Sharp rates for the subcritical parabolic-elliptic Keller-Segel model in the plane

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Outline

1. Preliminaries: functional inequalities, fast diffusion, Onofri, etc.
2. Keller-Segel model: subcritical and small mass results [A. Blanchet, JD, M. Escobedo, J. Fernández]
3. Keller-Segel model: inequalities and spectrum of the linearized operator [J. Campos, JD]
4. Keller-Segel model, a functional analysis approach [J. Campos, JD]

1 – Preliminaries: functional analysis

Flows, functional inequalities, functional spaces...

1. Fast diffusion, Gagliardo-Nirenberg and Onofri inequalities
2. Logarithmic Hardy-Littlewood-Sobolev and Onofri inequalities: duality, flows
3. Linearized fast diffusion and weighted L^2 spaces

Fast diffusion, Gagliardo-Nirenberg and Onofri inequalities

The fast diffusion equation

Consider the Fokker-Planck version of fast diffusion equation (FDE) with exponent $m \in (\frac{d-1}{d}, 1)$, $d \geq 3$

$$\frac{\partial u}{\partial t} = \Delta u^m + \nabla \cdot (x\, u) \quad t > 0\,, \quad x \in \mathbb{R}^d$$

with $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$ such that $u_0^m \in \mathrm{L}^1_+(\mathbb{R}^d)$ and $|x|^2\, u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

Any solution converges as $t \to \infty$ to the *Barenblatt profile*

$$u_\infty(x) = \left(C_M + \frac{1-m}{2m}\,|x|^2\right)^{\frac{1}{m-1}} \quad x \in \mathbb{R}^d$$

Asymptotic behaviour of the solutions of FDE

[J. Ralston, W.I. Newman] Define the relative entropy (or free energy) by

$$\mathcal{F}[u] := \frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m - u_\infty^m - m\,u_\infty^{m-1}(u-u_\infty)\right]\,dx$$

$$\frac{d}{dt}\mathcal{F}[u(t,\cdot)] = -\left(\tfrac{m}{m-1}\right)^2\int_{\mathbb{R}^d} u\,|\nabla u^{m-1} - \nabla u_\infty^{m-1}|^2\,dx =: -\mathcal{I}[u(t,\cdot)]$$

$$\mathcal{F}[u(t,\cdot)] \le \tfrac{1}{2}\,\mathcal{I}[u(t,\cdot)]$$

if m is in the range $(\frac{d-1}{d},1)$, thus showing that

$$\mathcal{F}[u(t,\cdot)] \le \mathcal{F}[u_0]\,e^{-2t}\quad \forall\, t\ge 0$$

With $p=\frac{1}{2m-1}$, the inequality $\mathcal{F}[u]\le\frac{1}{2}\,\mathcal{I}[u]$ can be rewritten in terms of $f=u^{m-1/2}$ as

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \le \mathsf{C}_{p,d}\,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

$f_\infty = u_\infty^{m-1/2}$ is optimal [M. del Pino, JD] [F. Otto] [D. Cordero-Erausquin, B. Nazaret, C. Villani]

Gagliardo-Nirenberg inequalities

Consider the following sub-family of Gagliardo-Nirenberg inequalities

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \leq \mathsf{C}_{p,d}\, \|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\, \|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)}$$

with $\theta = \theta(p) := \frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, $p = \frac{1}{2m-1}$

- $1 < p \leq \frac{d}{d-2}$ if $d \geq 3$, $\frac{d-1}{d} \leq m < 1$
- $1 < p < \infty$ if $d = 2$, $\frac{1}{2} < m < 1$

[M. del Pino, JD] equality holds in if $f = F_p$ with

$$F_p(x) = (1 + |x|^2)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

All extremal functions are equal to F_p up to a multiplication by a constant, a translation and a scaling

- When $p \to 1$, we recover the euclidean logarithmic Sobolev inequality in optimal scale invariant form [F. Weissler]
- If $d \geq 3$, the limit case $p = d/(d-2)$ corresponds to Sobolev's inequality [T. Aubin, G. Talenti]
- If $d = 2$ and $p \to \infty$...

