MEAN FIELD MODELS
IN GRAVITATION AND CHEMOTAXIS

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Gravitational Vlasov-Poisson System

also known as Jeans' equation or even "non collisional Boltzmann equation"

\[
\begin{cases}
\partial_t f + \mathbf{v} \cdot \nabla_x f - \nabla \Phi \cdot \nabla_v f = 0 \quad \text{(Vlasov)} \\
\Delta \Phi = \frac{\partial f}{\partial t} = \int_{\mathbb{R}^3} f(t, x, \mathbf{v}) d\mathbf{v} \quad \text{(Poisson)}
\end{cases}
\]

\[f = f(t, x, \mathbf{v}) \geq 0\] is the distribution function; defined in the phase space:

- time: \(t \geq 0\)
- position: \(x \in \mathbb{R}^3\)
- velocity: \(\mathbf{v} \in \mathbb{R}^3\)

Vlasov's equation \(\iff\) \[
\frac{d}{dt} f(t, X(t), V(t)) = 0 \quad \text{if } (X, V) \text{ obeys Newton's equations:} \\
X = V, \quad \dot{V} = -\nabla_x \Phi(t, x)
\]

By the Hamiltonian structure: \(\frac{d}{dt} \left[ \frac{1}{2} |V(t)|^2 + \Phi(t, X(t)) \right] = 0\) if \(\Phi = 0\)

Poisson's equation is the nonlinear (quadratic) term,

\(\Rightarrow\) a nonlocal term

\[\Phi(t, x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(t, \cdot)}{r} \, d\mathbf{v}\]
\[ \frac{\partial f}{\partial t} + \text{div}_x (v f) = \text{div}_v (\nabla \phi f) = 0 \quad \Rightarrow \quad f(t=0, x, v) = f_0(x, v) . \]

**Properties:**

(i) Conservation of Lebesgue's norm:

\[ S = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \beta(f) (t, x, v) \, dx \, dv = \int_{\mathbb{R}^3} \beta(f_0) \, dx \, dv - \text{does not depend on } t . \]

\( \Rightarrow \) Mass is conserved:

\[ M = \int_{\mathbb{R}^3} f \, dx \, dv = \int_{\mathbb{R}^3} f_0 \, dx \, dv - \text{does not depend on } t . \]

(ii) \( L^p \) norms are conserved:

\[ \| f(t, \cdot, \cdot) \|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)} = \| f_0 \|_{L^p(\mathbb{R}^6)} \quad 1 \leq p < \infty \]

\( \Rightarrow \) Uniform bound:

\[ \| f(t, \cdot, \cdot) \|_{L^\infty(\mathbb{R}^6)} \leq \| f_0 \|_{L^\infty(\mathbb{R}^6)} \]

(iii) Conservation of energy:

\[ E = \int_{\mathbb{R}^3} \left( \frac{v_1^2}{2} + \frac{1}{2} \int_{\mathbb{R}^3} f(t, x, v) \, \phi(x) \, dv \right) \, dx \]

\[ = \int_{\mathbb{R}^3} \left( \frac{v_1^2}{2} + \frac{1}{2} \int_{\mathbb{R}^3} \nabla \phi^2 \right) \, dx \]

(iii) Conservation of the free energy:

\[ F = E + T S \]

\( \Rightarrow \) Entropy (for some function \( \beta \))

\( \bar{T} \ell_{\beta}(T) \)
Cauchy Problem for the gravitational Vlasov-Poisson system

Two approaches:

1) Classical \( C^1 \) solutions, \( f_0 \) with compact support
   \[ \text{[Pfaffelmoser, Schaefer, Glassy]} \]

2) Weak solutions: \( 0 \leq f_0 \in L^1 \cap L^p + \text{moments} \)
   \[ \text{[Lions-Perthame]} \quad \text{(additional regularity is propagated)} \]

3) Renormalized solutions: \( f_0 \in L^1 \cap L^1 \log L^1 + E < \infty \quad [DiPerna-Lions] \)

Setting of this talk: \( f_0 \in L^1 \cap L^\infty \) (with compact support)
Then there exists a solution in \( L^\infty (t), L^1 \cap L^\infty (dx dv) \) such that

\[
\int f(t, x, v) \, dx \, dv = M = \| f_0 \|_{L^1},
\]

\[
\int \frac{1}{2} \left( |v|^2 + \phi(t, x) \right) f(t, x, v) \, dx \, dv \leq E = E[f_0]
\]

\[
\| f(t, \cdot, \cdot) \|_{L^\infty} \leq \| f_0 \|_{L^\infty} \quad \text{t.a.e.}
\]

