A nonlinear kinetic Fokker-Planck equation: fundamental solution, large time asymptotics, diffusion limit and gradient flow structure

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August 31, 2026

Nonlinear Diffusion and related PDEs in (non) Euclidean Settings
Birs 26w5543

Banff, August 30 - September 4, 2026

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- Scalings and the four-step method

Joint paper with Giovanni Brigati & Guillaume Carlier
The fundamental solution of a nonlinear kinetic Fokker-Planck equation
arXiv: 2603.26650

Fundamental solution of a nonlinear kinetic equation

We consider the nonlinear kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f^m \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \qquad \text{(NKE)}$$

The *pressure* variable

$$\mathsf{P} := \frac{m}{1-m}\, f^{m-1}$$

solves

$$\partial_t \mathsf{P} = (1-m)\, \mathsf{P}\, \Delta_v \mathsf{P} - \left|\nabla_v \mathsf{P}\right|^2 - v \cdot \nabla_x \mathsf{P}$$

Special (fundamental) solution: with $A = \frac{1+d-d\,m}{3-d+d\,m}$

$$\mathsf{P}_\star(t,x,v) = \beta(t) + \frac{(1+A)}{2\,(1-A)\,t} \left(\left| v - \frac{x}{(1-A)\,t} \right|^2 + A \left| \frac{x}{(1-A)\,t} \right|^2 \right)$$

$$f_\star(t,x,v) = \left(\frac{1-m}{m}\, \mathsf{P}_\star(t,x,v) \right)_+^{\frac{1}{m-1}}$$

$$m_1 := 1 - \frac{1}{d} < m < 1 \quad \text{or} \quad 1 < m < m_2 := 1 + \frac{1}{d}$$

Intermediate asymptotics

The *fundamental solution* $f_\star$ governs the large time behaviour of all solutions of (NKE) with initial datum f_0 such that

$$f_0 \in \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)\,, \quad \|f_0\|_{\mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$$
$$g_{\gamma_1}(x, v-x) \le f_0(x,v) \le g_{\gamma_2}(x, v-x)$$

for any $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, with $(1-m)\,\gamma_2 > (1-m)\,\gamma_1 > 0$ and

$$g_\gamma(x,v) := \left(\tfrac{1-m}{m}\,\left(\gamma + \tfrac{1+A}{2}\,|v|^2 + \tfrac{1+A}{2}\,A\,|x|^2\right)\right)_+^{\frac{1}{m-1}}$$

Theorem

With $d \ge 1$, $m \in (m_1, 1) \cup (1, m_2)$ and $1/2 < m < 3/2$ if $d = 1$,

$$\lim_{t \to +\infty} \|f(t,\cdot,\cdot) - f_\star(t,\cdot,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)} = 0$$

Level lines on the phase space

The level lines of the *fundamental solution* $f_\star$ are ellipses, which rotate and expand in the phase space $\mathbb{R}^d \times \mathbb{R}^d \ni (x, v)$ as t increases

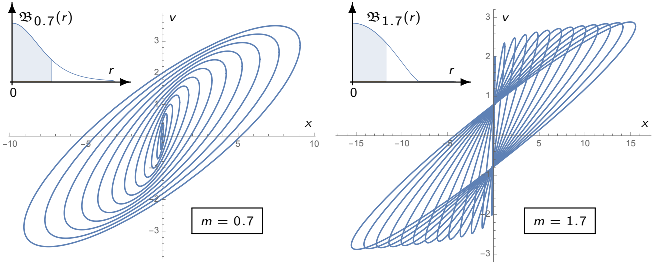

Figure: In dimension $d = 1$, the ellipse $\mathfrak{E}_m(t)$ is represented in the phase space $\mathbb{R}^d \times \mathbb{R}^d \ni (x; v)$ at times $t = 0.1$, $t = 0.6$, $1.1 \ldots 6.1$ for $m = 0.7$ (left) and $m = 1.7$ (right). In the upper left corners, the Barenblatt profiles $\mathfrak{B}_m(r) := \left(1 \pm r^2\right)_+^{1/(m-1)}$ are shown, with shaded areas corresponding to half of the mass. Here $\pm$ denotes the sign of $(1 - m)$ and r is such that $r^2 = A\,|x|^2 + |v|^2$: the function $f_\star(t, \cdot, \cdot)$ has compact support if $m > 1$

Self-similar variables

Intermediate asymptotics are replaced by estimates of convergence after a *time-dependent rescaling* based on *self-similar variables*

$$f(t,x,v)=R(t)^{-d\,(1+A)}\,g\left(\log R(t),\frac{x}{R(t)},\frac{v}{R(t)^A}-\frac{x}{R(t)}\right)$$

with initial datum $g(0,x,v)=g_0(x,v):=f_0(x,v+x)$ and

$$R(t)=\left(R_0^{1-A}+(1-A)\,t\right)^{\frac{1}{1-A}}\quad\text{and}\quad A=\frac{1+d-d\,m}{3-d+d\,m}$$

If f is a solution of (NKE), then g solves

$$\partial_t g+v\cdot\nabla_x g-A\,x\cdot\nabla_v g=\Delta_v g^m+(1+A)\,\nabla_v\cdot(v\,g)$$

Lemma

Let $d\geq1$. Then $g_\gamma\in\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}^d)$ if and only if $m\in(m_1,1)\cup(1,m_2)$. In that range, (NKE) admits a fundamental solution $f_\star$ corresponding to a stationary solution $g_\star:=\left(\frac{1-m}{m}\,\left(\gamma_\star+\frac{1+A}{2}\,|v|^2+\frac{1+A}{2}\,A\,|x|^2\right)\right)_+^{\frac{1}{m-1}}$

Mass, contraction and comparison

- The mass parameter
- Comparison, L^1-contraction and consequences
- Scalings and the four-step method