Onofri's inequality as a limit case

When $d=2$, Onofri's inequality can be seen as an endpoint case of the family of the Gagliardo-Nirenberg inequalities [JD], [del Pino, JD]

Proposition (JD)

Assume that $g \in \mathcal{D}(\mathbb{R}^d)$ *is such that* $\int_{\mathbb{R}^2} g\, d\mu = 0$ *and let* $f_p := F_p\left(1+\frac{g}{2p}\right)$. *With* $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$, *and* $d\mu(x) := \mu(x)\,dx$, *we have*

$$1 \le \lim_{p\to\infty} \mathsf{C}_{p,2} \frac{\|\nabla f_p\|_{\mathrm{L}^2(\mathbb{R}^2)}^{\theta(p)}\,\|f_p\|_{\mathrm{L}^{p+1}(\mathbb{R}^2)}^{1-\theta(p)}}{\|f_p\|_{\mathrm{L}^{2p}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla g|^2\,dx}}{\int_{\mathbb{R}^2} e^{g}\,d\mu}$$

The standard form of the euclidean version of Onofri's inequality is

$$\log\left(\int_{\mathbb{R}^2} e^{g}\,d\mu\right) - \int_{\mathbb{R}^2} g\,d\mu \le \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla g|^2\,dx$$

Logarithmic Hardy-Littlewood-Sobolev and Onofri inequalities: duality, flows

The logarithmic HLS inequality

The classical logarithmic Hardy-Littlewood-Sobolev (logHLS) in $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} n\,\log\left(\frac{n}{M}\right)\,dx+\frac{2}{M}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\,n(y)\,\log|x-y|\,dx\,dy+M\,(1+\log\pi)\geq 0$$

Equality is achieved by

$$\mu(x):=\frac{1}{\pi\,(1+|x|^2)^2}\quad\forall\,x\in\mathbb{R}^2$$

Notice that $-\Delta\log\mu=8\,\pi\,\mu$ can be inverted as

$$(-\Delta)^{-1}\mu=\frac{1}{8\,\pi}\,\log(\pi\,\mu)$$

With $M=8\,\pi$ and $n_\infty=8\,\pi\,\mu$ (logHLS) can be rewritten as

$$\underbrace{\int_{\mathbb{R}^2} n\,\log\left(\frac{n}{n_\infty}\right)\,dx}_{=:F_1^*[n]}\geq\underbrace{\frac{1}{2}\int_{\mathbb{R}^2}(n-n_\infty)\,(-\Delta)^{-1}(n-n_\infty)\,dx}_{=:F_2^*[n]}$$

The two-dimensional case: Legendre duality

Onofri's inequality

$$F_1[u] := \log\left(\int_{\mathbb{R}^2} e^u \, d\mu\right) \leq \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} u \, \mu \, dx =: F_2[u]$$

By duality: $F_i^*[v] = \sup\left(\int_{\mathbb{R}^2} v \, u \, d\mu - F_i[u]\right)$ we can relate Onofri's inequality with (logHLS)

Proposition (E. Carlen, M. Loss & V. Calvez, L. Corrias)

For any $v \in \mathrm{L}^1_+(\mathbb{R}^2)$ *with* $\int_{\mathbb{R}^2} v \, dx = 1$, *such that* $v \log v$ *and* $(1 + \log|x|^2)\, v \in \mathrm{L}^1(\mathbb{R}^2)$, *we have*

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v \, \log\left(\frac{v}{\mu}\right) dx - 4\,\pi \int_{\mathbb{R}^2} (v - \mu)\,(-\Delta)^{-1}(v - \mu) \, dx \geq 0$$

The two-dimensional case: (logHLS) and flows

[E. Carlen, J. Carrillo, M. Loss]

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx$$

is related to Gagliardo-Nirenberg inequalities if $v_t = \Delta\sqrt{v}$

- Alternatively, assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta \log\left(\frac{v}{\mu}\right) \quad t>0\,, \quad x\in\mathbb{R}^2$$

Proposition (JD)

If v is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\,dx = 1$, $v_0\log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ and $v_0 \log\mu \in \mathrm{L}^1(\mathbb{R}^2)$, then