Goal: Stability of stationary solutions ... which ones?
Minimize \( E \)? \( f_0 \): \( E \) must bounded from below...
Minimize: \[ E = \frac{E_{\text{kin}}(\phi)}{E_{\text{pot}}(\phi)} = \frac{\frac{1}{2} \int |\nabla \phi|^2 \, dx \, dv}{\frac{1}{2} \int |\nabla \phi|^2 \, dx} \]

on \( \Gamma_M = \{ f \in L^1_+ (R^6) : \int f \, dx \, dv = M, \| f \|_{L^\infty} \leq 1 \} \)

Why can we do that?

Hardy-Littlewood-Sobolev inequality:
\[ \int |\nabla \phi|^2 \, dx \leq C_{\text{HLS}} \| \phi \|^2_{L^{6/5}(R^3)} \]

Hölder: \( \| \phi \|_{L^{6/5}} \leq \| \phi \|_{L^1}^{7/12} \| \phi \|_{L^{5/3}}^{5/12} \)

Interpolation: \( \int f^{5/3} \, dx \leq C \| f \|_{L^\infty}^{2/3} \int f |v|^2 \, dx \, dv \leq 1 \)

\[ E_{\text{pot}}(\phi) = \frac{1}{2} \int |\nabla \phi|^2 \, dx \leq \frac{M}{16} \| \phi \|_{L^1}^{7/6} \| \phi \|_{L^{5/3}}^{1/3} \left( \int f |v|^2 \, dx \, dv \right)^{1/2} \]

\[ E(\phi) \geq E_{\text{kin}}(\phi) - \text{Const.} \left( E_{\text{kin}}(\phi) \right)^{1/2} \text{ on } \Gamma_M \]
Scaling: \[ f(\sigma x, \sigma^2) = f(x, v), \quad \sigma > 0 \]

\[ E(f^\sigma) = \sigma^{-2} E_{\text{kin}}(f) - \sigma E_{\text{Poisson}}(f) \geq - \frac{1}{4} \frac{E_{\text{Poisson}}(f)}{E_{\text{kin}}(f)} \quad (\text{optimization}) \]

Lemma: \[ I_M := \inf_{f \in C} E(f) = - \frac{1}{4} C M^{7/3} \]

\[ \text{where} \quad C = \text{the constant of the interpolation inequality} \]

Some steps of the minimization procedure:

1. \[ I_M = M^{7/3} I_4 \]
2. \[ \lambda = M - \int_{|x| < R} dx \int_{\mathbb{R}^3} dv \quad f \in C(M) \]

\[ E(f) - I_M \geq - \frac{7}{3} \left( \frac{I_M^3}{M^2} + \frac{1}{4 \pi R} \right) (M-1) \lambda \]

If \( f \) is radially symmetric

Theorem: Any minimizing sequence \( f_n \) for \( I_{M} \) in \( \Gamma_M \), with radial non-increasing spatial densities \( f_{\text{rad}} \), converges to a minimizer \( f_M \) with support in \( B_{R_0} \times \mathbb{R}^3 \), \( R_0 = \frac{3 M^2}{28 \pi |I_M|} \)

Moreover \[ f_M(x, u) = \Pi \left\{ \frac{1}{2} v^2 + \Phi_R - \frac{7}{3} \frac{I_M}{M} u^2 \leq 0 \right\} (x, u) \]

is uniquely determined by a nonlinear Poisson equation.

All minimizers are equal to \( f_M \) up to a translation w.r.t. \( x \).
An orbital stability result.

Define the distance:

\[ d(g, h) = E(g) - E(h) + \int_{\mathbb{R}^3} |\nabla \Phi_g - \nabla \Phi_h|^2 \, dx \]

Denote by \( f^* \) the radially symmetric non-increasing rearrangement of \( f \) with respect to the variable \( x \).