The mass parameter

Let f be an arbitrary nonnegative function in $\mathcal{C}(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}^d))$, $M>0$ a real number and

$$f_M(t,x,v):=M\,f\left(M^{2\,\zeta}\,t,M^{\zeta}\,x,M^{-\,\zeta}\,v\right)\ \text{with}\ \zeta:=-\,\frac{1}{4}\,(1-m)$$

Lemma

With the above notation, f_M solves (NKE) *if and only if f solves* (NKE)

The condition $\|g_\star\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}^d)}=1$ determines $\gamma=\gamma_\star$ such that

$$\frac{2^d}{\left(\sqrt{A}\,(1-A)\right)^d}\left|\frac{1-m}{m}\right|^{\frac{1}{m-1}}\gamma_\star^{\,d\,\frac{m-m_1}{m-1}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(1\pm|z|^2\right)_+^{\frac{1}{m-1}}\,dz=1$$

L^1-contraction and Maximum Principle

Lemma

Let $d \geq 1$ and $m \in (m_1, 1) \cup (1, m_2)$. If f_1 and f_2 solve (NKE) *in $\mathcal{C}\big(\mathbb{R}^+; \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)\big)$ in the sense of distributions, then $t \mapsto \iint_{\mathbb{R}^d \times \mathbb{R}^d} \big(f_2(t,x,v) - f_1(t,x,v)\big)_+ \,dx\,dv$ is nonincreasing*

Here $(\ )_+$ denotes the positive part

Corollary

Let $d \geq 1$ and $m \in (m_1, 1) \cup (1, m_2)$. If f_1 and $f_2 \in \mathcal{C}\big(\mathbb{R}^+; \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)\big)$ solve (NKE) *with initial data $f_{1,0}$ and $f_{2,0}$ such that $f_{1,0} \leq f_{2,0}$ a.e., then $f_1(t,\cdot,\cdot) \leq f_2(t,\cdot,\cdot)$ a.e. for all $t \in \mathbb{R}^+$*

Uniqueness and mass conservation

Reminder: $g_\gamma := \left(\frac{1-m}{m}\left(\gamma + \frac{1+A}{2}\,|v|^2 + \frac{1+A}{2}\,A\,|x|^2\right)\right)_+^{\frac{1}{m-1}}$

Proposition

(NKE) *has at most one solution* f *in* $\mathcal{C}\big(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}^d)\big)$ *with initial datum* f_0 *if* f *and* g *are related by the self-similar change of variables and*

$$g_{\gamma_1}(x,v) \le g(t,x,v) \le g_{\gamma_2}(x,v)$$

Moreover $\|f(t,\cdot,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}^d)} = \|f_0\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}^d)}$ *for any* $t\ge 0$

Any solution $f\in\mathcal{C}\big(\mathbb{R}^+;\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}^d)\big)$ of (NKE) can be rewritten by *solving the characteristics* of the free transport operator as

$$\partial_t h = \big(\Delta_v - 2\,t\,\nabla_x\cdot\nabla_v + t^2\,\Delta_x\big)\,h^m$$

where $h(t,x,v) = f(t,x+t\,v,v)$

Existence

Proposition

Under our assumptions, (NKE) has a solution g in $\mathcal{C}\left(\mathbb{R}^+; \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)\right)$

To $g(t,x,v) = G\left(\sqrt{A}\,t, A^{1/4}\,x, A^{-1/4}\,v\right)$ such that

$$\partial_t G + v \cdot \nabla_x G - x \cdot \nabla_v G = \frac{1}{A}\,\Delta_v G^m + \frac{1+A}{\sqrt{A}}\,\nabla_v \cdot (v\,G)$$

A *splitting scheme* with $G(t,\cdot) = S_n(t)\,G_0(\cdot)$, $0 \le t \le \frac{1}{n}$, $n \in \mathbb{N} \setminus \{0\}$

$$\begin{aligned} &\frac{\partial G}{\partial t} = 2\left(\frac{1}{A}\,\Delta_v G^m + \frac{1+A}{\sqrt{A}}\,\nabla_v \cdot (v\,G)\right) && \text{if} \quad 0 \le t \le \frac{1}{2\,n}\,, \\ &\frac{\partial G}{\partial t} + 2\left(v \cdot \nabla_x G - x \cdot \nabla_v G\right) = 0 && \text{if} \quad \frac{1}{2\,n} \le t \le \frac{1}{n} \end{aligned}$$

- extend with $(k,t) \in \mathbb{N} \times [0, 1/n]$ by $G_n\left(t + \frac{k}{n}, x, v\right) = S_n(t)\,G_n\left(\frac{k}{n}, x, v\right)$
- conclude with Arzelà-Ascoli

Scalings and the four-step method

A possible strategy for proving Theorem 1 is to adapt the *four-step method* using

$$f_\star(t,x,v) = t^{-d\,\frac{1+A}{1-A}}\,f_\star\left(1, t^{-\frac{1}{1-A}}\,x, t^{-\frac{A}{1-A}}\,v\right)$$

for all $t>0$, based on the L^1-preserving scale invariance $f \mapsto f_\lambda$ with

$$f_\lambda(t,x,v) := \lambda^4\,f\left(\lambda^{2\,(m-m_1)}\,t,\ \lambda^{m-m_3}\,x,\ \lambda^{m_2-m}\,v\right)$$

with $m_1 = 1-1/d$ and $m_2 = 1+1/d$ and $m_3 := 1-3/d$

- However further regularity properties (and hypoelliptic techniques) are needed for compactness issues, in order to identify $f_\star$ as the unique possible limit as $t\to+\infty$
- How much can we relax the condition on the behaviour of $f_0 \in \mathrm{L}^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ as $|x|^2+|v|^2\to\infty$?

