Uniqueness and symmetry based on nonlinear flows

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A result of uniqueness on a classical example

On the sphere $\mathbb{S}^d$, let us consider the positive solutions of

$$-\Delta u + \lambda\, u = u^{p-1}$$

$p \in [1,2) \cup (2,2^*]$ if $d \geq 3$, $2^* = \frac{2\,d}{d-2}$

$p \in [1,2) \cup (2,+\infty)$ if $d = 1,\, 2$

Theorem

If $\lambda \leq d$, $u \equiv \lambda^{1/(p-2)}$ is the unique solution

[Gidas & Spruck, 1981], [Bidaut-Véron & Véron, 1991]

Bifurcation point of view

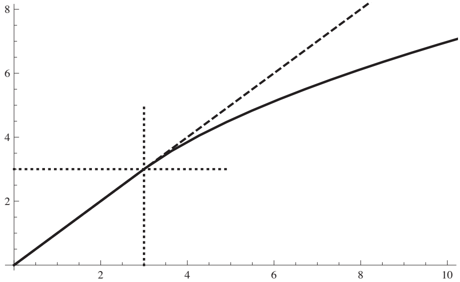


Figure: $(p-2)\,\lambda \mapsto (p-2)\,\mu(\lambda)$ *with* $d=3$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > \frac{d}{p-2}$$

▷ The inequality holds with $\mu(\lambda) = \lambda = \frac{d}{p-2}$ [Bakry & Emery, 1985] [Beckner, 1993], [Bidaut-Véron & Véron, 1991, Corollary 6.1]

Inequalities without weights and fast diffusion equations: optimality and uniqueness of the critical points

- The Bakry-Emery method (compact manifolds)

▷ The Fokker-Planck equation
▷ The Bakry-Emery method on the sphere: a parabolic method
▷ The Moser-Trudiger-Onofri inequality (on a compact manifold)

- Fast diffusion equations on the Euclidean space (without weights)

▷ Euclidean space: Rényi entropy powers
▷ Euclidean space: self-similar variables and relative entropies
▷ The role of the spectral gap

The Fokker-Planck equation

The linear Fokker-Planck (FP) equation

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla \phi)$$

on a domain $\Omega \subset \mathbb{R}^d$, with no-flux boundary conditions

$$(\nabla u + u \nabla \phi) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

is equivalent to the Ornstein-Uhlenbeck (OU) equation

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \phi \cdot \nabla v =: \mathcal{L}\, v$$

(Bakry, Emery, 1985), (Arnold, Markowich, Toscani, Unterreiter, 2001)
With mass normalized to 1, the unique stationary solution of (FP) is

$$u_s = \frac{e^{-\phi}}{\int_\Omega e^{-\phi}\,dx} \quad \Longleftrightarrow \quad v_s = 1$$

The Bakry-Emery method

With $d\gamma = u_s\,dx$ and v such that $\int_\Omega v\,d\gamma = 1$, $q \in (1,2]$, the q-*entropy* is defined by

$$\mathcal{E}_q[v] := \frac{1}{q-1}\int_\Omega \left(v^q - 1 - q\,(v-1)\right)d\gamma$$

Under the action of (OU), with $w = v^{q/2}$, $\mathcal{I}_q[v] := \frac{4}{q}\int_\Omega |\nabla w|^2\,d\gamma$,

$$\frac{d}{dt}\mathcal{E}_q[v(t,\cdot)] = -\,\mathcal{I}_q[v(t,\cdot)] \quad \text{and} \quad \frac{d}{dt}\Big(\mathcal{I}_q[v] - 2\,\lambda\,\mathcal{E}_q[v]\Big) \le 0$$

$$\text{with} \quad \lambda := \inf_{w \in H^1(\Omega,d\gamma)\setminus\{0\}} \frac{\int_\Omega \left(2\,\frac{q-1}{q}\,\|\mathrm{Hess}\,w\|^2 + \mathrm{Hess}\,\phi : \nabla w \otimes \nabla w\right)d\gamma}{\int_\Omega |\nabla w|^2\,d\gamma}$$

Proposition

(Bakry, Emery, 1984) (JD, Nazaret, Savaré, 2008) *Let Ω be convex. If $\lambda > 0$ and v is a solution of (OU), then $\mathcal{I}_q[v(t,\cdot)] \le \mathcal{I}_q[v(0,\cdot)]\,e^{-2\,\lambda\,t}$ and $\mathcal{E}_q[v(t,\cdot)] \le \mathcal{E}_q[v(0,\cdot)]\,e^{-2\,\lambda\,t}$ for any $t \ge 0$ and, as a consequence,*

$$\mathcal{I}_q[v] \ge 2\,\lambda\,\mathcal{E}_q[v] \quad \forall\, v \in \mathrm{H}^1(\Omega, d\gamma)$$