**Theorem** For any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[ d(f, f_{\delta}) < \delta \Rightarrow d(f^{\ast}(t, \ldots), f_{\delta}) < \varepsilon \quad \forall t \geq 0 \]

**Disclaimer**

1) Many contributions to orbital stability of gravitational systems:

   René, Wolansky, Gru, Soler, Sánchez, Schaeffer, Bouchard, Léonard, Richats, Raphaël

2) Many other aspects of the study of these models + more realistic models

   relativistic effects / dispersion effects (not well understood in the gravitational case) / moments estimates / etc.
Stationary solutions of the Vlasov-Poisson system (with constraints)

\[ f_\infty(x,v) = g \left( \frac{1}{2} |v|^2 + \phi(x) \right) \]

Solves the Vlasov equation

Let \( G(\phi) = \int_{\mathbb{R}^3} g \left( \frac{1}{2} |v|^2 + \phi \right) \, dv = 4\pi \int_0^{+\infty} s^2 \, g \left( \frac{s^2}{2} + \phi \right) \, ds \)

\[ \Rightarrow \]

We have a solution of the system if we can find a solution of the nonlinear Poisson equation

\[ \Delta \phi = G(\phi + \mu) = \rho \quad \text{in} \quad \mathbb{R}^3 \]

under the mass constraint:

\[ M = \int_{\mathbb{R}^3} \rho \, dx = \int_{\mathbb{R}^3} G(\phi + \mu) \, dx \]

\( \Rightarrow \)

This implicitly determines \( \mu = \mu[\phi] \), a "Lagrangian" associated to the system.

\[ \Rightarrow \]

A possible strategy is to minimize the functional:

\[ \phi \mapsto \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx + \int_{\mathbb{R}^3} G(\phi + \mu[\phi]) \, dx \]

where \( \mu[\phi] \) is such that:

\[ \int_{\mathbb{R}^3} G(\phi + \mu[\phi]) \, dx = M \]
A dual formulation: minimize the free energy

\[ F(f) = E(f) + S(f) \] under the mass constraint \( \int f \, dx = \mu \) (temperature = 1)

Energy: \( E(f) = \int f(v^2 + \phi_f) \, dx \, dv \)

Entropy: \( S(f) = \int \beta(f) \, dx \, dv \) (Cassimir energy)

A minimizer (a critical point) satisfies

\[ \beta'(\phi) + \frac{|v|^2}{2} + \phi \phi_{\phi} + \mu = 0 \]

(at least on each connected component of the support of \( \phi \))

\[ \phi_{\phi} = g \left( \frac{|v|^2}{2} + \phi + \mu \right) \text{ where } g(s) = \left( \beta' \right)^{-1}(s) \]

\[ \beta \leftrightarrow g: \text{ a Legendre transform (up to a sign)} \]

Example: 1) \( \beta(s) = s \log s \) : \( g(s) = e^{-s} \) a major drawback \( \phi_{\phi} \notin L^1 \)

2) Polytropic gases: \( g(s) = (-s)^{k} \) for \( k \geq 1 \), compactly supported + experimental "evidence"
Some Remarks

- Polytopic gases, globular clusters, special stationary solutions, etc.: [Binney - Tremaine 1987]

- How generic are the stationary solutions of the form
  \[ f_{\infty}(\xi, \nu) = g \left( \frac{\nu^2}{\xi} + \phi_{\infty} + \nu \right) \]?

  \[ \Rightarrow \text{Jeans' theorem} \]

- Orbital stability is a very weak notion:
  \* We have as many stationary solutions as we have functions of \( g \) (or \( \beta \)).
  \* Only constraints are: monotonicity of \( g \) \( \iff \) convexity of \( \beta \).
    \* Compactness in interpolating inequality needed to control \( E_{pot} \).
  \* Stability always holds up to translations/Galilean transforms.
  \* Orbital stability holds with respect to all stationary solutions simultaneously.
II. THE KELLER-SEGEL SYSTEM: EXISTENCE, BLOW-UP, MEASURE VALUED SOLUTIONS

The (Patlak-) Keller-Segel model for chemotaxis:

\[
\begin{align*}
\begin{cases}
\partial_t \rho &= \text{div}_x \left( \rho \nabla \phi + \nabla \rho \right) & t > 0, \\
\Delta \phi &= \rho & x \in \mathbb{R}^2
\end{cases}
\end{align*}
\]

- Dimensionalized form + the parabolic-elliptic form
- Chemotactic sensitivity = 1, cell diffusivity = 1, chemical diffusivity = 1, reaction rate = 0.