Entropy methods in self-similar variables

- Moments and relative entropy estimates
- Convergence in L^1 and intermediate asymptotics
- Decay rates of the spatial density

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Moments and relative entropy estimates

$g_\star$ has finite mass for any $m > m_1 = \frac{d-1}{d}$
$g_\star$ admits a moment $\left(|x|^2 + |v|^2\right)$ only if $m > \widetilde{m}_1 = \frac{d}{d+1} > m_1$
The missing range is covered by considering the difference with $g_\star$

Lemma

Under our assumptions, the solution g is such that

$$\sup_{t \geq 0} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(|x|^2 + |v|^2\right) |g(t,x,v) - g_\star(x,v)| \, dx \, dv < \infty$$

As $\left(|x|^2 + |v|^2\right) \to \infty$, the right-hand side is integrable iff $m > m_1$

$$|g(t,x,v) - g_\star(x,v)| \leq g_{\gamma_2}(x,v) - g_{\gamma_1}(x,v) = O\left(\left(|x|^2 + |v|^2\right)^{\frac{2-m}{m-1}}\right)$$

Relative entropy functional as in [Newman1984], [Ralston1984]

$$\mathcal{E}[g] := \mathcal{H}[g] - \mathcal{H}[g_\star]$$

with $\mathcal{H}[g] := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{g^m}{m-1} + \frac{1+A}{2}\left(|v|^2 + A\,|x|^2\right) g\right) dx \, dv$

Relative entropy production

We shall write $\mathcal{E}(t) = \mathcal{E}[g(t,\cdot,\cdot)]$
If $m < 1$, notice that

$$\mathcal{E}[g] = \frac{1}{m-1} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(g^m - g_\star^m - m\, g_\star^{m-1}\left(g - g_\star\right)\right) dx\, dv$$

In the range $m_1 < m \le \widetilde{m}_1$, the relative entropy can be understood as the functional acting on $w = g/g_\star$ given by

$$w \mapsto \frac{1}{m-1} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(w^m - 1 - m\left(w - 1\right)\right) g_\star^m \, dx\, dv$$

Cf. [BBDGV, 2009] for details

Lemma

If g is a solution, then

$$\frac{d}{dt}\mathcal{E}(t) = -\iint_{\mathbb{R}^d\times\mathbb{R}^d} g\left|\nabla_v \mathrm{Q} - \nabla_v \mathrm{Q}_\star\right|^2 dx\, dv$$

with pressure variables $\mathrm{Q} := \frac{m}{1-m}\, g^{m-1}$ *and* $\mathrm{Q}_\star := \frac{m}{1-m}\, g_\star^{m-1}$

Convergence in L^1 ? Sketch of the proof (preliminaries)

The proof of Theorem 1 has to be completed by identifying the limit as $t \to +\infty$ of the solution g: we have to prove that

$$\lim_{t\to+\infty} \|g(t,\cdot,\cdot) - g_\star\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \mathcal{E}(t) = 0$$

if g solves the equation

$$\lim_{t\to+\infty} \int_t^{+\infty} \iint_{\mathbb{R}^d\times\mathbb{R}^d} g\,\left|\nabla_v \mathrm{Q} - \nabla_v \mathrm{Q}_\star\right|^2 dx\,dv\,ds = 0$$

Let us define the *spatial density* $\rho_g(t,x) := \int_{\mathbb{R}^d} g(t,x,v)\,dv$ and the corresponding *local equilibrium* by

$$\widetilde{g}(t,x,v) := \left(\mu(t,x) + \tfrac{1-m}{2\,m}\,(1+A)\,|v|^2\right)_+^{\frac{1}{m-1}}$$

where $\mu(\tau,x) = \mu_1\,(\rho_g(\tau,x))^{k-1}$, $\frac{1}{k-1} = \frac{d}{2} + \frac{1}{m-1}$, is found by inverting

$$\rho_g(t,x) = \int_{\mathbb{R}^d} \left(\mu(t,x) + \frac{1-m}{2\,m}\,(1+A)\,|v|^2\right)_+^{\frac{1}{m-1}} dv$$

Sketch of the proof (continued)

Gagliardo-Nirenberg + generalized Csiszár-Kullback inequalities

$$\int_{\mathbb{R}^d} g \left|\nabla_v \mathsf{Q} - \nabla_v \mathsf{Q}_\star\right|^2 dv \geq \mathcal{C}\, \frac{\|g(t,x,\cdot) - \widetilde{g}(t,x,\cdot)\|^2_{\mathrm{L}^1(\mathbb{R}^d)}}{(\rho_g(t,x))^{2-k}}$$

With $|g - \widetilde{g}| = \rho_g^{1-k/2} \cdot |g - \widetilde{g}|/\rho_g^{1-k/2}$ + Cauchy-Schwarz, we obtain

$$\lim_{t\to\infty} \int_t^{t+1} \|g(s,\cdot,\cdot) - \widetilde{g}(s,\cdot,\cdot)\|^2_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}^d)}\, ds = 0$$

Convergence in L^1 to a stationary solution ?

