x), \quad \alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2 \frac{d - \gamma}{\beta + 2 - \gamma}
\]

we claim that Inequality (CKN) can be rewritten for a function \( \nu(|x|^{\alpha-1} x) = w(x) \) as

\[
\|\nu\|_{L^{2p,d-n}(\mathbb{R}^d)} \leq K_{\alpha,n,p} \|D_{\alpha} \nu\|_{L^{2,d-n}(\mathbb{R}^d)} \|\nu\|_{L^{p+1,d-n}(\mathbb{R}^d)}^{1-q} \quad \forall \nu \in H^{p}_{d-n,d-n}(\mathbb{R}^d)
\]

with the notations \( s = |x|, \quad D_{\alpha} \nu = (\alpha \frac{\partial \nu}{\partial s}, \frac{1}{s} \nabla \omega \nu) \) and

\[
d \geq 2, \quad \alpha > 0, \quad n > d \quad \text{and} \quad p \in (1, p_{\ast}]
\]

By our change of variables, \( w_{\ast} \) is changed into

\[
\nu_{\ast}(x) := (1 + |x|^2)^{-1/(p-1)} \quad \forall x \in \mathbb{R}^d
\]
The derivative of the generalized Rényi entropy power functional is

\[ G[u] := \left( \int_{\mathbb{R}^d} u^m \, d\mu \right)^{\sigma^{-1}} \int_{\mathbb{R}^d} u |D_\alpha P|^2 \, d\mu \]

where \( \sigma = \frac{2}{d} \frac{1}{1-m} - 1 \). Here \( d\mu = |x|^{n-d} \, dx \) and the pressure is

\[ P := \frac{m}{1-m} u^{m-1} \]

Looking for an optimal function in (CKN) is equivalent to minimize \( G \) under a mass constraint.
With $L_\alpha = -D_\alpha^* D_\alpha = \alpha^2 (u'' + \frac{n-1}{s} u') + \frac{1}{s^2} \Delta_\omega u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = L_\alpha u^m$$

critical case $m = 1 - 1/n$; subcritical range $1 - 1/n < m < 1$

The key computation is the proof that

$$-\frac{d}{dt} G[u(t, \cdot)] \left( \int_{\mathbb{R}^d} u^m \, d\mu \right)^{1-\sigma} \geq (1 - m) (\sigma - 1) \int_{\mathbb{R}^d} u^m \left| L_\alpha P - \frac{\int_{\mathbb{R}^d} u |D_\alpha P|^2 \, d\mu}{\int_{\mathbb{R}^d} u^m \, d\mu} \right|^2 \, d\mu$$

$$+ 2 \int_{\mathbb{R}^d} \left( \alpha^4 \left(1 - \frac{1}{n}\right) \left| P'' - \frac{P'}{s} \right|^2 + \frac{2\alpha^2}{s^2} \left| \nabla_\omega P' - \frac{\nabla_\omega P}{s} \right|^2 \right) u^m \, d\mu$$

$$+ 2 \int_{\mathbb{R}^d} \left( (n - 2) \left(\alpha_{FS}^2 - \alpha^2\right) |\nabla_\omega P|^2 + c(n, m, d) \frac{|\nabla_\omega P|^4 \alpha^2}{P^2} \right) u^m \, d\mu =: \mathcal{H}[u]$$

for some numerical constant $c(n, m, d) > 0$. Hence if $\alpha \leq \alpha_{FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if $P$ is an affine function of $|x|^2$, which proves the symmetry result. A quantifier elimination problem (Tarski, 1951)?
This method has a hidden difficulty: integrations by parts! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if $u$ solves the Euler-Lagrange equation, we test by $L_\alpha u^m$

$$0 = \int_{\mathbb{R}^d} dG[u] \cdot L_\alpha u^m \, d\mu \geq \mathcal{H}[u] \geq 0$$

$\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by $|x|^\gamma \text{div} \left(|x|^{-\beta} \nabla w^{1+p}\right)$ the equation

$$\frac{(p-1)^2}{p(p+1)} w^{1-3p} \text{div} \left(|x|^{-\beta} w^{2p} \nabla w^{1-p}\right) + |\nabla w^{1-p}|^2 + |x|^{-\gamma} (c_1 w^{1-p} - c_2) = 0$$
Fast diffusion equations with weights: large time asymptotics

- The entropy formulation of the problem
- [Relative uniform convergence]
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here $v$ solves the Fokker-Planck type equation

$$v_t + |x|^{\gamma} \nabla \cdot \left[ |x|^{-\beta} v \nabla (v^{m-1} - |x|^{2+\beta-\gamma}) \right] = 0 \quad (WFDE-FP)$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret
When symmetry holds, (CKN) can be written as an entropy – entropy production inequality

\[
\frac{1-m}{m} (2 + \beta - \gamma)^2 \mathcal{E}[v] \leq \mathcal{I}[v]
\]

and equality is achieved by \( \mathcal{B}_{\beta, \gamma}(x) := (1 + |x|^{2+\beta-\gamma})^{\frac{1}{m-1}} \)

Here the free energy and the relative Fisher information are defined by

\[
\mathcal{E}[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left( v^m - \mathcal{B}_{\beta, \gamma}^m - m \mathcal{B}_{\beta, \gamma}^{m-1} (v - \mathcal{B}_{\beta, \gamma}) \right) \frac{dx}{|x|^{\gamma}}
\]

\[
\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} - \nabla \mathcal{B}_{\beta, \gamma}^{m-1} \right|^2 \frac{dx}{|x|^{\beta}}
\]