Initial data: \( \rho_0 \in L^1 \cap L^\infty (\mathbb{R}^2) \), \( \int_{\mathbb{R}^2} \rho_0 \|x\|^2 \, dx < \infty \).

Blow-up?:

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \|x\|^2 \rho (t, x) \, dx = \int_{\mathbb{R}^2} \|x\|^2 \Delta \rho + \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} 2x \cdot \frac{x-y}{|x-y|^2} \rho(t, x) \rho(t, y) \, dx \, dy
\]

\[
= 4M \left( 1 - \frac{M}{8\pi} \right) \geq 0 \quad \text{Here we use:} \quad \phi = \frac{1}{2\pi} \log |1 + \rho|
\]

\[ M > 8\pi : \begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^2} \|x\|^2 \rho (t, x) \, dx < 0 \quad \text{blow-up in finite time!}
\end{align*}\]
Existence theory.

\[ \frac{d}{dt} \int_{\mathbb{R}^2} \psi \log(\psi) \, dx = -4 \int_{\mathbb{R}^2} |\nabla \psi|^2 - \int D\phi \, \psi \, dx \]

\[ = -4 \int_{\mathbb{R}^2} |\nabla \psi|^2 + \int \psi^2 \, dx = \text{RHS} \]

Let us the Cagliendo - Nirenberg - Sobolev inequality:

\[ \| M \|_{L^p(\mathbb{R}^2)} \leq C_{\text{CNS}}^{(p)} \| D\psi \|_{L^2(\mathbb{R}^2)} \frac{1}{p} \| \psi \|_{L^2(\mathbb{R}^2)}^{1 - \frac{2}{p}} \]

with \( p = 4 \), \( u = \sqrt{p} \)

\[ \int \psi^2 \, dx \leq \left( C_{\text{CNS}}^{(4)} \right)^2 \int |\nabla \psi|^2 \cdot M \]

\[ \text{RHS} < 0 \text{ if } M \leq 4 \left( C_{\text{CNS}}^{(4)} \right)^{-2} \approx 1.862 \times 4\pi < 8\pi \]

\[ \approx 7.448\pi \]
A sharper approach: consider the free energy \( F = \int p \log p \, dx - \frac{1}{2} \int p \varphi \, dx \)
and compute its time derivative:

\[
\frac{dF}{dt} = - \int p \left( \frac{\partial p}{\partial t} + \Delta \varphi \right) \, dx < 0
\]

[J.D. - Penthame] Observe that \( F \) is bounded from below if \( M \leq 8\pi \).

Logarithmic Hardy-Littlewood-Sobolev inequality [Carlen-les] [Bednorz]

Let \( f \in L^1_+(\mathbb{R}^2) \) be such that \( \int \log f \, dx \leq 1 \) and \( \int \log (1 + |x|) \, dx 
\)

If \( \int f \, dx = 1 \), then

\[
F = (1 - \Theta) \int p \log p \, dx + \Theta \left\{ \int p \log p \, dx + \frac{1}{4\pi \Theta} \int \int f(x) f(y) \log |x-y| \, dx \, dy \right\} \geq 1 + \log \Theta
\]

\[
\Theta \in (0,1) \Rightarrow M \in (0, \pi)
\]

\[
\Theta \in (0,1) \Leftrightarrow M \in (0, \frac{\pi}{8})
\]

Lax Carleman identity.

[Blanchet - J.D. - Penthame] regularization, gain of regularity, intermediate asymptotics...

[Goudon]
Summary:

1) $M < 8\pi$: Existence of a global solution, which vanishes as $t \to \infty$.

2) $M > 8\pi$: Blow-up in finite time

3) $M = 8\pi$: Global solution [Biler-Karch-Lemrargot-Nadolgia], [Blanchet-Carrillo-Flaschka]

We are interested in case 2): at blow-up time, mass concentrates in jumps (point) and new dynamics have to be established [Herrero-Velázquez], [Velázquez]

[Disclaimer: many related papers...]