A proof of the interpolation inequality by the *carré du champ* method

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

$p \in [1,2) \cup (2,2^*]$ if $d \ge 3$, $2^* = \frac{2\,d}{d-2}$

$p \in [1,2) \cup (2,+\infty)$ if $d = 1,\ 2$

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

[Bakry & Emery, 1985] *carré du champ* method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and observe that $\frac{d}{dt}\mathcal{E}_p[\rho] = -\mathcal{I}_p[\rho]$,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

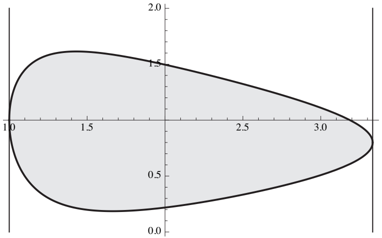
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^{\#}$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

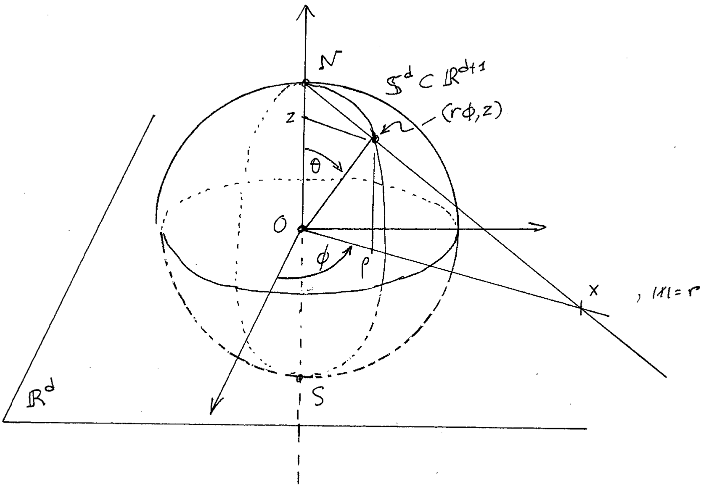
(Demange), (JD, Esteban, Kowalczyk, Loss): for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0$$



(p, m) *admissible region*, $d = 5$

Cylindrical coordinates, Schwarz symmetrization, stereographic projection...



... and the ultra-spherical operator

Change of variables $z=\cos\theta$, $v(\theta)=f(z)$, $d\nu_d:=\nu^{\frac{d}{2}-1}\,dz/Z_d$, $\nu(z):=1-z^2$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\,f:=(1-z^2)\,f''-d\,z\,f'=\nu\,f''+\frac{d}{2}\,\nu'\,f'$$

which satisfies $\langle f_1,\mathcal{L}\,f_2\rangle=-\int_{-1}^{1}f_1'\,f_2'\;\nu\;d\nu_d$

Proposition

Let $p\in[1,2)\cup(2,2^*]$, $d\ge1$. For any $f\in\mathrm{H}^1([-1,1],d\nu_d)$,

$$-\langle f,\mathcal{L}\,f\rangle=\int_{-1}^{1}|f'|^2\;\nu\;d\nu_d\ge d\,\frac{\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2}$$

The heat equation $\frac{\partial g}{\partial t} = \mathcal{L}\, g$ for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\, \frac{|f'|^2}{f}\, \nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1} |f'|^2\, \nu\, d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f, \mathcal{L}\, f\rangle = \langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\,\left\langle \frac{|f'|^2}{f}\,\nu, \mathcal{L}\, f\right\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\,d\,\mathcal{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1} |f'|^2\, \nu\, d\nu_d + 2\,d\int_{-1}^{1} |f'|^2\, \nu\, d\nu_d$$

$$= -\,2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\, f''}{f}\right)\nu^2\, d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\, f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} = 2^\# < \frac{2\,d}{d-2} = 2^*$$

The elliptic point of view (nonlinear flow)

$\frac{\partial u}{\partial t} = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$, $\kappa = \beta\,(p-2)+1$

$$-\mathcal{L}\;u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\;d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

The two terms cancel and we are left only with

$$\int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\;d\nu_d = 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban

- Extension to compact Riemannian manifolds of dimension 2...

