MEAN FIELD MODELS IN GRAVITATION AND CHEMOTAXIS

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$\boxed{I}$ GRAVITATIONAL VLASOV-POISSON SYSTEM

also known as Jeans' equation or even "non collisional Boltzmann equation"

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x \phi\cdot\nabla_v f = 0 & \text{(Vlasov)} \\ \Delta\phi = \rho_f(t,x) := \int_{\mathbb{R}^3} f(t,x,v)\,dv & \text{(Poisson)} \end{cases}$$

$f = f(t,x,v) \geq 0$ is the distribution function; defined on the phase space:

time: $t \geq 0$; position: $x \in \mathbb{R}^3$; velocity: $v \in \mathbb{R}^3$

Vlasov's equation $\Longleftrightarrow$ $\frac{d}{dt} f(t, X(t), V(t)) = 0$ if (X,V) obeys Newton's equations:

$\dot{X} = V$, $\dot{V} = -\nabla_x \phi(t,X)$: the characteristics

By the Hamiltonian structure: $\frac{d}{dt}\left[\frac{1}{2}|V(t)|^2 + \phi(t,X(t))\right] = 0$ if $\partial_t\phi = 0$

Poisson's equation ↳ the nonlinear (quadratic) term.

↳ a non local term

$$\phi_f(t,x) = -\frac{1}{4\pi|\cdot|} *_x \rho_f(t,\cdot)$$

$$\partial_t f + \mathrm{div}_x (v f) - \mathrm{div}_v (\nabla_x \phi f) = 0 , \quad f(t=0,x,v) = f_0(x,v) .$$

Properties: (i) conservation of Lebesgue's norms:

$$S = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \beta(f)(t,x,v)\, dx\, dv = \iint \beta(f_0)\, dx\, dv$$

does not depend on t.

↳ **Mass is conserved:**

$$M = \iint f\, dx\, dv = \iint f_0\, dx\, dv \text{ does not depend on } t .$$

↳ **L^p norms are conserved** $1 \le p < \infty$

$$\| f(t,\cdot,\cdot) \|_{L^p(\mathbb{R}^3_x \times \mathbb{R}^3_v)} = \| f_0 \|_{L^p(\mathbb{R}^6)}$$

↳ **uniform bound:**

$$\| f(t,\cdot,\cdot) \|_{L^\infty(\mathbb{R}^6)} \le \| f_0 \|_{L^\infty(\mathbb{R}^6)}$$

(ii) **Conservation of energy:**

$$E = \iint f(t,x,v) \frac{|v|^2}{2}\, dx\, dv + \frac{1}{2} \iint f(t,x,v)\, \phi(t,x)\, dx\, dv$$

$$= \iint_{\mathbb{R}^6} \frac{|v|^2}{2} f\, dx\, dv - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx$$

(iii) **Conservation of the free energy:** $F = E + \underset{\uparrow \text{ temperature } (T=1)}{T} \overset{\nwarrow \text{ entropy (for some function } \beta)}{S}$

Cauchy Problem for the gravitational Vlasov-Poisson system.

Two approaches :

1) classical (C^1) solutions, f_0 with compact support
[Pfaffelmoser], [Schaeffer], [Glassey]

2) weak solutions : $0 \leq f_0 \in L^1 \cap L^p$ + moments
[Lions - Perthame] (additional regularity is propagated)

[3) renormalized solutions: $f_0 \in L^1 \cap L^1 \log L^1 + E < \infty$ [DiPerna - Lions]]

Setting of this talk : $f_0 \in L^1_+ \cap L^\infty$ (with compact support)
Then there exists a solution in $L^\infty(dt, L^1 \cap L^\infty(dx\,dv))$ such that

$$\iint f(t,x,v)\,dx\,dv = M = \|f_0\|_{L^1}.$$

$$\iint \frac{1}{2}\left(|v|^2 + \phi_f(t,x)\right) f(t,x,v)\,dx\,dv \leq E = E[f_0]$$

$$\|f(t,\cdot,\cdot)\|_{L^\infty} \leqslant \|f_0\|_{L^\infty} \quad t \text{ a.e.}$$

Goal: stability of stationary solutions ... which ones?
Minimize E? Pb : E is not bounded from below...