A $\to$ "volume filling effects" [Hillen-Painter], [Velázquez], [Dolak-Schmeiser]

B $\to$ "finite sampling radius" [Hillen, Painter, Schmeiser]

C $\to$ density dependent chemotactic sensitivities [Velázquez]

D $\to$ kinetic models + diffusive limits [Chalub, Markowich, Perthame]

Case C: formal solution is made of an absolutely continuous (wrt Lebesgue) measure + Dirac point masses

But: dynamics of the KS system depends on the regularization!
A suitable regularization:
\[ \partial_t \varphi^\varepsilon = \text{div}_x \left( \nabla \varphi^\varepsilon + \varphi^\varepsilon \nabla \Phi^\varepsilon \right) \]
\[ \Phi^\varepsilon(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \left( |x-y| + \varepsilon \right) \varphi(y) \, dy \]

Proposition: \[ \| \varphi^\varepsilon \|_{L^1(\mathbb{R}^2)} = \| \varphi_0 \|_{L^1} = M \]
\[ \| \varphi^\varepsilon(\cdot, t) \|_{L^\infty(\mathbb{R}^2)} \leq C \left( 1 + \frac{1}{\varepsilon^2} \right) \]

Take a test function \( \gamma \in C_0^\infty(\mathbb{R}^2) \):
\[ \int \gamma \varphi^\varepsilon \nabla \Phi^\varepsilon \, dx = \frac{1}{4\pi} \int \int \frac{(\gamma(x) - \gamma(y))(x-y)}{|x-y| \vert |x-y| + \varepsilon \vert} \varphi^\varepsilon(x) \varphi^\varepsilon(y) \, dx \, dy \]
\[ \left| \int \gamma \varphi^\varepsilon \nabla \Phi^\varepsilon \, dx \right| \leq \frac{M^2}{4\pi} \| \gamma \|_{1,\infty} \]

Follow Pauvard's approach [Pauvard 02: defect measures...]
Consider \( \varphi^\varepsilon \) and \( \Phi^\varepsilon(x) = \int K^\varepsilon(x-y) \varphi^\varepsilon(y) \, dy \)
\[ M^\varepsilon(x) = \frac{\pi \varepsilon^2}{|x|^2 (|x|^2 + 3)} \]

as measures.
Lemma [Paupaua]

\( f^\varepsilon, m^\varepsilon \) are tightly bounded locally in \( t \).
\( f^\varepsilon \) is tightly equicontinuous in \( t \).

Limits:
\[
p^\varepsilon \to p
\]
\[
m^\varepsilon \to m = \nu + \iint K(x - y) f(t, x) f(t, y) \, dx \, dy
\]
deficit measure:
\[
K(x) = \begin{cases} \frac{x}{|x|^3} & \text{if} \\ K(0) = 0 \end{cases}
\]

The atomic support:
\[
S_{\nu}(p(t)) = \{ x \in \mathbb{R}^2 : f(t)(t, x) > 0 \}
\]
is an admissible set.

Lemma [Poupaud]

\( \nu \) is symmetric, nonnegative, and satisfies:
\[
t(\nu(t, x)) \leq \sum_{a \in S_{\nu}(p(t))} \int (\nu(t, a) \delta(x - a))
\]

Theorem [J.D. Schmeiser]

The limit \((f, \nu)\) is a measure valued of
\[
\frac{\partial}{\partial t} p + \nabla_x \cdot (f \nu - Dp) = 0
\]
where:
\[
\int_0^T \int_{\mathbb{R}^2} \nu \cdot \frac{\partial}{\partial t} f(t, x) \, dx \, dt = -\frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2} \int_0^x (\xi(t, x) - \nu(t, y)) K(x - y) \, dy \, dx
\]
\[
\quad - \frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2} f(t, x) \, dx \, dt
\]
A strong formulation (found) \[ \mathcal{M} \rho(t,x) = \overline{\rho}(t,x) + \sum_n M_n(t) \; \overline{S}(x-x_n(t)) \] (X)

Then \[ v = \sum_n 4\pi M_n \; \text{Id} \; \overline{S}(x-x_n(t)) \]

N.B. \( 6\pi \nu = 8\pi M_n \leq M_n^2 \) \( \Rightarrow M_n \geq 8\pi \)

and \[ \partial_t \overline{\rho} = \text{div} \left( \nabla \overline{\rho} + \overline{\rho} \; \nabla \overline{\Phi} \right) + \frac{1}{2\pi} \nabla \overline{\rho} \; \sum_n M_n \; \frac{x-x_n}{|x-x_n|^2} \]

\[ M_n = \overline{\rho}(t, x_n(t)) \; M_n(t) \]

\[ x_n = -\nabla \overline{\Phi}(t, x_n(t)) - \frac{1}{2\pi} \sum_{m \neq n} M_m \; \frac{x_n-x_m}{|x_n-x_m|^2} \]