Assume $m < 1$ and take any sequence $(t_n)_{n\in\mathbb{N}}$ such that $0 < t_n \to \infty$, set $g_n(t,x,v) := g(t_n + t, x, v)$, $\widetilde{g}_n(t,x,v) := \widetilde{g}(t_n + t, x, v)$, $\rho_n := \rho_{g_n}$
Averaging lemmas + boundedness of $\rho_n^R(t,x) := \int_{\mathbb{R}^d} g_n(t,x,v)\,\psi_R(v)\,dv$ in $\mathrm{H}^{1/4}\big((0,1)\times\mathbb{R}^d\big)$ + a.e. and $\mathrm{L}^1\big((0,1)\times\mathbb{R}^d\big)$ convergence of $(\rho_n)_{n\in\mathbb{N}}$ to some limit ρ_∞: the limit $g_\infty = \widetilde{g}_\infty = g_\star$ solves

$$\partial_t g_\infty + v\cdot\nabla_x g_\infty - A\,x\cdot\nabla_v g_\infty = 0$$

Intermediate asymptotics in various norms

The convergence is not limited to L^1

By Hölder interpolation: general *intermediate asymptotics*

Corollary

Under our assumptions, if f solves (NKE) *with initial datum f_0 and $p \in [1, \infty)$, then*

$$\lim_{t \to +\infty} t^{d \, \frac{p-1}{p} \, \frac{1+A}{1-A}} \, \| f(t,x,v) - f_\star(t,x,v) \|_{\mathrm{L}^p(\mathbb{R}^d \times \mathbb{R}^d)} = 0$$

Consequences, linearization and formal results

- Original variables: decay of the spatial density
- Formal diffusion limit
- Scalings, translations and Galilean invariance
- Linearization, eigenfunctions and asymptotic rates

A classical estimate on the spatial density

Here we consider the *spatial density* $\rho_f(t,x)=\int_{\mathbb{R}^d} f(t,x,v)\,dv$ and assume $m>\widetilde{m}_1$ (finite moments)

Decay of the spatial density in original variables ?

Lemma

If $g\in\mathrm{L}^1_+\left(\mathbb{R}^d\times\mathbb{R}^d,|v|^2\,dx\,dv\right)\cap\mathrm{L}^\infty\left(\mathbb{R}^d\times\mathbb{R}^d,dx\,dv\right)$, *then we have*

$$\|\rho_g\|_{\mathrm{L}^{1+\frac{2}{d}}(\mathbb{R}^d)}\le\mathcal{C}_d\,\|g\|^{\frac{2}{d+2}}_{\mathrm{L}^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\,g(x,v)\,dx\,dv\right)^{\frac{d}{d+2}}$$

with $\mathcal{C}_d:=2^{d/(d+2)}\,\frac{d+2}{2\,d}\,\left|\mathbb{S}^{d-1}\right|^{2/(d+2)}$

Proof. Notice $\rho_g(x)\le\left|\mathbb{S}^{d-1}\right|\,\|g\|_{\mathrm{L}^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\,\frac{R^d}{d}+\frac{1}{R^2}\int_{\mathbb{R}^d}|v|^2\,g(x,v)\,dv$

- Minimize the r.h.s. on $R>0$
- Take the power $(d+2)/d$ of both sides of the estimate
- Integrate w.r.t. x

Fast diffusion case: kinetic energy

- Fast diffusion case $m<1$. Take $\gamma=\gamma_\star \iff \iint_{\mathbb{R}^d\times\mathbb{R}^d} g_\star\,dx\,dv=1$

$\mathcal{E}[g]=Z_m\iint_{\mathbb{R}^d\times\mathbb{R}^d}\phi(g/g_\star)\,g_\star^m\,dx\,dv$ with

$$\phi(s):=\frac{Z_m^{-1}}{m-1}\left(s^m-1-m\,(s-1)\right),\quad Z_m=\iint_{\mathbb{R}^d\times\mathbb{R}^d} g_\star^m\,dx\,dv$$

The function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is invertible, with generalized inverse ψ

Lemma (Jensen's inequality)

If $m\in(\widetilde{m}_1,1)$, then we have

$$(1+A)\,\frac{1-m}{2\,m}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\,g\,dx\,dv\le Z_m\,\psi\left(Z_m^{-1}\,\mathcal{E}[g]\right)$$

for any $g\in\mathrm{L}^1_+\big(\mathbb{R}^d\times\mathbb{R}^d,(1+|v|^2)\,dx\,dv\big)\cap\mathrm{L}^m\big(\mathbb{R}^d\times\mathbb{R}^d,dx\,dv\big)$

- Porous medium case $m>1$ is simpler

$$\frac{1}{2}\,(1+A)\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^2\,g\,dx\,dv\le\mathcal{E}[g]+\mathcal{H}[g_\star]$$

Decay rates of the spatial density

If g is a solution, we learn that $\|\rho_g(t,\cdot)\|_{L^{1+2/d}(\mathbb{R}^d)}$ is uniformly bounded in terms of g_0

Corollary

Under our assumptions, if f is a solution of (NKE) *with $f_0 \in L^1_+(\mathbb{R}^d \times \mathbb{R}^d, (|x|^2+|v|^2)\,dx\,dv) \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, then for any $t>0$, we have*

$$\|\rho_f(t,\cdot)\|_{L^{1+\frac{2}{d}}(\mathbb{R}^d)} \le \mathcal{C}[f_0]\,\big(1+(1-A)\,t\big)^{-\frac{3-d+d\,m}{(d+2)\,(m-m_1)}}$$

for some explicit constant $\mathcal{C}[f_0]$

We estimate $\rho_g = \int_{\mathbb{R}^d} g\,dv$
by $\|g\|_{\mathrm{L}^\infty(\mathbb{R}^d\times\mathbb{R}^d)}$ if $|v|\le R$ and by $g\,|v|^2/R^2$ if $|v|>R$

$$\rho_g(x) \le \left|\mathbb{S}^{d-1}\right| \|g\|_{\mathrm{L}^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\,\frac{R^d}{d} + \frac{1}{R^2}\int_{\mathbb{R}^d} |v|^2\,g(x,v)\,dv\,.$$