[J.D., Sánchez, Soler '04]

Minimize: $E = \underbrace{E_{KIN}(f)}_{\frac{1}{2}\iint f|v|^2 dx\,dv} - \underbrace{E_{POT}(f)}_{\frac{1}{2}\int|\nabla\phi|^2 dx}$

on $\Gamma_M = \{ f \in L^1_+(\mathbb{R}^6) : \iint f\,dx\,dv = M,\ \|f\|_{L^\infty} \leq 1 \}$

<u>Why can we do that?</u>

Hardy-Littlewood-Sobolev inequality

$$\int|\nabla\phi|^2 dx \leq C_{HLS} \|\underbrace{\rho}\|^2_{L^{6/5}(\mathbb{R}^3)}$$

Hölder: $\|\rho\|_{L^{6/5}} \leq \underbrace{\|\rho\|^{7/12}_{L^1}}_{=M} \underbrace{\|\rho\|^{5/12}_{L^{5/3}}}$

Interpolation: $\int_{\mathbb{R}^3} \rho^{5/3} dx \leq C \cdot \underbrace{\|f\|^{2/3}_{L^\infty}}_{\leq 1} \cdot \underbrace{\iint f|v|^2 dx\,dv}_{2E_{KIN}(f)}$

$$\boxed{E_{POT}(f) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\phi|^2 dx \leq \mathcal{C}\, \|f\|^{7/6}_{L^1(\mathbb{R}^3)} \|f\|^{1/3}_{L^\infty} \left(\iint f|v|^2 dx\,dv\right)^{1/2}}$$

$E(f) \geq E_{KIN}(f) - \text{Const.}\left(E_{KIN}(f)\right)^{1/2}$ on Γ_M

Scaling: $f^{\sigma}(x,v) = f(\sigma x, \frac{v}{\sigma})$, $\sigma > 0$

$$E(f^{\sigma}) = \sigma^2 E_{KIN}(f) - \sigma E_{POT}(f) \geq -\frac{1}{4}\frac{E_{POT}^2(f)}{E_{KIN}(f)} \quad \text{(optimize on } \sigma>0\text{)}$$

Lemma: $$I_M := \inf_{f \in \Gamma_M} E(f) = -\frac{1}{4}\,\mathcal{C}\,M^{7/3}$$

$\uparrow$ the optimal constant of the interpolation inequality

Some steps of the minimization procedure:

- $I_M = M^{7/3} I_1$
- $\lambda = M - \int_{|x|<R} dx \int_{\mathbb{R}^3} dv\, f \in (0, M]$

$$E(f) - I_M \geq -\frac{7}{3}\left(\frac{I_M}{M^2} + \frac{1}{4\pi R}\right)(M-\lambda)\lambda$$

if ρ_f is radially symmetric

Theorem: Any minimizing sequence (f_n) for I_M in Γ_M, with radial nonincreasing spatial densities (ρ_{f_n}), converges to a minimizer f_M with support in $\underset{\ni x}{B_{R_0}} \times \underset{\ni v}{\mathbb{R}^3}$, $R_0 = \frac{3M^2}{28\pi |I_M|}$.

Moreover $$f_M(x,v) = \mathbb{1}_{\left\{\frac{1}{2}|v|^2 + \phi_{f_M} - \frac{7}{3}\frac{I_M}{M} \leq 0\right\}}(x,v)$$
is uniquely determined by a nonlinear Poisson equation.

All minimizers are equal to f_M up to a translation w.r.t. x.

An orbital stability result.

Define the distance :

$$d(g,h) = E(g) - E(h) + \int_{\mathbb{R}^3} |\nabla\phi_g - \nabla\phi_h|^2 dx$$

Denote by f^* the radially symmetric non increasing rearrangement of f with respect to the variable x.

Theorem For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(f_0, f_M) < \delta \Rightarrow d(f^*(t,\cdot,\cdot), f_M) < \varepsilon \ \forall t \geq 0$$

Disclaimer 1) Many contributions to orbital stability of gravitational systems :

Rein, Wolansky, Guo, Soler, Sanchez, Schaeffer, Burchard...
Lemou - Mehats - Raphaël

2) Many other aspects of the study of these models + more realistic models
relativistic effects / dispersion effects (not well understood in the gravitational case) / moments estimates / etc.

Stationary solutions of the Vlasov-Poisson system (with constrains)

$$f_\infty(x,v) = g\left(\frac{|v|^2}{2} + \phi(x) + \mu\right)$$

Solves the Vlasov equation

Let $G(\phi) := \int_{\mathbb{R}^3} g\left(\frac{|v|^2}{2} + \phi\right) dv = 4\pi \int_0^{+\infty} s^2\, g\left(\frac{s^2}{2} + \phi\right) ds$

$\Longrightarrow$ We have a solution of the system if we can find a solution of the nonlinear Poisson equation $\quad \Delta\phi = G(\phi+\mu) = \rho \quad \mathbb{R}^3$

under the mass constraint : $\quad M = \int_{\mathbb{R}^3} \rho\, dx = \int_{\mathbb{R}^3} G(\phi+\mu)$.