Notice that the total mass is conserved: \[ \frac{d}{dt} \left( \int \overline{\rho} \; dx + \sum M_n(t) \right) = 0 \]

Open question: is it true that \( \rho \) takes the form (X)?
Further remarks and open questions

- If \( \rho \) satisfies (\( R \)):
  \[
  \frac{1}{2\pi} \int_{\mathbb{R}^2} \rho |x|^2 \, dx = \bar{M} \left( 4 - \frac{M}{2\pi} - \frac{M - \bar{M}}{2\pi} \right) - \frac{1}{2\pi} \sum_{a \neq b} \delta_{a,b} \tag{A.13}
  \]
  \[< 0 \]

  \( \Rightarrow \) two pure Dirac massos \( (\tilde{\rho} = 0) \) merge in finite time.

  \( \Rightarrow \) There is such that
  \[
  \frac{1}{2\pi} \int_{\mathbb{R}^2} \tilde{\rho} q \, dx = (q - 1) (C_{\text{conv}}) \int_{\mathbb{R}^2} q \left( \frac{|x|}{\sqrt{t - y^2}} \right) \, dx \leq 0 \text{ if } \bar{M} \leq \bar{\bar{M}}.
  \]

Conjecture: if \( \text{supp} (\bar{\rho}) \) is bounded and \( M > 8\pi \), \( \rho (t,x) = M \xi (x - x_0) \) for large \( t \).

\[ x_0 = \frac{1}{M} \int_{\mathbb{R}^2} x \rho (x) \, dx \]

**Blow-up profile?**

\[ F_\varepsilon [\rho] = \int_{\mathbb{R}^2} \rho \log \rho \, dx + \frac{\varepsilon^2}{4\pi} \int_{\mathbb{R}^2} \log \left( \frac{|x - y| + \varepsilon}{\varepsilon} \right) \rho (x) \rho (y) \, dx \]

Choose \( a \in \mathbb{R}^2 \), define

\[ R (\varepsilon \bar{\rho}) = \varepsilon^2 \rho (a + \varepsilon \bar{\rho}) \]

\[ F_\varepsilon [\rho] = (2M - \frac{M^2}{4\pi}) \log (1 + \varepsilon^2) + \frac{1}{\varepsilon^2} F_\varepsilon (R) \]

If \( \rho (t,x) \) is a solution of the regularized problem, then bounded from below only \( 8\pi t \)!

\[
\int_{\mathbb{R}^2} R (\tilde{\rho}) \, dx = \frac{1}{8\pi} \int_{\mathbb{R}^2} \frac{|y|^2 - 1}{|x - y| + 1} R (\tilde{\rho}) \, dy \leq \frac{1}{8\pi} \left( \int \int R (\tilde{\rho}) \, dx \right)^2 \quad \text{either on } \int \int R (\tilde{\rho}) \, dx \]

or

\[ \text{on } \int \int R (\tilde{\rho}) \, dx \]
FROM KINETIC TO DIFFUSIVE MODELS

Many results on diffusive limits have been achieved by various authors
[Boyard], [Caudin], [Heller], [Degond] etc.

2 approaches to derive nonlinear drift-diffusion equations

1) start from a particle (stochastic particle) description and
take an appropriate "hydrodynamic" type limit.

2) build a hierarchy of models and derive diffusive
models from kinetic theory.

[Chavanis]
[Chavanis-Lemou]
[J.D. Ben Abdallah]
[J.D. Frankowich, Olg. Schmeiser]

A BGK type collision kernel, with non-maxwellian local gibbs state:

\[ \dot{f} + v \cdot \nabla f - \nabla \cdot \Phi \cdot \nabla f = G - f = Q(f) \]

Assume \( \Phi \) is given, for simplicity.

\[ Q(f) \]

Collision kernel

Local Gibbs state
"Local Gibbs state"

\[ G_f = g \left( \frac{m^2}{2} + \Phi(x) - \mu(t,x) \right) \]

1) **Kernel**

\[ Q(f) = 0 \iff f = G_f \]

If additionally \( f \) is stationary, we recover stationary solutions

\[ f = \text{free like in Part I} \]

A case of special interest in astrophysics:

\[ g(s) = (-s)^k \]

(phytropic gas).