Optimize on $R=R(x)>0$: with $\mathcal{C}_d := 2^{d/(d+2)}\,\frac{d+2}{2\,d}\,\left|\mathbb{S}^{d-1}\right|^{2/(d+2)}$

$$|\rho_g(x)| \le \mathcal{C}_d\,\|g\|_{\mathrm{L}^\infty(\mathbb{R}^d\times\mathbb{R}^d)}^{\frac{2}{d+2}}\left(\int_{\mathbb{R}^d} |v|^2\,g(x,v)\,dv\right)^{\frac{d}{d+2}}$$

If $g\in\mathrm{L}^1_+\big(\mathbb{R}^d\times\mathbb{R}^d,|v|^2\,dx\,dv\big)\cap\mathrm{L}^\infty\big(\mathbb{R}^d\times\mathbb{R}^d,dx\,dv\big)$, then we have

$$\|\rho_g\|_{\mathrm{L}^{1+\frac{2}{d}}(\mathbb{R}^d)} \le \mathcal{C}_d\,\|g\|_{\mathrm{L}^\infty(\mathbb{R}^d\times\mathbb{R}^d)}^{\frac{2}{d+2}}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2\,g(x,v)\,dx\,dv\right)^{\frac{d}{d+2}}$$

Formal diffusion limit / overdamped regime

- With $\varepsilon \to 0$ and $2\,(d\,m - d + 1)\,\eta = 3$,

$$f(t,x,v) = \left(\frac{\varepsilon^{2\,\eta}}{R(\varepsilon\,t)}\right)^d h_\varepsilon\left(\varepsilon^2\,\tau(\varepsilon\,t), \varepsilon^{\eta+1}\,x, \frac{\varepsilon^{\eta-1}\,v}{R(\varepsilon\,t)}\right)$$

$$R(s) = (1 + s/\alpha)^\alpha \quad \text{with } 1/\alpha = d\,m - d + 2 = d\,(m - m_c)$$

so that $(\tau,x,v) \mapsto h_\varepsilon(\tau,x,v)$ solves

$$\varepsilon\,\frac{d\tau}{ds}(s)\,\partial_\tau h_\varepsilon + R(s)\,v\cdot\nabla_x h_\varepsilon = \frac{\sigma(s)}{\varepsilon}\left(\Delta_v h_\varepsilon^m + \nabla_v\cdot(v\,h_\varepsilon)\right)$$

$\sigma(s) := (1 + s/\alpha)^{-1}$, $\tau = \tau(s)$ to be chosen, and $s = \varepsilon\,t$

- Formal *Hilbert expansion*

$$h_\varepsilon(\tau,x,v) \approx H(\tau,x,v) := \left(\mu(\tau,x) + \tfrac{1-m}{2\,m}\,|v|^2\right)_+^{\frac{1}{m-1}}$$

where $\quad \rho(\tau,x) = \displaystyle\int_{\mathbb{R}^d} \left(\mu(\tau,x) + \tfrac{1-m}{2\,m}\,|v|^2\right)_+^{\frac{1}{m-1}} dv = \left(\frac{\mu(\tau,x)}{\mu_1}\right)^{\frac{d}{2}+\frac{1}{m-1}}$

$\mu(\tau,x) = \mu_1\,\left(\rho(\tau,x)\right)^{k-1}$, $k = 1 + 2\,\alpha\,(m-1)$

- *local mass conservation* : $\rho_\varepsilon(\tau,x) := \int_{\mathbb{R}^d} h_\varepsilon(\tau,x,v)\,dv$ solves

$$\frac{1}{R}\,\frac{d\tau}{ds}\,\partial_\tau\rho_\varepsilon + \nabla_x\cdot j_\varepsilon = 0$$

- flux $j_\varepsilon(\tau,x) := \frac{1}{\varepsilon}\int_{\mathbb{R}^d} v\,h_\varepsilon(\tau,x,v)\,dv$

$$\frac{\varepsilon}{R}\,\frac{d\tau}{ds}\,\partial_\tau j_\varepsilon + \nabla_x\cdot\int_{\mathbb{R}^d} v\otimes v\,h_\varepsilon(\tau,x,v)\,dv = -\frac{\sigma}{R}\,j_\varepsilon$$

Standard heuristics apply: we drop the $O(\varepsilon)$ term

$$-\frac{\sigma(s)}{R(s)}\,j_\varepsilon(\tau,x) \approx \nabla_x\cdot\int_{\mathbb{R}^d} v\otimes v\,H(\tau,x,v)\,dv = \nu_1\,\nabla_x\left(\rho(\tau,x)^k\right)$$

for some explicit numerical constant $\nu_1 > 0$

- $\rho_\varepsilon \approx \rho$ solves the nonlinear diffusion equation

$$\partial_\tau \rho = \Delta_x \rho^k \quad \text{where} \quad \frac{1}{k-1} = \frac{1}{m-1} + \frac{d}{2}$$

in the limit as $\varepsilon \to 0$ if we choose the time scale $s \mapsto \tau(s)$ such that

$$\frac{d\tau}{ds} = \nu_1 \, \frac{R(s)^2}{\sigma(s)}$$

with the condition $\tau(0) = 0$, *i.e.*,

$$\tau(t) = \tfrac{\alpha\,\nu_1}{2\,(1+\alpha)} \left((1 + t/\alpha)^{2\,(1+\alpha)} - 1 \right)$$

- *Self-similar Barenblatt-Pattle solution*, with $1/\beta = d\,(k-1) + 2$

$$\rho(\tau, x) = (\tau/\beta)^{-d\,\beta}\, \rho_\star\big((\tau/\beta)^{-\beta}\, x\big)$$

▷ Assuming that α is positive, that is, $m > m_c$, we notice that $k-1$ and $m-1$ have the same sign because $k - 1 = 2\,\alpha\,(m-1)$

▷ This formal computation requires at least a second moment

$$m > \widetilde{m}_1 = d/(d+1) \Longleftrightarrow k > d/(d+2)$$

From invariances to linearization and asymptotic modes

A preliminary question: does the notion of fundamental solution make sense for non-centred initial data ?