$\Longrightarrow$

We shall also denote by $\mathfrak{R}$ the Ricci tensor, by $\mathrm{H}_g u$ the Hessian of u and by

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

the trace free Hessian. Let us denote by $\mathrm{M}_g u$ the trace free tensor

$$\mathrm{M}_g u := \nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2$$

We define

$$\lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M}) \setminus \{0\}} \frac{\displaystyle\int_{\mathfrak{M}} \Big[\|\, \mathrm{L}_g u - \tfrac{1}{2}\,\mathrm{M}_g u \,\|^2 + \mathfrak{R}(\nabla u, \nabla u) \Big]\, e^{-u/2}\, d v_g}{\displaystyle\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, d v_g}$$

Theorem

Assume that $d=2$ and $\lambda_\star>0$. If u is a smooth solution to

$$-\frac{1}{2}\,\Delta_g u+\lambda=e^u$$

then u is a constant function if $\lambda\in(0,\lambda_\star)$

The Moser-Trudinger-Onofri inequality on $\mathfrak{M}$

$$\frac{1}{4}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})}+\lambda\int_{\mathfrak{M}} u\,d\,v_g\ge\lambda\,\log\left(\int_{\mathfrak{M}} e^u\,d\,v_g\right)\quad\forall\,u\in\mathrm{H}^1(\mathfrak{M})$$

for some constant $\lambda>0$. Let us denote by λ_1 the first positive eigenvalue of $-\Delta_g$

Corollary

If $d=2$, then the MTO inequality holds with $\lambda=\Lambda:=\min\{4\,\pi,\lambda_\star\}$. Moreover, if Λ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than Λ

The flow

$$\frac{\partial f}{\partial t} = \Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}$$

$$\mathcal{G}_\lambda[f] := \int_{\mathfrak{M}} \|\,\mathrm{L}_g f - \tfrac{1}{2}\,\mathrm{M}_g f\,\|^2\,e^{-f/2}\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla f,\nabla f)\,e^{-f/2}\,d\,v_g - \lambda\int_{\mathfrak{M}} |\nabla f|^2\,e^{-f/2}\,d\,v_g$$

Then for any $\lambda \le \lambda_\star$ we have

$$\frac{d}{dt}\mathcal{F}_\lambda[f(t,\cdot)] = \int_{\mathfrak{M}} \left(-\tfrac{1}{2}\,\Delta_g f + \lambda\right)\left(\Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}\right)d\,v_g = -\,\mathcal{G}_\lambda[f(t,\cdot)]$$

Since $\mathcal{F}_\lambda$ is nonnegative and $\lim_{t\to\infty}\mathcal{F}_\lambda[f(t,\cdot)] = 0$, we obtain that

$$\mathcal{F}_\lambda[u] \ge \int_0^\infty \mathcal{G}_\lambda[f(t,\cdot)]\,dt$$

Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

On the Euclidean space $\mathbb{R}^2$, given a general probability measure μ does the inequality

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \ge \lambda\left[\log\left(\int_{\mathbb{R}^d} e^u\,d\mu\right)-\int_{\mathbb{R}^d} u\,d\mu\right]$$

hold for some $\lambda > 0$? Let

$$\Lambda_\star := \inf_{x\in\mathbb{R}^2}\frac{-\,\Delta\log\mu}{8\,\pi\,\mu}$$

Theorem

Assume that μ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda < \Lambda_\star$ and the inequality holds with $\lambda = \Lambda_\star$ if equality is achieved among radial functions

Euclidean space: Rényi entropy powers and fast diffusion

- The Euclidean space without weights

▷ Rényi entropy powers, the entropy approach without rescaling: (Savaré, Toscani): scalings, nonlinearity and a concavity property inspired by information theory

The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial v}{\partial t} = \Delta v^m$$

with initial datum $v(x, t = 0) = v_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} v_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, v_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t, x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathcal{B}_\star\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, v^{m-1}$$

If v solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1) \, \mathsf{p} \, \Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m} \left(\frac{1}{d} + m - 1 \right) = \frac{2}{d} \, \frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The variation of the Fisher information

Lemma

If v solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $1 - \frac{1}{d} \le m < 1$, then

$$\mathsf{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} v\,|\nabla \mathsf{p}|^2\,dx = -\,2 \int_{\mathbb{R}^d} v^m \left(\|\mathrm{D}^2 \mathsf{p}\|^2 + (m-1)\,(\Delta \mathsf{p})^2 \right) dx$$

Explicit arithmetic geometric inequality

$$\|\mathrm{D}^2 \mathsf{p}\|^2 - \frac{1}{d}\,(\Delta \mathsf{p})^2 = \left\| \mathrm{D}^2 \mathsf{p} - \frac{1}{d}\,\Delta \mathsf{p}\,\mathrm{Id} \right\|^2$$

.... there are no boundary terms in the integrations by parts ?