2) **Localization conservation**

\[ 0 = \int_{\mathbb{R}^3} Q(f) \, dx \]

determines

\[ \mu(t,x) = \phi(x) + \bar{\mu}(t,x) \]

Let \( G(\phi) = \int_{\mathbb{R}^3} \Phi(x) \, dx = 4\pi \int_{\mathbb{R}} \int_{\mathbb{R}^2} g(\sqrt{t^2 + x^2}) \, dx \, dt \)

\[ \bar{\mu}(t,x) = \bar{\mu}(\rho) \] is implicitly determined by the condition:

\[ G(\bar{\mu}(\rho)) = \rho. \]

*Parabolic rescaling:*

\[ \varepsilon^2 \partial_t f + \varepsilon (v \cdot \nabla f - \nabla v \cdot \nabla f) = G_f - f^\varepsilon, \quad f^\varepsilon_{\varepsilon \to 0} = f_0 \]

**Theorem [JD - Prokaryotic - Schrödinger]** Under conditions(...)

\[ f^\varepsilon \xrightarrow{L^1_{t,x}} f_0 = g \left( \frac{1}{2} (v^2 - \mu(\rho(t,x))) \right) \]

where \( \rho \) is a solution:

\[ \partial_t \rho = \Delta (\nu(\rho)) + \nu \nabla (\rho \nabla \Phi), \quad \rho(t=0,x) = \int f^\varepsilon (x,v) \, dv \]

\[ \nu(\rho) = \int_0^\rho \bar{\mu}'(s) \, ds \]
Formal asymptotics:
\[ \varepsilon^2 \partial_x f + \varepsilon (\nu \partial_x f - \partial_x \Phi \partial_x f) = \mathcal{G}_f - f \]

\[ f = \sum_{k \geq 0} \varepsilon^k f_k \]

Identify order by order in \( \varepsilon \):

\( O(1) \):
\[ \mathcal{G}^0 = f^0 \quad \Rightarrow \quad f^0 = \varrho \left( \frac{1}{2} \nu \Gamma^0 \right), \quad \bar{\mu}^0 = \mu^0 - \varphi \]

\( O(\varepsilon) \):
\[ \nu \partial_x f^0 - \partial_x \Phi \partial_x f^0 = \mathcal{G}^1 - f^1 \]

Integrate with respect to \( \nu \):
\[ \int \nu f^1 \, d\nu = \int [\partial_x \mathcal{G}^1 - \partial_x (\nu \partial_x \mathcal{G}^0)] \, d\nu \]
\[ = -f^0 \partial_x \mu^0 \]

\( O(\varepsilon^2) \):
\[ \partial_t f^0 + \nu \partial_x f^1 - \partial_x \Phi \partial_x f^1 = \mathcal{G}^2 - f^2 \]

Integrate wrt \( \nu \) (local mass conservation):
\[ \partial_t f^0 = \text{div}_x (\varrho^0 \partial_x \mu^0) = \Delta \left( \varrho^0 (\Phi^0) + \nabla \Phi^0 \nabla \Phi \right) \]
Free energy \[ F = \iint_{\mathbb{R}^3 \times \Omega^2} \left( \frac{\nabla^2}{2} + \phi(x) \right) f \, dx \, dv + \int_{\Omega^2} \beta(t) \, dx \, dv \]

\[ \frac{dF}{dt} = \frac{1}{\varepsilon} \iint_{\mathbb{R}^3 \times \Omega^2} \left( \frac{\nabla^2}{2} + \phi(x) + \beta'(t) \right) Q(t) \]

\[ (\beta')^{-1} (s) = g(s) \]

\[ \frac{dF}{dt} = \frac{1}{\varepsilon} \iint_{\mathbb{R}^3 \times \Omega^2} (\beta'(t) - \beta'(G_t)) (t - G_t) \, dx \, dv \]

Method: use free energy to bound (locally) the first and the second moments

Two examples:

1) \( g(s) = e^{-s} \)

\( \beta(s) = \text{slugs} \) : linear BGK model

\( v(e) = e^{\frac{-s}{2}} \) : linear drift-diffusion eq.