↳ this implicitely determines $\mu = \mu[\phi]$, a "Lagrange m" associated to the straint.

$\Longrightarrow$ A possible strategy is to minimize the functional:

$$\phi \longmapsto \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2 dx + \int_{\mathbb{R}^3} \mathcal{G}\left(\phi + \mu[\phi]\right) dx \; [\phi]$$

where $\mu[\phi]$ is such that : $\int_{\mathbb{R}^3} G(\phi+\mu[\phi])$.

A "dual formulation" : minimize the free energy

$F[f] = E[f] + S[f]$ under the mass constraint $\iint f \, dx\, dv = M$

(temperature $= 1$)

Energy : $E(f) = \frac{1}{2} \iint f(|v|^2 + \phi_f)\, dx\, dv$

"Entropy" : $S(f) = \iint \beta(f)\, dx\, dv$ (Casimir energy).

A minimizer (a critical point) satisfies

$$\beta'(f_\infty) + \frac{|v|^2}{2} + \phi_{f_\infty} + \mu = 0$$

Now a true Lagrange multiplier

(at least on each connected component of the support of f_∞)

$$\hookrightarrow f_\infty = g\left(\frac{|v|^2}{2} + \phi_{f_\infty} + \mu\right) \text{ where } g(s) = (\beta')^{-1}(-s)$$

to be appropriately defined.

$\beta \longleftrightarrow g$: a Legendre transform (up to a sign)

Examples : 1) $\beta(s) = s \log s$: $g(s) = e^{-s}$

a major drawback $f_\infty \notin L^1$!

2) Polytropic gases : $g(s) = (-s)_+^k$ $f_\infty \in L^1$, compactly supported

\+ experimental "evidences"

Some Remarks

- Polytropic gases, globular clusters, special stationary solutions, etc.:

  [Binney - Tremaine '87]

- How generic are the stationary solutions of the form

$$f_\infty(x,v) = g\left(\frac{|v|^2}{2} + \phi_{f_\infty} + \lambda\right) \quad ?$$

$\Longrightarrow$ Jeans' "theorem"

- Orbital stability is a very weak notion:

  * we have as many stationary solutions as we have functions g (or β

    (only constraints are: monotonicity of $g \Longleftrightarrow$ convexity of β

    + compactness in the interpolation inequality needed to control E_{POT})

  * stability always holds up to translations / galilean transforms

  * orbital stability holds with respect to all stationary solutions simultaneously!

$\boxed{\text{II}}$. THE KELLER-SEGEL SYSTEM : EXISTENCE, BLOW-UP, MEASURE VALUED SOLUTIONS

The (Patlak-) Keller-Segel model for chemotaxis:

$$\begin{cases} \partial_t \rho = \operatorname{div}_x (\rho \nabla \phi + \nabla \rho) & t \geq 0 \\ \Delta \phi = \rho & x \in \mathbb{R}^2 \end{cases}$$

adimensionalized form + the parabolic-elliptic form
+ chemotactic sensitivity = 1, cell diffusivity = 1,
chemical diffusivity = 1, reaction rate = 0.

initial data : $\rho_0 \in L^1_+ \cap L^\infty(\mathbb{R}^2)$, $\int_{\mathbb{R}^2} \rho_0 |x|^2 dx < \infty$.

Blow-up?

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \rho(t,x) dx = \int |x|^2 \Delta \rho + \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} 2x \frac{x-y}{|x-y|^2} \rho(t,x) \rho(t,y) dx dy$$

$$= 4M \left(1 - \frac{M}{8\pi}\right)$$

Here we use:

$$\phi = \frac{1}{2\pi} \log|\,| * \rho$$

$$\nabla \phi = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \rho(y) dy$$

$M > 8\pi$: $\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \rho(t,x) dx < 0$

blow-up in finite time!

Existence theory.

[Jäger-Luckhaus] $\frac{d}{dt}\int_{\mathbb{R}^2} \rho(t,x)\log\rho(t,x)\,dx = -4\int_{\mathbb{R}^2}|\nabla\sqrt{\rho}|^2 - \int \nabla\phi\,\nabla\rho\,dx$

$$= -4\int_{\mathbb{R}^2}|\nabla\sqrt{\rho}|^2 + \int \rho^2\,dx$$

$$= \mathrm{RHS}$$

↳ Use the Gagliardo-Nirenberg-Sobolev inequality:

$$\|u\|^2_{L^p(\mathbb{R}^2)} \leq C^{(p)}_{GNS}\,\|\nabla u\|^{2-\frac{4}{p}}_{L^2(\mathbb{R}^2)}\,\|u\|^{\frac{4}{p}}_{L^2(\mathbb{R}^2)},$$

with $p=4$, $u=\sqrt{\rho}$

$$\int \rho^2\,dx \leq \left(C^{(4)}_{GNS}\right)^2 \int|\nabla\sqrt{\rho}|^2 \cdot M$$

$$\mathrm{RHS} \leq 0 \text{ if } M \leq 4\left(C^{(4)}_{GNS}\right)^{-2} \approx \underbrace{1.862\ldots \times 4\pi}_{\approx 7.448\pi} < 8\pi$$