Non-centred Dirac distributions as initial data...

- Translations $(x,v) \mapsto (x-x_0,v)$
- Galilean transformations $(t,x,v) \mapsto (x-t\,v_0,v-v_0)$

Proposition

The function $t \mapsto f_\star(t,x-x_0-t\,v_0,v-v_0)$ solves (NKE) *with measure initial datum $\delta(x-x_0,v-v_0)$ for any $(x_0,v_0) \in \mathbb{R}^d \times \mathbb{R}^d$*

In self-similar variables, these transformations give rise to specific asymptotic behaviours of the solutions.

Can we identify asymptotic behaviours through modes of the linearized evolution operator using the invariances of the equation ?

Linearized evolution operator and dissipation

Take $g(t,x,v)=g_\star(x,v)+\varepsilon\,h(t,x,v)$ and $\varepsilon\to 0$

$$\partial_t h=\mathcal{L}\,h$$

where the *linearized operator* $\mathcal{L}$ is defined by

$$\mathcal{L}\,h:=m\,\Delta_v\big(g_\star^{m-1}\,h\big)+(1+A)\,\nabla_v\cdot(v\,h)-v\cdot\nabla_x h+A\,x\cdot\nabla_v h$$

We assume that $\|g_\star\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}=1$ and recall that

$$m\,\big(g_\star(x,v)\big)^{m-1}:=\Big((1-m)\,\big(\gamma_\star+\tfrac{1+A}{2}\,|v|^2+\tfrac{1+A}{2}\,A\,|x|^2\big)\Big)_+$$

▷ *Accretivity*

$$\frac{d}{dt}\iint_{\mathbb{R}^d\times\mathbb{R}^d}|h|^2\,g_\star^{m-2}\,dx\,dv=-2\,m\iint_{\mathbb{R}^d\times\mathbb{R}^d}g_\star\,\big|\nabla_v\big(g_\star^{m-2}\,h\big)\big|^2\,dx\,dv$$

▷ The case $m>1$ raises various difficulties due to the compactness of the support of $g_\star$ and is omitted here

Kernel

Lemma

Let $d \geq 1$ and $m \in (m_1, 1)$

$$\mathrm{Spec}(\mathcal{L}) \subset \left\{\lambda \in \mathbb{C} \,:\, \mathrm{Re}(\lambda) \leq 0\right\}$$

and the kernel of $\mathcal{L}$ is generated by $h_0 := g_\star^{2-m}$

▷ Hint: take a derivative with respect to the mass parameter γ of the solution

$$g_\gamma(x,v) := \left(\tfrac{1-m}{m}\ \left(\gamma + \tfrac{1+A}{2}\,|v|^2 + \tfrac{1+A}{2}\,A\,|x|^2\right)\right)_+^{\frac{1}{m-1}}$$

of the stationary equation

$$v \cdot \nabla_x g - A\,x \cdot \nabla_v g = \Delta_v g^m + (1+A)\,\nabla_v \cdot (v\,g) = 0$$

We can identify real eigenvalues of $(-\mathcal{L})$ by constructing special solutions based on the invariances of (NKE) such that $\mathcal{L}\, h + \lambda\, h = 0$ and

$$g(t,x,v) - g_\star(x,v) = e^{-\lambda\, t}\, h(x,v) + o\left(e^{-\lambda\, t}\right) \quad \text{as} \quad t \to +\infty$$

- *A mode based on scaling.* Take $f_0(x,v) = f_\star(t_0,x,v)$ with $t_0 > 0$

$$h_1(x,v) = \tfrac{B}{m}\left(\gamma_\star + (B + A\,C)\,|v|^2 + C\,x\cdot v + A\,(B-C)\,|x|^2\right) h_0(x,v)$$

with $B = (1+A)/2$, $1/C = d\,(1-m)$ and eigenvalue $\lambda_1 = 1 - A$

- *A mode based on translations with respect to v in the phase space* $\lambda_2 = A$ and $h_{2,i}(x,v) := (v_i - x_i)\, h_0(x,v)$
- *A mode based on translations with respect to x in the phase space* $\lambda_3 = 1$ and $h_{3,i}(x,v) := (v_i - A\, x_i)\, h_0(x,v)$

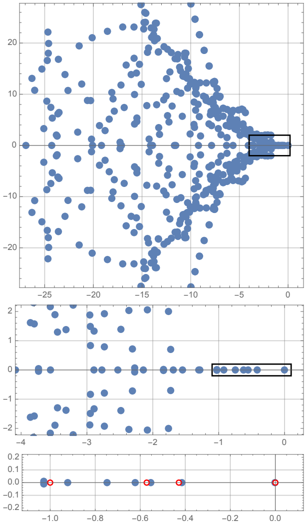
▷ *Conjecture.* For some values of m, $\mathcal{L}$ is sectorial and

$$\mathrm{Spec}(\mathcal{L}) \setminus \{0\} \subset \left\{\lambda \in \mathbb{C} \,:\, \mathrm{Re}(\lambda) \le -\min\{A, 1-A\}\right\}$$

▷ Simpler questions: $\sup\left\{\mathrm{Re}(\lambda) \,:\, \lambda \in \mathrm{Spec}(\mathcal{L}) \setminus \{0\}\right\} < 0$? Is there some $\lambda_\star > 0$ such that $\limsup_{t\to+\infty} e^{\lambda_\star t}\,\mathcal{E}(t) < +\infty$? Which space ?