$\Longrightarrow$

The concavity property

Theorem

[Toscani-Savaré] *Assume that* $m \geq 1 - \frac{1}{d}$ *if* $d > 1$ *and* $m > 0$ *if* $d = 1$. *Then* $F(t)$ *is increasing,* $(1-m)\,\mathsf{F}''(t) \leq 0$ *and*

$$\lim_{t\to+\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t\to+\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

[Dolbeault-Toscani] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \geq \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

if $1 - \frac{1}{d} \leq m < 1$. Hint: $v^{m-1/2} = \frac{w}{\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}}$, $q = \frac{1}{2\,m-1}$

Euclidean space: self-similar variables and relative entropies

- In the Euclidean space, it is possible to characterize the optimal constants using a spectral gap property

Self-similar variables and relative entropies

The large time behavior of the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ is governed by the source-type *Barenblatt solutions*

$$v_\star(t,x) := \frac{1}{\kappa^d (\mu\, t)^{d/\mu}}\, \mathcal{B}_\star\left(\frac{x}{\kappa\,(\mu\, t)^{1/\mu}}\right) \quad \text{where} \quad \mu := 2 + d\,(m-1)$$

where $\mathcal{B}_\star$ is the Barenblatt profile (with appropriate mass)

$$\mathcal{B}_\star(x) := \left(1 + |x|^2\right)^{1/(m-1)}$$

A time-dependent rescaling: self-similar variables

$$v(t,x) = \frac{1}{\kappa^d\, R^d}\, u\Big(\tau, \frac{x}{\kappa\, R}\Big) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}\,, \quad \tau(t) := \tfrac{1}{2}\,\log\left(\frac{R(t)}{R_0}\right)$$

Then the function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla\cdot\Big[u\,\big(\nabla u^{m-1} - 2\,x\big)\Big] = 0$$

Free energy and Fisher information

- The function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \Big[u \left(\nabla u^{m-1} - 2\,x \right) \Big] = 0$$

- (Ralston, Newman, 1984) Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{E}[u] := \int_{\mathbb{R}^d} \left(-\frac{u^m}{m} + |x|^2 u \right) \, dx - \mathcal{E}_0$$

- Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{E}[u] = -\,\mathcal{I}[u]\,, \quad \mathcal{I}[u] := \int_{\mathbb{R}^d} u \left| \nabla u^{m-1} + 2\,x \right|^2 \, dx$$

Without weights: relative entropy, entropy production

- *Stationary solution:* choose C such that $\|u_\infty\|_{\mathrm{L}^1} = \|u\|_{\mathrm{L}^1} = M > 0$

$$u_\infty(x) := \left(C + |x|^2\right)_+^{-1/(1-m)}$$

- *Entropy – entropy production inequality* (del Pino, JD)

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[u] \geq 4\,\mathcal{E}[u]$$

$p = \frac{1}{2m-1}$, $u = w^{2p}$: (GN) $\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$

Corollary

(del Pino, JD) *A solution u with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies* $\mathcal{E}[u(t,\cdot)] \leq \mathcal{E}[u_0]\,e^{-4\,t}$

A computation on a large ball, with boundary terms

$$\frac{\partial u}{\partial \tau}+\nabla \cdot\Big[u\left(\nabla u^{m-1}-\,2\,x\right)\Big]=0\quad \tau>0\,,\quad x\in B_R$$

where B_R is a centered ball in $\mathbb{R}^d$ with radius $R>0$, and assume that u satisfies zero-flux boundary conditions

$$\left(\nabla u^{m-1}-\,2\,x\right)\cdot\frac{x}{|x|}=0\quad \tau>0\,,\quad x\in\partial B_R\,.$$

With $z(\tau,x):=\nabla \mathrm{Q}(\tau,x):=\nabla u^{m-1}-\,2\,x$, the *relative Fisher information* is such that

$$\frac{d}{d\tau}\int_{B_R} u\,|z|^2\,dx+4\int_{B_R} u\,|z|^2\,dx$$

$$+\,2\,\tfrac{1-m}{m}\int_{B_R} u^m\left(\left\|\mathrm{D}^2\mathrm{Q}\right\|^2-\,(1-m)\,(\Delta \mathrm{Q})^2\right)dx$$

$$=\int_{\partial B_R} u^m\left(\omega\cdot\nabla |z|^2\right)d\sigma\le 0$$ (by Grisvard's lemma)

Spectral gap: sharp asymptotic rates of convergence

Assumptions on the initial datum v_0

(H1) $V_{D_0} \le v_0 \le V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \ge 3$ *and* $m \le m_*$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

Theorem

(Blanchet, Bonforte, JD, Grillo, Vázquez) *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \ne m_* := \frac{d-4}{d-2}$, *the entropy decays according to*