A sharper approach: consider the free energy $F := \int_{\mathbb{R}^2} \rho \log \rho \, dx - \frac{1}{2}\int_{\mathbb{R}^2} \rho \phi \, dx$ and compute its time derivative:

$$\frac{dF}{dt} = -\int \rho \left| \frac{\nabla \rho}{\rho} + \nabla \phi \right|^2 dx < 0 .$$

[J.D. - Perthame] Observe that F is bounded from below if $M \leq 8\pi$.

Logarithmic Hardy-Littlewood-Sobolev inequality [Carlen-Loss] [Beckner]

Let $f \in L^1_+(\mathbb{R}^2)$ be such that $f \log f \in L^1$ and $f \log(1+|x|^2) \in L^1(\mathbb{R}^2)$.

If $\int_{\mathbb{R}^2} f \, dx = M$, then

$$\boxed{\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log|x-y| \, dx\, dy \geq M(1+\log \pi)}$$

$$F = \underbrace{(1-\theta) \int_{\mathbb{R}^2} \rho \log \rho \, dx}_{\theta \in (0,1) \iff M \in (0, 8\pi)} + \theta \left\{ \int \rho \log \rho \, dx + \frac{1}{4\pi\theta} \iint \rho(x) \rho(y) \log|x-y| \, dx \, dy \right\}$$

$$= \frac{2}{M} \iff \frac{M}{8\pi} = \theta$$

↑ use Carleman identity.

[Blanchet - J.D. - Perthame] regularization, gain of regularity, intermediate asymptotics...
[Goudon]

Summary :

1) $M < 8\pi$: Existence of a global solution, which vanishes as $t \to \infty$.

2) $M > 8\pi$: Blow-up in finite time

3) $M = 8\pi$: global solution [Biler-Karch-Laurençot-Nadzieja]
[Blanchet-Carrillo-Masmoudi]

$\Longrightarrow$ We are interested in case 2) : at blow-up time, mass concentrates in a point (formal) and new dynamics have to be established

[Herrero-Velázquez] [Velázquez]

[Disclaimer : many related papers...]

Ⓐ ↳ "volume filling effects" [Hillen-Painter] (Velázquez] [Dolak-Schmeiser]

Ⓑ ↳ "finite sampling radius" [Hillen, Painter, Schmeiser]

Ⓒ ↳ density dependent chemotactic sensitivities [Velázquez]

Ⓓ ↳ kinetic models + diffusive limits [Chalub, Markowich, Perthame]

Case Ⓒ : formal solution is made of an absolutely continuous (wrt Lebesgue) measure

+ Dirac point masses

But : dynamics of the KS system depends on the regularization!

Coll. C. Schmeiser

A simple regularization:

$$\partial_t \rho^\varepsilon = \mathrm{div}_x \left(\nabla \rho^\varepsilon + \rho^\varepsilon \nabla \phi^\varepsilon\right)$$

$$\phi^\varepsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\left(|x-y| + \varepsilon\right) \rho(y)\, dy$$

Properties: $\|\rho^\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^2)} = \|\rho_0\|_{L^1} = M$

$$\|\rho^\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le C\left(1 + \frac{1}{\varepsilon^2}\right)$$

Take a test function $\psi \in C_0^\infty(\mathbb{R}^2)$:

$$\int \psi \rho^\varepsilon \nabla \phi^\varepsilon dx = \frac{1}{4\pi} \iint \frac{\left(\psi(x) - \psi(y)\right)(x-y)}{|x-y| \left(|x-y| + \varepsilon\right)} \rho^\varepsilon(x) \rho^\varepsilon(y)\, dx\, dy$$

$$\left| \int \psi \rho^\varepsilon \nabla \phi^\varepsilon dx \right| \le \frac{M^2}{4\pi} \|\varphi\|_{1,\infty}$$

Follow Poupaud's approach (Poupaud '02: defect measures...)

Consider ρ^ε and $m^\varepsilon(t,x) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} K^\varepsilon(x-y)\, \rho^\varepsilon(t,x)\, \rho^\varepsilon(t,y)\, dx\, dy$

as measures.

$$K^\varepsilon(x) = \frac{x^{\otimes 2}}{|x| \left(|x| + \varepsilon\right)}$$

a matrix valued function

Lemma [Poupaud] $\rho^\varepsilon, m^\varepsilon$ are tightly bounded locally wrt t
ρ^ε is tightly equicontinuous in t.