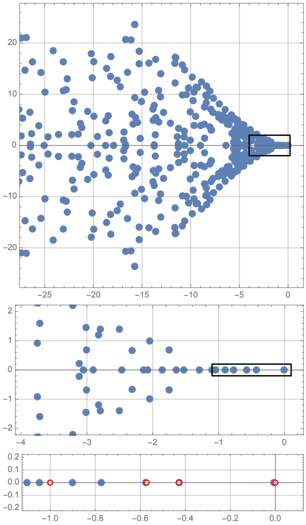
Dirichlet BC in the rectangle $[-18, 18] \times [-28, 28]$

$m = 0.8$, $d = 1$



Dirichlet BC in the ellipse with same volume

$m = 0.8,\ d = 1$



Further results

- Gradient flows
- Existence and diffusion limits

Nonlinear kinetic Fokker–Planck equations as gradient flows of the free energy

Joint paper with Giovanni Brigati, Guillaume Carlier & Filippo Quattrocchi

Nonlinear kinetic Fokker-Planck equations as gradient flows of the free energy

arXiv: 2606.16315

Nonlinear kinetic Fokker–Planck equation

Nonlinear kinetic Fokker–Planck equation ($m > 1$, porous medium case only)

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \big(f\,(\nabla_v P_m(f) + v)\big) \qquad \text{(kFP)}$$

time $t \geq 0$, phase space $\Gamma := \mathbb{R}^d \times \mathbb{R}^d \ni (x, v)$, *pressure variable* $P_m(f)$

$$P_m(f) := \frac{m}{m-1}\, f^{m-1}$$

$P_1(f) := \log f$ (case $m = 1$): linear Vlasov–Fokker–Planck equation corresponding to the second-order Langevin stochastic equation

Free energy and generalized *Fisher information*

$$\mathcal{F}_m(f) := \iint_\Gamma \left(\frac{1}{m}\, P_m(f) + \frac{1}{2}\,|v|^2\right)\, f\,dx\,dv$$

$$\mathcal{I}_m(f) := \iint_\Gamma |\nabla_v P_m(f) + v|^2\; f\,dx\,dv$$

are such that, for smooth solution with sufficient decay properties

$$\frac{d}{dt}\mathcal{F}_m\big(f(t,\cdot,\cdot)\big) = -\,\mathcal{I}_m(f)$$

Transport by Vlasov, discrepancy and speed

Newton's equations associated to the force field $F_t(x,v)$

$$x'(t) = v(t)\,, \quad v'(t) = F_t\big(x(t), v(t)\big)$$

Vlasov equation

$$\partial_t \mu_t + v \cdot \nabla_x \mu_t + \nabla_v \cdot (F_t\, \mu_t) = 0 \quad \text{in} \quad \mathcal{D}'\big((0,T)\times\Gamma\big) \qquad \text{(V)}$$

$\mathcal{P}_2(\Gamma)$: set of probability measures μ with $\iint_\Gamma \big(|x|^2+|v|^2\big)\, d\mu < \infty$
Discrepancy between μ and $\nu \in \mathcal{P}_2(\Gamma)$

$$\mathsf{d}_T^2(\mu,\nu) := \inf_{(\mu_t,F_t)_{t\in[0,T]}} T \int_0^T \|F_t\|^2_{\mathrm{L}^2(\mu_t)}\, dt\,, \quad \mu_0 = \mu\,, \quad \mu_T = \nu$$

[Brigati, Maas, Quattrocchi] Optimisers are d_T-regular curves of measures $(\mu_t)_t$ and one can define a *speed* by

$$|\mu_t'|_{\mathsf{d}} := \lim_{h\to 0_+} \frac{\mathsf{d}_h(\mu_t, \mu_{t+h})}{h}$$

Chain rule for the free energy and gradient flow

For a smooth function f of (V)

$$\nabla_v \cdot \big(f\, \nabla_v P_m(f)\big) = \Delta_v f^m\,, \quad \sqrt{f}\, \nabla_v P_m(f) = \tfrac{m}{m-1/2}\, \nabla_v \big(f^{m-1/2}\big)$$

$$\mathcal{I}_m(f) = \iint_\Gamma \left(\left(\tfrac{2\,m}{2\,m-1}\right)^2 \left|\nabla_v \big(f^{m-1/2}\big)\right|^2 + |v|^2\, f - 2\,d\, f^m \right) dx\, dv$$

Weak solutions $0 \le f \in \mathrm{L}^1_{\rm loc}\big(\mathbb{R}^+; \mathrm{L}^1(1+|v|^2,\Gamma)\big) \cap \mathrm{L}^m_{\rm loc}\big(\mathbb{R}^+; \mathrm{L}^m(\Gamma)\big)$

Theorem

Let $m \in [1,3/2]$ and $T > 0$. If $\mu_t = f(t,\cdot,\cdot)$ solves (V) in $\mathcal{P}_2(\Gamma)$ with $\int_0^T \big(\mathcal{F}_m(\mu_t) + \mathcal{I}_m(\mu_t)\big)\, dt + \int_0^T \|F_t\|^2_{\mathrm{L}^2(\mu_t)}\, dt < \infty$, for $0 < a < b < T$ a.e.

$$\mathcal{F}_m(\mu_b) - \mathcal{F}_m(\mu_a) = \int_a^b \iint_\Gamma F_t \cdot \big(\nabla_v P_m(\mu_t) + v\big)\, \mu_t\, dx\, dv\, dt$$

$$\ge -\frac{1}{2} \int_a^b \big(|\mu_t'|_{\rm d}^2 + \mathcal{I}_m(\mu_t)\big)\, dt$$

with equality iff f solves (kFP) on $[0,T]$, in which case $|\mu_t'|_{\rm d} = \mathcal{I}_m(\mu_t)$