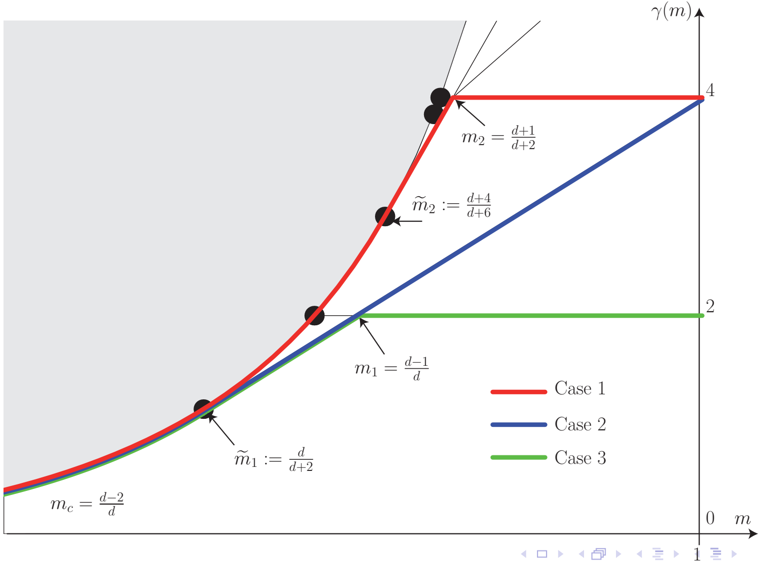
$$\mathcal{E}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\, t \ge 0$$

where $\Lambda_{\alpha,d} > 0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)\,, \int_{\mathbb{R}^d} f d\mu_{\alpha-1} = 0$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

Spectral gap and best constants



Caffarelli-Kohn-Nirenberg, symmetry and symmetry breaking results, and weighted nonlinear flows

▷ The critical Caffarelli-Kohn-Nirenberg inequality
[JD, Esteban, Loss]

[▷ A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities]
[JD. Esteban, Loss, Muratori]

▷ Large time asymptotics and spectral gaps

▷ Optimality cases

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

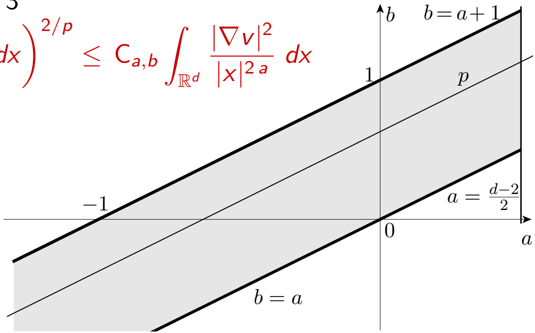
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \text{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d = 3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a \le b \le a+1$ if $d \ge 3$

$a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$

and $a < a_c := (d-2)/2$

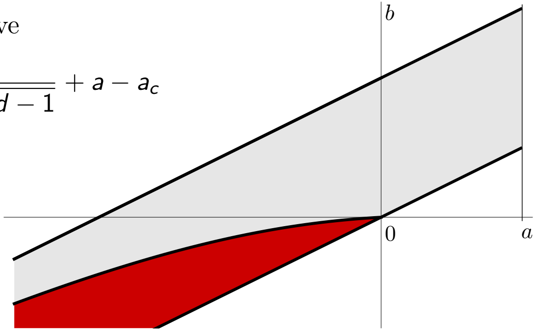
$$p = \frac{2\,d}{d - 2 + 2\,(b-a)}$$

(Glaser, Martin, Grosse, Thirring (1976))
(Caffarelli, Kohn, Nirenberg (1984))
[F. Catrina, Z.-Q. Wang (2001)]

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c-a)^2 + d - 1}} + a - a_c$$



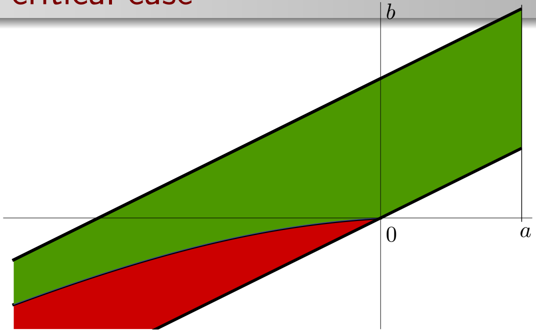
[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]
The functional

$$C^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v = v_\star$

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (2016)]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega) = r^{a-a_c}\,\varphi(s,\omega) \quad \text{with} \quad r=|x|\,, \quad s=-\log r \quad \text{and} \quad \omega=\frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the *subcritical* interpolation inequality

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}\ge\mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}\quad\forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda:=(a_c-a)^2$, $\mathcal{C}=\mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda)=\frac{1}{\mathsf{C}_{a,b}}\quad\text{with}\quad a=a_c\pm\sqrt{\Lambda}\quad\text{and}\quad b=\frac{d}{p}\pm\sqrt{\Lambda}$$

Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_\star(s,\omega) = (\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$\mathrm{H}^1(\mathcal{C}) \ni \varphi \mapsto \|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}$$

under a constraint on $\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}$

$\varphi_\star$ *is not* optimal for (CKN) if the Pöschl-Teller operator

$$-\partial_s^2 - \Delta_\omega + \Lambda - \varphi_\star^{p-2} = -\partial_s^2 - \Delta_\omega + \Lambda - \frac{1}{(\cosh s)^2}$$

has *a negative eigenvalue*, i.e., for $\Lambda > \Lambda_1$ (explicit)