Limits: $\rho^\varepsilon \longrightarrow \rho$

$$m^\varepsilon \longrightarrow m = \underset{\substack{\uparrow \\ \text{defect measure.}}}{\nu} + \iint K(x-y)\,\rho(t,x)\,\rho(t,y)$$

$$\nwarrow \begin{cases} K(x) = \dfrac{x^{\otimes 2}}{|x|} \text{ if} \\ K(0)=0 \end{cases}$$

The atomic support: $S_{at}(\rho(t)) := \{a \in \mathbb{R}^2 : \rho(t)(\{a\}) > 0\}$ is an at nble set.

Lemma [Poupaud] ν is symmetric, nonnegative, and satisfies.

$$\operatorname{tr}\left(\nu(t,x)\right) \le \sum_{a \in S_{at}(\rho(t))} \left(\rho(t)(\{a\})\right) \delta(x-a)$$

Theorem [JD-Schmeiser] the limit (ρ, ν) is a measure valn of

$$\partial_t \rho + \operatorname{div}_x \left(j(\rho,\nu) - \nabla\rho\right) = 0$$

where: $\forall \psi \in C^1_b((0,T)\times\mathbb{R}^2)$

$$\int_0^T \int_{\mathbb{R}^2} \psi \; j(\rho,\nu)\; dx\, dt = -\frac{1}{4\pi} \int_0^T \iint (\psi(t,x) - \psi(t,y))\, K(x-y)\, \rho\, dx\, dy\; dt$$
$$- \frac{1}{4\pi} \int_0^T \int \nu(t,x)$$

A strong formulation (formal)

continuous (!)

$$\text{If} \quad \rho(t,x) = \bar\rho(t,x) + \sum_n M_n(t)\, \delta(x - x_n(t)) \qquad (*)$$

$$\text{then} \quad \nu = \sum_n 4\pi M_n \, \mathrm{Id}\, \delta(x - x_n(t))$$

N.B. $\operatorname{tr} \nu = 8\pi M_n \leq M_n^2$
$\Longrightarrow M_n \geq 8\pi$

and

$$\partial_t \bar\rho = \operatorname{div}\left(\nabla\bar\rho + \bar\rho \nabla\bar\Phi\right) + \frac{1}{2\pi} \nabla\bar\rho \cdot \sum_n M_n \frac{x - x_n}{|x - x_n|^2}$$

$$\dot M_n = \bar\rho(t, x_n(t))\, M_n(t)$$

$$\dot x_n = -\nabla\bar\Phi(t, x_n(t)) - \frac{1}{2\pi} \sum_{m \neq n} M_m \frac{x_n - x_m}{|x_n - x_m|^2}$$

Notice that the total mass is conserved: $\frac{d}{dt}\left(\int \bar\rho \, dx + \sum_n M_n(t)\right) = 0$.

Open question: is it true that ρ takes the form $(*)$?

Further remarks and open questions

- If ρ satisfies $(*)$: $\frac{d}{dt}\int_{\mathbb{R}^2} \rho |x|^2\,dx = \bar{M}\left(4 - \frac{M}{2\pi} - \frac{M-\bar{M}}{2\pi}\right) - \frac{1}{2\pi}\sum_{\substack{a,b\in S_{at}\\ a\neq b}} \Bigl(\text{[illegible]}\Bigr)$

$$< 0$$

↳ two pure Dirac masses $(\bar{\rho}\equiv 0)$ merge in finite time

↳ $\exists\ \bar{M}_{crit}$ such that

$$\frac{d}{dt}\int \bar{\rho}^{\,q}\,dx \le (q-1)\left(C_{GNS}^{(2(1+\frac{1}{q}))}\right)^{2(1+\frac{1}{q})} \int \left|\nabla \bar{\rho}^{\,q/2}\right|^2 t)).M_1$$

$$\le 0 \text{ if } \bar{M} \le \bar{M}_c$$

Large time asymptotics of the measure valued solution

<u>Conjecture:</u> if $\operatorname{supp}(\rho_0)$ is bounded and $M > 8\pi$, $\rho(t,x) = M\,\delta_{x_0}(x)$ for t large e

$$x_0 = \frac{1}{M}\int_{\mathbb{R}^2} x\,\rho_0(x)\,dx$$

- <u>Blow-up profile?</u> $F_\varepsilon[\rho] = \int_{\mathbb{R}^2} \rho\log\rho\,dx + \frac{1}{4\pi}\iint \log(|x-y|+\varepsilon)\,\rho(x)\,\rho(y)\,d$

Choose $a\in\mathbb{R}^2$, define $R(\xi) = \varepsilon^2\rho(a+\varepsilon\xi)$

$$F_\varepsilon[\rho] = \left(2M - \frac{M^2}{4\pi}\right)\log\left(\frac{1}{\varepsilon}\right) + \underbrace{F_1(R)}_{}$$

bounded from below only 8π! (and there exists a mi).