- (kFP) is the gradient flow of the free energy for our topology

Minimising-movement scheme and maximal dissipation

A JKO type scheme $\mu_0^{(h)} = \mu_0 \in \mathcal{P}_2(\Gamma)$ with $\mathcal{F}_m(\mu_0) < \infty$

$$\begin{aligned} &\mu_0^{(h)} = \mu_0 \in \mathcal{P}_2(\Gamma) \quad \text{with} \quad \mathcal{F}_m(\mu_0) < \infty \\ &\mu_{k+1}^{(h)} \in \arg\min_{\nu \in \mathcal{P}_2(\Gamma)} \left(\mathcal{F}_m(\nu) + \frac{1}{2\,h}\, \mathrm{d}_h^2\left(\mu_k^{(h)}, \nu\right)\right) \end{aligned} \tag{JKO}$$

Piecewise constant interpolation: $\mu_t^{(h)}$

Theorem

Let $m \geq 1$

- *(JKO) is well-posed for any $h \in (0,1)$ and $k \in \mathbb{N}$*
- $\left(\mu_t^{(h)}\right)_h$ *relatively compact in* $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{L}^1(\Gamma))$
- *Up to the extraction of a sequence, any limit* $\mu = \lim_{h\to 0_+} \mu^{(h)}$

solves (kFP) in $\mathcal{D}'\left(\mathbb{R}^+ \times \Gamma\right)$ *with initial datum* μ_0

Existence and diffusion limits

Joint paper with Emeric Bouin & Antoine Mellet
Nonlinear kinetic Fokker-Planck equations: existence and diffusion limits
Preprint hal-05730128

Nonlinear kinetic Fokker–Planck equation: existence

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \Delta_v f^m + \nabla_v \cdot (v\, f) \quad \text{with} \quad t > 0\,,\ x \in \Omega\,,\ v \in \mathbb{R}^d$$

Moments and the free energy functional

$$\mathcal{M}_{a,b}[f] := \iint_{\Omega \times \mathbb{R}^d} \left(|x|^a + |v|^b\right) f_{\rm in}(x,v)\, dx\, dv$$

$$\mathcal{F}[f] := \iint_{\Omega \times \mathbb{R}^d} \left(\frac{f^m}{m-1} + \frac{|v|^2}{2}\, f\right) dx\, dv$$

$$m_c := (d-2)/d,\ m_0 := d/(d+2) \text{ and } m_1 := (d-1)/d$$

Theorem

Let $\Omega = \mathbb{T}^d$ or $\mathbb{R}^d$ with $d \ge 1$ and assume that $m \in (m_c, 1) \cup (1, +\infty)$. For any initial condition $f_{\rm in} \in \mathrm{L}^1_+ \cap \mathrm{L}^\infty(\Omega \times \mathbb{R}^d)$ such that $\mathcal{M}_{a,b}[f_{\rm in}]$ is finite for some specific a and b, Equation (kFP) has a unique solution $f \in \mathrm{L}^\infty_{\rm loc}\big(\mathbb{R}^+; \mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega \times \mathbb{R}^d)\big)$ with $\nabla_v f^{q/2} \in \mathrm{L}^2_{\rm loc}\big(\mathbb{R}^+ \times \Omega \times \mathbb{R}^d\big)$ for all $q > m$. Moreover, if $m > m_0$, the solution f satisfies the entropy dissipation inequality

$$\mathcal{F}[f(t)] + \int_0^t \iint_{\Omega \times \mathbb{R}^d} f \left| v + \frac{m}{m-1} \nabla_v f^{m-1} \right|^2 dx\, dv\, dt \le \mathcal{F}[f_{\rm in}]$$

Diffusion limit

$$\varepsilon^2 \frac{\partial f}{\partial t} + \varepsilon\, v \cdot \nabla_x f = \Delta_v f^m + \nabla_v \cdot (v\, f) \quad \text{with} \quad t>0\,,\ x \in \Omega\,,\ v \in \mathbb{R}^d \qquad (\mathrm{kFP}_\varepsilon)$$

... limit as $\varepsilon \to 0_+$? Local equilibrium with spatial density ρ

$$G[\rho](v) := \left(\mu_1\, \rho^{k-1} - \frac{m-1}{2\,m}\,|v|^2\right)_+^{\frac{1}{m-1}}$$

Theorem

Let $\Omega = \mathbb{T}^d$ or $\mathbb{R}^d$ with $d \geq 1$ and assume that $m \in (m_1, 1) \cup (1, +\infty)$. Let $f_{\rm in} \in \mathrm{L}^1(\Omega \times \mathbb{R}^d)$ be such that $\mathcal{F}[f_{\rm in}]$ is finite and $0 \leq f_{\rm in} \leq G[M]$ a.e. for some constant $M>0$ and, if $\Omega = \mathbb{R}^d$, $\mathcal{M}_{a,0}[f_{\rm in}]$ is finite for some $a \in (0,1)$ if $m<1$ and $a=2$ if $m>1$. For all $T>0$, the solution f_ε to $(\mathrm{kFP}_\varepsilon)$ with initial condition $f_{\rm in}$ converges strongly in $\mathrm{L}^1\big((0,T) \times \Omega \times \mathbb{R}^d\big)$ to $G[\rho]$ and $\rho_\varepsilon := \int_{\mathbb{R}^d} f_\varepsilon(\cdot,\cdot,v)\,dv$ converges strongly in $\mathrm{L}^1\big((0,T) \times \Omega\big)$ to the solution ρ to

$$\partial_t \rho = \nu_1\, \Delta \rho^k \quad \text{in } [0,T] \times \Omega \quad \text{where} \quad \frac{1}{k-1} = \frac{1}{m-1} + \frac{d}{2}$$

with initial condition $\rho_{\rm in} = \int_{\mathbb{R}^d} f_{\rm in}(\cdot, v)\,dv$

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/
▷ Lectures

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Thank you for your attention !