The variational problem on the cylinder

$$\Lambda \mapsto \mu(\Lambda) := \min_{\varphi \in \mathrm{H}^1(\mathcal{C})} \frac{\|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda \, \|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}}$$

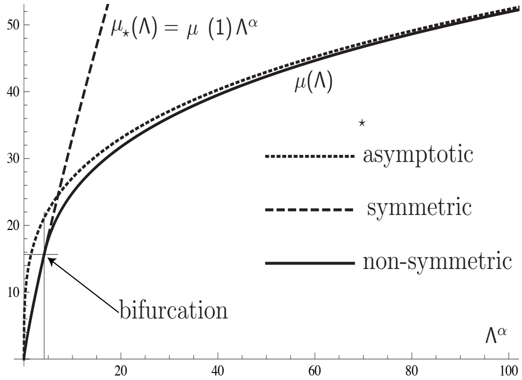
is a concave increasing function

Restricted to symmetric functions, the variational problem becomes

$$\mu_\star(\Lambda) := \min_{\varphi \in \mathrm{H}^1(\mathbb{R})} \frac{\|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \Lambda \, \|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\varphi\|^2_{\mathrm{L}^p(\mathbb{R}^d)}} = \mu_\star(1)\, \Lambda^\alpha$$

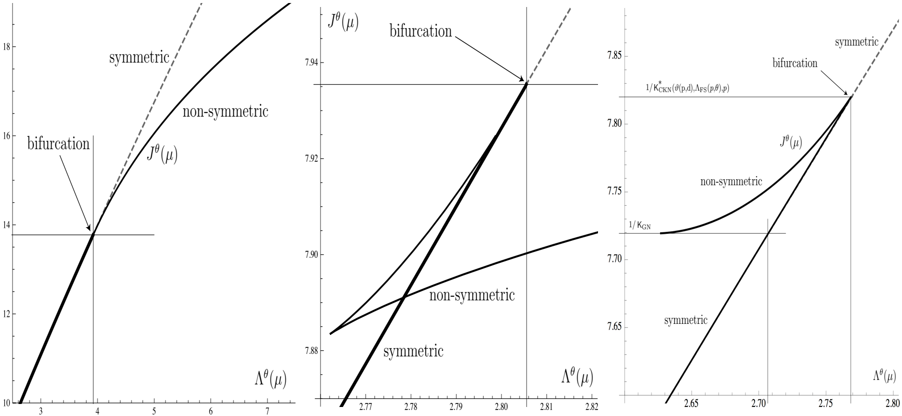
Symmetry means $\mu(\Lambda) = \mu_\star(\Lambda)$
Symmetry breaking means $\mu(\Lambda) < \mu_\star(\Lambda)$

Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point* Λ_1 *computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as predicted by F. Catrina and Z.-Q. Wang*

what we have to prove / discard...



When the local criterion (linear stability) differs from global results in a larger family of inequalities (center, right)...

The uniqueness result and the strategy of the proof

The elliptic problem: rigidity

The symmetry issue can be reformulated as a uniqueness (rigidity) issue. An optimal function for the inequality

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

solves the (elliptic) Euler-Lagrange equation

$$-\nabla\cdot\left(|x|^{-2a}\,\nabla v\right) = |x|^{-bp}\,v^{p-1}$$

(up to a scaling and a multiplication by a constant). Is any nonnegative solution of such an equation equal to

$$v_\star(x) = \left(1+|x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}}$$

(up to invariances) ? On the cylinder

$$-\,\partial_s^2\,\varphi - \partial_\omega\varphi + \Lambda\,\varphi = \varphi^{p-1}$$

Up to a normalization and a scaling

$$\varphi_\star(s,\omega) = (\cosh s)^{-\frac{1}{p-2}}$$

Symmetry in one slide: 3 steps

- A change of variables: $v(|x|^{\alpha-1}\,x)=w(x)$, $\mathsf{D}_\alpha v=\left(\alpha\,\frac{\partial v}{\partial s},\frac{1}{s}\,\nabla_\omega v\right)$

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\le\mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta}\quad\forall\,v\in\mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