If $\rho^\varepsilon(t,x)$ is a solution of the regularized problem, then

$$\underset{\longrightarrow 0}{\cancel{\varepsilon^2\partial_t R}} = \operatorname{div}_\xi\left(\nabla_\xi R + R\,\nabla\phi_1[R]\right)$$

$$\int_{\mathbb{R}^2} R(\xi)\,d\xi = \frac{1}{8\pi}\iint \frac{|\xi-\eta|}{|\xi-\eta|+1}\,R(\xi)\,R(\eta)\,d\xi\,d\eta \le \frac{1}{8\pi}\left(\int R(\xi)\,d\xi\right)^2$$

Either or $\int_{\mathbb{R}} 8\pi$

$\boxed{\text{III}}$ FROM KINETIC TO DIFFUSIVE MODELS

Many results on diffusive limits have been achieved by various authors [Poupaud], [Goudon][Mellet][Degond] etc.

2 approaches to derive nonlinear drift-diffusion équations

1) start from a particle (stochastic particles) description and take an appropriate "hydrodynamic type" limit.

2) build a hierarchy of models and derive diffusive models from kinetic theory.

[Chavanis] [Chavanis-Laurençot]
[JD - Ben Abdallah]
[JD - Markowich - Ölz - Schmeiser]

A BGK type collision kernel, with non-maxwellian local Gibbs state:

$$\partial_t f + v\cdot\nabla_x f - \underbrace{\nabla_x}\Phi\cdot\nabla_v f = \underbrace{G_f - f}_{\text{collision kernel}} =: Q(f)$$

Ⓗ assume Φ is given, for simplicity

local Gibbs state

"Local Gibbs state"

$$G_f = g\left(\frac{|v|^2}{2} + \phi(x) - \mu(t,x)\right)$$

1) **Kernel**

$$Q(f) = 0 \iff f = G_f$$

If additionally f is stationary, we recover stationary solutions $f = f_\infty$ like in Part I

A case of special interest in astrophysics: $g(s) = (-s)_+^k$ (polytropic gases).

2) **Local mass conservation:**

$$0 = \int_{\mathbb{R}^3} Q(f)\, dv \quad \text{determines} \quad \mu(t,x) = \phi(x) + \bar{\mu}(t,x)$$

let $G(\phi) = \int_{\mathbb{R}^3} g\left(\frac{|v|^2}{2} + \phi\right) dv = 4\pi \int_0^{+\infty} s^2\, g\left(\frac{s^2}{2} + \phi\right) ds$

$\bar{\mu}(t,x) = \bar{\mu}(\rho)$ is implicitly determined by the condition:

$$G(\bar{\mu}(\rho)) = \rho.$$

A parabolic rescaling:

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon\left(v \cdot \nabla_x f^\varepsilon - \nabla_x \phi\, \nabla_v f^\varepsilon\right) = G_{f^\varepsilon} - f^\varepsilon, \qquad f^\varepsilon|_{t=0} = f_0$$

Theorem [JD - Markowich - Ölz - Schmeiser] Under conditions (...)

$$f^\varepsilon \xrightarrow{L^1_{loc}} f^0 = g\left(\tfrac{1}{2}|v|^2 - \bar{\mu}(\rho(t,x))\right)$$

where ρ is a solution of $\partial_t \rho = \Delta(\nu(\rho)) + \mathrm{div}_x(\rho \nabla \phi)$, $\rho(t=0,x) = \int f_0(x,v)\, dx\, dv$

$$\nu(\rho) = \int_0^\rho s\, \bar{\mu}'(s)\, ds$$

Formal asymptotics : $\varepsilon^2 \partial_t f + \varepsilon (v \cdot \nabla_x f - \nabla_x \phi \nabla_v f) = G_f - f$

$$f = \sum_{k \geq 0} \varepsilon^k f^k$$

Identify orders by order in ε :

$O(1)$: $G^0 = f^0 \Longrightarrow f^0 = g\left(\frac{|v|^2}{2} - \bar{\mu}(\varphi^0)\right)$, $\bar{\mu}(\varphi^0) = \mu^0 - \nabla\phi$