If \( v \) solves the Fokker-Planck type equation

\[
v_t + |x|^{\gamma} \nabla \cdot \left[ |x|^{-\beta} v \nabla (v^{m-1} - |x|^{2+\beta-\gamma}) \right] = 0 \quad \text{(WFDE-FP)}
\]

then

\[
\frac{d}{dt} \mathcal{E}[v(t, \cdot)] = -\frac{m}{1-m} \mathcal{I}[v(t, \cdot)]
\]
The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\text{ess}}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions.
Global vs. asymptotic estimates

Estimates on the global rates. When symmetry holds (CKN) can be written as an entropy – entropy production inequality

\[(2 + \beta - \gamma)^2 \mathcal{E}[\nu] \leq \frac{m}{1 - m} \mathcal{I}[\nu]\]

so that

\[\mathcal{E}[\nu(t)] \leq \mathcal{E}[\nu(0)] e^{-2(1-m)\Lambda_* t} \quad \forall \ t \geq 0 \quad \text{with} \quad \Lambda_* := \frac{(2 + \beta - \gamma)^2}{2(1 - m)}\]

Optimal global rates. Let us consider again the entropy – entropy production inequality

\[\mathcal{K}(M) \mathcal{E}[\nu] \leq \mathcal{I}[\nu] \quad \forall \ \nu \in L^{1,\gamma}(\mathbb{R}^d) \quad \text{such that} \quad \|\nu\|_{L^{1,\gamma}(\mathbb{R}^d)} = M,\]

where \(\mathcal{K}(M)\) is the best constant: with \(\Lambda(M) := \frac{m}{2} (1 - m)^{-2} \mathcal{K}(M)\)

\[\mathcal{E}[\nu(t)] \leq \mathcal{E}[\nu(0)] e^{-2(1-m)\Lambda(M) t} \quad \forall \ t \geq 0\]
Linearization and optimality

Joint work with M.J. Esteban and M. Loss
Linearization and scalar products

With \( u_\varepsilon \) such that

\[
u_\varepsilon = B_* \left( 1 + \varepsilon f B_*^{1-m} \right) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\varepsilon \, dx = M_*
\]

at first order in \( \varepsilon \to 0 \) we obtain that \( f \) solves

\[
\frac{\partial f}{\partial t} = \mathcal{L} f \quad \text{where} \quad \mathcal{L} f := (1 - m) B_*^{m-2} |x|^{\gamma} D_\alpha^* \left( |x|^{-\beta} B_* D_\alpha f \right)
\]

Using the scalar products

\[
\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1 f_2 B_*^{2-m} |x|^{-\gamma} \, dx \quad \text{and} \quad \langle\langle f_1, f_2 \rangle \rangle = \int_{\mathbb{R}^d} D_\alpha f_1 \cdot D_\alpha f_2 B_* |x|^{-\beta} \, dx
\]

we compute

\[
\frac{1}{2} \frac{d}{dt} \langle f, f \rangle = \langle f, \mathcal{L} f \rangle = \int_{\mathbb{R}^d} f (\mathcal{L} f) B_*^{2-m} |x|^{-\gamma} \, dx = - \int_{\mathbb{R}^d} |D_\alpha f|^2 B_* |x|^{-\beta} \, dx
\]

for any \( f \) smooth enough: with \( \langle f, \mathcal{L} f \rangle = - \langle\langle f, f \rangle \rangle \)

\[
\frac{1}{2} \frac{d}{dt} \langle\langle f, f \rangle \rangle = \int_{\mathbb{R}^d} D_\alpha f \cdot D_\alpha (\mathcal{L} f) B_* |x|^{-\beta} \, dx = - \langle\langle f, \mathcal{L} f \rangle \rangle
\]
Linearization of the flow, eigenvalues and spectral gap

Now let us consider an eigenfunction associated with the smallest positive eigenvalue $\lambda_1$ of $\mathcal{L}$

$$-\mathcal{L} f_1 = \lambda_1 f_1$$

so that $f_1$ realizes the equality case in the Hardy-Poincaré inequality

$$\langle \langle g, g \rangle \rangle := -\langle g, \mathcal{L} g \rangle \geq \lambda_1 \|g - \bar{g}\|^2, \quad \bar{g} := \langle g, 1 \rangle / \langle 1, 1 \rangle \quad (P1)$$

$$-\langle \langle g, \mathcal{L} g \rangle \rangle \geq \lambda_1 \langle \langle g, g \rangle \rangle \quad (P2)$$

Proof by expansion of the square

$$-\langle \langle (g - \bar{g}), \mathcal{L} (g - \bar{g}) \rangle \rangle = \langle \mathcal{L} (g - \bar{g}), \mathcal{L} (g - \bar{g}) \rangle = \|\mathcal{L} (g - \bar{g})\|^2$$

- (P1) is associated with the symmetry breaking issue
- (P2) is associated with the carré du champ method

The optimal constants / eigenvalues are the same

Key observation: $\lambda_1 \geq 4 \iff \alpha \leq \alpha_{FS} := \sqrt{\frac{d-1}{n-1}}$
Three references

- Lecture notes on *Symmetry and nonlinear diffusion flows... a course on entropy methods* (see webpage)

- [JD, Maria J. Esteban, and Michael Loss] *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs* ... the elliptic point of view: arXiv: 1711.11291

- [JD, Maria J. Esteban, and Michael Loss] *Interpolation inequalities, nonlinear flows, boundary terms, optimality and linearization*... the parabolic point of view
These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
▷ Lectures

The papers can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/list/
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Thank you for your attention !