- Concavity of the Rényi entropy power: with $\mathcal{L}_\alpha=-\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha=\alpha^2\left(u''+\frac{n-1}{s}\,u'\right)+\frac{1}{s^2}\,\Delta_\omega\,u$ and $\frac{\partial u}{\partial t}=\mathcal{L}_\alpha u^m$

$$-\,\frac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\int_{\mathbb{R}^d}u^m\,d\mu\right)^{1-\sigma}$$
$$\ge(1-m)\,(\sigma-1)\int_{\mathbb{R}^d}u^m\left|\mathcal{L}_\alpha\mathsf{P}-\frac{\int_{\mathbb{R}^d}u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d}u^m\,d\mu}\right|^2d\mu$$
$$+\,2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\frac{1}{n}\right)\left|\mathsf{P}''-\frac{\mathsf{P}'}{s}-\frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2+\frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}'-\frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right)u^m\,d\mu$$
$$+\,2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\mathrm{FS}}^2-\alpha^2\right)|\nabla_\omega\mathsf{P}|^2+c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right)u^m\,d\mu$$

- Elliptic regularity and the Emden-Fowler transformation: justifying the integrations by parts

$\Longrightarrow$

Proof of symmetry (1/3: changing the dimension)

We rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight $|x|^{n-d}$ which is the same in all norms. With $\beta = 2\,a$ and $\gamma = b\,p$,

$$v(|x|^{\alpha-1}\,x) = w(x)\,, \quad \alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d - \gamma}{\beta + 2 - \gamma}$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta} \quad \forall\,v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

with the notations $s = |x|$, $\mathsf{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$ and

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall\,x \in \mathbb{R}^d$$

The strategy of the proof (2/3: Rényi entropy)

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathsf{D}_\alpha \mathsf{P}|^2 \, d\mu$$

where $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d} \, dx$ and the pressure is

$$\mathsf{P} := \frac{m}{1-m} \, u^{m-1}$$

Looking for an optimal function in (CKN) is equivalent to minimize $\mathcal{G}$ under a mass constraint

With $\mathsf{L}_\alpha = -\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha = \alpha^2\left(u'' + \frac{n-1}{s}\,u'\right) + \frac{1}{s^2}\,\Delta_\omega\,u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathsf{L}_\alpha\,u^m$$

critical case $m = 1 - 1/n$; subcritical range $1 - 1/n < m < 1$

The key computation is the proof that

$$\begin{aligned}
&-\tfrac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\textstyle\int_{\mathbb{R}^d} u^m\,d\mu\right)^{1-\sigma}\\
&\ge (1-m)\,(\sigma-1)\int_{\mathbb{R}^d} u^m\left|\mathsf{L}_\alpha\mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d} u^m\,d\mu}\right|^2 d\mu\\
&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\tfrac{1}{n}\right)\left|\mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}' - \frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right) u^m\,d\mu\\
&+2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\mathrm{FS}}^2 - \alpha^2\right)|\nabla_\omega\mathsf{P}|^2 + c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right) u^m\,d\mu =: \mathcal{H}[u]
\end{aligned}$$

for some numerical constant $c(n,m,d) > 0$. Hence if $\alpha \le \alpha_{\mathrm{FS}}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result. *A quantifier elimination problem (Tarski, 1951) ?*

(3/3: elliptic regularity, boundary terms)

This method has a hidden difficulty: integrations by parts ! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if u solves the Euler-Lagrange equation, we test by $\mathsf{L}_\alpha u^m$

$$0=\int_{\mathbb{R}^d} \mathrm{d}\mathcal{G}[u]\cdot\mathsf{L}_\alpha u^m\,d\mu\ge\mathcal{H}[u]\ge 0$$

$\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by $|x|^\gamma\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w^{1+p}\right)$ the equation

$$\frac{(p-1)^2}{p\,(p+1)}\,w^{1-3p}\,\mathrm{div}\left(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\right)+|\nabla w^{1-p}|^2+|x|^{-\gamma}\left(c_1\,w^{1-p}-c_2\right)=0$$

Fast diffusion equations with weights: large time asymptotics

- The entropy formulation of the problem
- [Relative uniform convergence]
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \nabla \cdot \Big[|x|^{-\beta} v \nabla \big(v^{m-1} - |x|^{2+\beta-\gamma} \big) \Big] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\frac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{E}[v]\le\mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}(x):=\left(1+|x|^{2+\beta-\gamma}\right)^{\frac{1}{m-1}}$
Here the *free energy* and the *relative Fisher information* are defined by

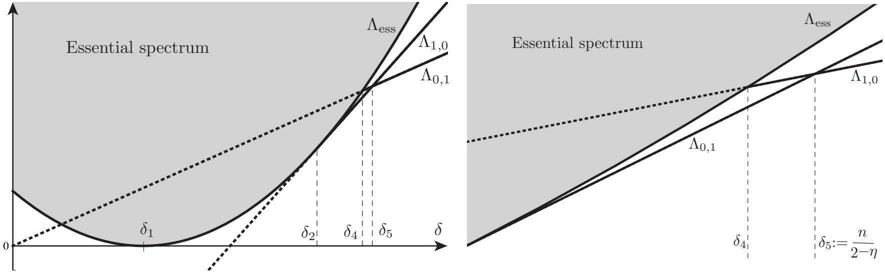
$$\mathcal{E}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\,\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\,\frac{dx}{|x|^\beta}$$