$O(\varepsilon)$; $v \cdot \nabla_x f^0 - \nabla_x \phi \cdot \nabla_v f^0 = G^1 - f^1$

integrate with respect to v :

$$\int v f^1 \, dv = \int (v \cdot \nabla_x \mu^0 \quad g'\left(\tfrac{1}{2}|v|^2 + \phi - \mu^0\right)) v \, dv$$

$$= - \rho^0 \nabla_x \mu^0$$

$O(\varepsilon^2)$: $\partial_t f^0 + v \cdot \nabla_x f^1 - \nabla_x \phi \nabla_v f^1 = G^2 - f^2$

integrate wrt v (local mass conservation).

$$\partial_t \rho^0 = \mathrm{div}_x (\varphi^0 \nabla \mu^0) = \Delta \left(\nu(\rho^0)\right) + \nabla(\varphi^0 \nabla \phi)$$

Free energy $$F = \iint_{\mathbb{R}^3\times\mathbb{R}^3}\left(\frac{|v|^2}{2}+\phi(x)\right) f \,dx\,dv + \iint \beta(f)\,dx\,dv$$

$$\varepsilon\partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f = \frac{1}{\varepsilon}(G_f - f) = \varepsilon^{-1} Q(f)$$

$$\frac{d}{dt}F = \frac{1}{\varepsilon}\iint\left(\frac{|v|^2}{2}+\phi(x)+\beta'(f)\right) Q(f)$$

$$(\beta')^{-1}(-s) = g(s)$$

$$\frac{dF}{dt} = -\frac{1}{\varepsilon}\iint\left(\beta'(f)-\beta'(G_f)\right)\left(f-G_f\right)dx\,dv$$

Method: use free energy to bound (locally) the first and the second moments
use the div-curl lemma to conclude that $V(\rho^\varepsilon)\longrightarrow V(\rho^0)$.

Two examples: 1) $g(s)=e^{-s}$ $\beta(s)=s\log s$: linear BGK model $G_f=\rho\frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{3/2}}$

$V(\rho)=\rho$: linear drift-diffusion eq.

$$\partial_t\rho = \Delta\rho + \mathrm{div}_x(\rho\nabla\phi)$$

2) polytropic gas case: $g(s)=(-s)_+^k$

$$V(\rho) = C_k\,\rho^{\frac{k+5/2}{k+3/2}} = C_k\,\rho^m, \quad m\in\left(1,\tfrac{5}{3}\right)$$

(PM) $\partial_t\rho = \Delta\rho^m + \mathrm{div}_x(\rho\nabla\phi)$ porous medium type equation.

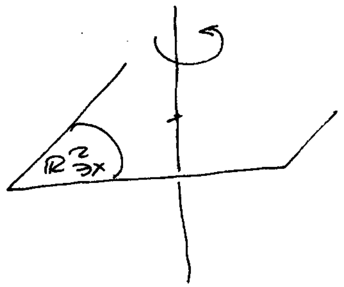
N.B. Compatibility of generalized entropies : $F[G_f] = \mathcal{F}[\rho]$ is an entropy for (PM).

POLYTROPES : FLAT ROTATING GRAVITATIONAL SYSTEMS

$$\partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2\omega v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f = G_f - f$$

In the rotating coordinate axis: $\phi = -\frac{1}{4\pi|x|} \ast_x \int_{\mathbb{R}^2} f \, dv$ $\qquad x \in \mathbb{R}^2$

$$g(s) = \left(\frac{-s}{k+1}\right)_+^k \qquad \bar{\mu}(\rho) = -(k+1)\left(\frac{\rho}{2\pi}\right)^{\frac{1}{k+1}}$$



$$G(s) = 2\pi \left(\frac{-s}{k+1}\right)_+^{k+1}$$

Free energy: $F = \iint \left(\frac{1}{2}|v|^2 f - \frac{1}{2}\omega^2 |x|^2 f + \frac{1}{2}\phi_f f + \beta(f)\right) dx\, dv$

$$\frac{dF}{dt} = \iint (G_f - f) \qquad (\beta'(f) - \beta'(G_f))\, dx\, dv \leq 0 .$$