\( G_t = f e^{\frac{-s}{2}} \)

2) polytropic gas \( \omega \)

\( g(s) = (s)^k \)

\( v(e) = C_k e^{\frac{k+5/2}{4}} \) = \( C_k \rho^m \), \( m \in \left( 1, \frac{5}{3} \right) \)

\( \rho \, \eta = \Delta \rho^m + \text{div}_x (\rho \nabla \psi) \) porous medium type equation.

N.B. Compatibility of generalized entropies: \( F[G_t] = F[\rho] \) is an entropy for (PM).
POLYTROPES: Flat Rotating Gravitational Systems

\[ \Delta f + \nabla \cdot \nabla f = \omega^2 x \cdot \nabla f + 2 \omega v \nabla \cdot \nabla f - \nabla \phi \cdot \nabla f = Gf - f \]

In the rotating coordinate axis:

\[ \phi = -\frac{1}{4\pi |x|} \int_{\mathbb{R}^2} f \, d\nu \]

\[ x \in \mathbb{R}^2 \]

\[ g(s) = \left( \frac{s}{s_{k+1}} \right)^{k+1} \]

\[ \tilde{\nu}(\rho) = - (k+1) \left( \frac{\rho}{2\pi} \right)^{k/2} \]

\[ G(s) = 2\pi \left( \frac{s}{s_{k+1}} \right)^{k+1} \]

Free energy:

\[ F = \iint \left( \frac{1}{2} |v|^2 f - \frac{1}{2} \omega^2 x^2 f + \frac{1}{2} \phi f \right) \, dx \, d\nu \]

\[ \frac{df}{d\nu} = \iint (Gf - f) \left( \beta'(f) - \beta''(f) \right) \, dx \, d\nu \leq 0 \]

\[ \implies \text{take a diffusion limit} \]
(2) \[ \nabla \Phi = \text{div}_x \left( \nabla \mu(x) - \omega^2 \sigma \Phi + \rho \nabla_x \phi \right) \]

\[ \phi = -\frac{1}{4\pi \chi_1} \cdot \rho \]

\[ \mu(x) = \frac{(2\pi)^{1/2}}{\Gamma(m)} \phi \]

\[ m = \frac{k+2}{k+1} \]

\[ \rho = \left( \frac{\omega^2 \chi_1}{2} - \phi + \mu \right)^{\frac{1}{m-1}} \]

\[ |x| \text{ not too large} \]

\[ \Phi = \frac{\omega^2}{2} \chi_1^2 \]

Theorem (J. D. Fernández)

There exists \( \omega_\ast = \omega_\ast (\chi_1) \) such that (2) has a "localized" minimizer (stationary solution) iff \( \omega \leq \omega_\ast \).

Functional to be minimized:

\[ F[\Phi] = \kappa \int \rho^m - \frac{\omega^2}{2} \int \rho \chi_1^2 \| \Phi \|_{L^2}^2 \int \frac{\rho(x) \rho(y)}{|x-y|} \]
Multi-Bump Solutions

Point particle picture

Continuum mechanics picture

A Lyapunov-Schmidt approach (use a scaling invariance).
To any stationary configuration of points, associate a critical point
(with eventually high Morse index)
Some open questions

- Large time behaviour of gravitational systems:
  \[ t \to \infty \] What is left (locally)? (up to galilean transforms/translators)

- How to measure meta-stability in gravitation?
  is there an equivalent of the Ehrenfest time (Schrödinger equations)

- Higher order critical points: orbital stability in gravitation
  \[ \to \] diffusive models: use Wasserstein's distance?
  \[ \text{(lack of convexity)} \]
  \[ \to \] kinetic models: control of the tails/supports!

- "Symmetry breaking" : how to reduce from 3 D to 2D... 1D
  \[ \to \] No good symmetry results in mean discs annuli
  \[ \text{field theory (like Gidas-Ni-Nirenberg's result)} \]
  \[ \text{dimensional aspects!} \]

- Blow-up analysis (chemotaxis): comparable to the nonlinear Schrödinger equation?
  \[ \to \] gradient flows w.r.t. Wasserstein? unclear.
  \[ \to \] renormalization of the free energy: which description of the blowing-up states?