If v solves the *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[\,|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0 \qquad \text{(WFDE-FP)}$$

then $\dfrac{d}{dt}\mathcal{E}[v(t,\cdot)]=-\dfrac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$



The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\rm ess}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions

$\Longrightarrow$

Global vs. asymptotic estimates

- *Estimates on the global rates.* When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{E}[v] \le \frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{E}[v(t)] \le \mathcal{E}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t} \quad \forall\,t\ge 0 \quad \text{with} \quad \Lambda_\star := \frac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

- *Optimal global rates.* Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{E}[v] \le \mathcal{I}[v] \quad \forall\,v\in \mathrm{L}^{1,\gamma}(\mathbb{R}^d) \quad \text{such that} \quad \|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)} = M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M) := \frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{E}[v(t)] \le \mathcal{E}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t} \quad \forall\,t\ge 0$$

Linearization and optimality

Joint work with M.J. Esteban and M. Loss

Linearization and scalar products

With u_ε such that

$$u_\varepsilon = \mathcal{B}_\star \left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\varepsilon\, dx = M_\star$$

at first order in $\varepsilon \to 0$ we obtain that f solves

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f \quad \text{where} \quad \mathcal{L}\, f := (1-m)\, \mathcal{B}_\star^{m-2}\, |x|^\gamma\, \mathsf{D}_\alpha^* \left(\, |x|^{-\beta}\, \mathcal{B}_\star\, \mathsf{D}_\alpha\, f\right)$$

Using the scalar products

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1\, f_2\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx \quad \text{and} \quad \langle\!\langle f_1, f_2 \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D}_\alpha\, f_1 \cdot \mathsf{D}_\alpha\, f_2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx$$

we compute

$$\frac{1}{2}\, \frac{d}{dt}\, \langle f, f \rangle = \langle f, \mathcal{L}\, f \rangle = \int_{\mathbb{R}^d} f\, (\mathcal{L}\, f)\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx = -\int_{\mathbb{R}^d} |\mathsf{D}_\alpha\, f|^2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx$$

for any f smooth enough: with $\langle f, \mathcal{L}\, f \rangle = -\,\langle\!\langle f, f \rangle\!\rangle$

$$\frac{1}{2}\, \frac{d}{dt} \langle\!\langle f, f \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D}_\alpha\, f \cdot \mathsf{D}_\alpha\, (\mathcal{L}\, f)\, \mathcal{B}_\star\, |x|^{-\beta}\, dx = -\,\langle\!\langle f, \mathcal{L}\, f \rangle\!\rangle$$

Linearization of the flow, eigenvalues and spectral gap

Now let us consider an eigenfunction associated with the smallest positive eigenvalue λ_1 of $\mathcal{L}$

$$-\mathcal{L}\, f_1 = \lambda_1\, f_1$$

so that f_1 realizes the equality case in the *Hardy-Poincaré inequality*

$$\langle\!\langle g, g \rangle\!\rangle := -\,\langle g, \mathcal{L}\, g\rangle \geq \lambda_1\, \| g - \bar{g} \|^2\,, \quad \bar{g} := \langle g, 1\rangle \,/\, \langle 1, 1\rangle \tag{P1}$$

$$-\,\langle\!\langle g, \mathcal{L}\, g \rangle\!\rangle \geq \lambda_1\, \langle\!\langle g, g \rangle\!\rangle \tag{P2}$$

Proof by expansion of the square
$-\,\langle\!\langle (g-\bar{g}), \mathcal{L}\,(g-\bar{g}) \rangle\!\rangle = \langle \mathcal{L}\,(g-\bar{g}), \mathcal{L}\,(g-\bar{g})\rangle = \|\mathcal{L}\,(g-\bar{g})\|^2$

- (P1) is associated with the symmetry breaking issue
- (P2) is associated with the *carré du champ* method

The optimal constants / eigenvalues are the same

- Key observation: $\lambda_1 \geq 4 \quad \Longleftrightarrow \quad \alpha \leq \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$

Three references

- Lecture notes on *Symmetry and nonlinear diffusion flows...* a course on entropy methods (see webpage)

- [JD, Maria J. Esteban, and Michael Loss] *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs* ... the elliptic point of view: arXiv: 1711.11291

- [JD, Maria J. Esteban, and Michael Loss] *Interpolation inequalities, nonlinear flows, boundary terms, optimality and linearization...* the parabolic point of view
Journal of elliptic and parabolic equations, 2: 267-295, 2016.

These slides can be found at

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Thank you for your attention !