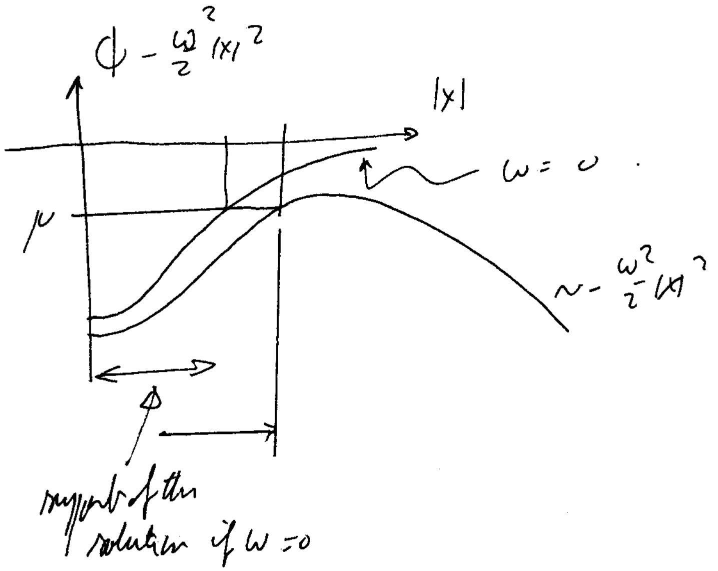
$\Longrightarrow$ take a diffusion limit.

(2) $\partial_t \rho = \mathrm{div}_x \left(\nabla \nu(\rho) - \omega^2 x \rho + \rho \nabla_x \phi \right)$

$\uparrow$ centrifugal force term

$\phi = -\frac{1}{4\pi |x|} * \rho \qquad \nu(\rho) = \frac{(2\pi)^{1-m}}{m} \rho^m$

$m = \frac{k+2}{k+1}$



$$\rho = \left(\frac{\omega^2}{2}|x|^2 - \phi + \mu \right)_+^{\frac{1}{m-1}}$$ $|x|$ not too large

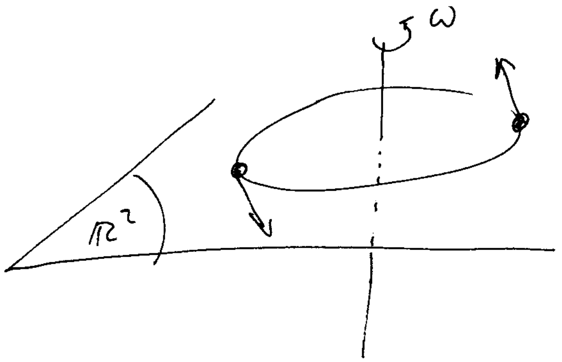
<u>Theorem</u> [J. D.-Fernandez]

There exists $\omega_* = \omega_*(M)$ such that (2) has a "localized" minimizer (stationary solution) iff $\omega \leq \omega_*$.

Functional to be minimized:

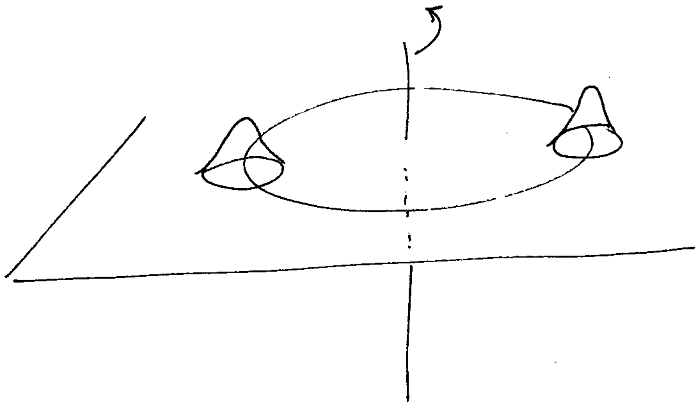
$$F[\rho] = K_m \int_{\mathbb{R}^2} \rho^m - \frac{\omega^2}{2} \int_{\mathbb{R}^2} \rho |x|^2 dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|} dx\, dy$$

MULTI-BUMP SOLUTIONS [Coll. del Pino]



Point particles picture

↓



Continuum mechanics picture

A Lyapunov-Schmidt approach (use a scaling invariance).
To any stationary configuration of points associate a critical point
(with eventually high Morse index)

Some open questions

- Large time behaviour of gravitational systems:
  $t \to +\infty$ What is left (locally)? (up to galilean transforms / translations)
- How to measure metastability in gravitation?
  is there an equivalent of the Ehrenfest time (Schrödinger equations)
- Higher order critical points: orbital stability in gravitation
  → diffusive models: use Wasserstein's distance?
  (lack of convexity)
  → kinetic models: control of the tails / supports !
- "Symmetry breaking": how to reduce from 3 D to $\underset{\substack{\uparrow \\ \text{disks}}}{2D} \ldots \underset{\substack{\uparrow \\ \text{annuli}}}{1D}$
  → No good symmetry results in mean field theory (like Gidas-Ni-Nirenberg's result).
  → dimensional aspects !
- Blow-up analysis (chemotaxis): comparable to the nonlinear Schrödinger equation?
  → gradient flows w.r.t. Wasserstein? unclear.
  → renormalization of the free energy: which description of the blowing-up solutions?