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu \geq F_2[u]-F_1[u]$$

with $\log(v/\mu) = u/2$

Linearized fast diffusion and weighted L^2 spaces

Linearization and weighted L^2 spaces

Back to the fast diffusion equation $\frac{\partial u}{\partial t} = \Delta u^m + \nabla \cdot (x\, u)$ we investigate the linear regime $u = u_\infty (1 + \varepsilon\, f\, u_\infty^{1-m})$ as $\varepsilon \to 0$

$$\frac{\partial f}{\partial t} = -\mathcal{L}\, f := u_\infty^{m-2}\, \nabla \cdot (u_\infty\, \nabla f)$$

Use $L^2(\mathbb{R}^d,\, u_\infty^{2-m}\, dx)$ [A. Blanchet, M. Bonforte, JD, G. Grillo, J.L. Vázquez]

$$\mathcal{F}[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - u_\infty^m - m\, u_\infty^{m-1} (u - u_\infty) \right] dx \sim \frac{m}{2} \int_{\mathbb{R}^d} |f|^2\, u_\infty^{2-m}\, dx$$

$$\frac{d}{dt} \mathcal{F}[u(t,\cdot)] = -\,\mathcal{I}[u(t,\cdot)] \sim -\int_{\mathbb{R}^d} |\nabla f|^2\, u_\infty\, dx = -\,\langle f, \mathcal{L}\, f \rangle$$

$$\mathcal{F}[u(t,\cdot)] \le \tfrac{1}{2}\, \mathcal{I}[u(t,\cdot)] \quad \Longrightarrow \int_{\mathbb{R}^d} |f|^2\, u_\infty^{2-m}\, dx \le \frac{1}{\Lambda} \int_{\mathbb{R}^d} |\nabla f|^2\, u_\infty\, dx$$

for any $m < 1$, $m \neq (d-4)/(d-2)$ *i.e.* Hardy-Poincaré inequality (under zero average condition on f) and... $\Lambda = 1$ if $m \in [(d-1)/d, 1)$

Linearized fast diffusion: spectrum

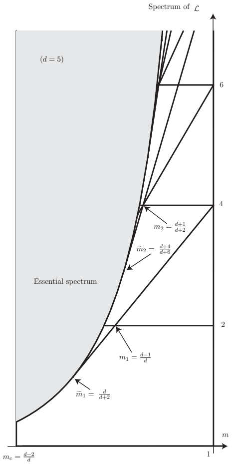


Figure: Spectrum of $\mathcal{L}$, appropriately normalized

2 – Keller-Segel model: subcritical and small mass results

The parabolic-elliptic Keller and Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2\,,\ t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2\,,\ t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|\, u(t,y)\, dy$$

and observe that

$$\nabla v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\, u(t,y)\, dy$$

Mass conservation: $\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^2} u(t,x)\, dx = 0$

Blow-up

$M=\int_{\mathbb{R}^2} n_0\,dx>8\pi$ and $\int_{\mathbb{R}^2}|x|^2\,n_0\,dx<\infty$: blow-up in finite time

a solution u of

$$\frac{\partial u}{\partial t}=\Delta u-\nabla\cdot(u\,\nabla v)$$

satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2\,u(t,x)\,dx$$

$$=-\underbrace{\int_{\mathbb{R}^2} 2\,x\cdot\nabla u\,dx}_{-4\,M}+\frac{1}{2\pi}\underbrace{\iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{2x\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\,dx\,dy}_{\frac{(x-y)\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\,dx\,dy}$$

$$=4\,M-\frac{M^2}{2\pi}<0\quad\text{if}\quad M>8\pi$$

Existence and free energy

$M = \int_{\mathbb{R}^2} n_0\, dx \leq 8\pi$: global existence [W. Jäger, S. Luckhaus], [JD, B. Perthame], [A. Blanchet, JD, B. Perthame], [A. Blanchet, J.A. Carrillo, N. Masmoudi]

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u\,(\nabla(\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u\; dx - \frac{1}{2}\int_{\mathbb{R}^2} u\, v\; dx$$

satisfies

$$\frac{d}{dt} F[u(t,\cdot)] = -\int_{\mathbb{R}^2} u\,\left|\nabla(\log u) - \nabla v\right|^2\, dx$$

(log HLS) inequality [E. Carlen, M. Loss]:
F is bounded from below if $M < 8\pi$

The existence setting

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u\,\nabla v) & x \in \mathbb{R}^2\,,\ t>0 \\ -\Delta v = u & x \in \mathbb{R}^2\,,\ t>0 \\ u(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)\;, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)\;, \quad M := \int_{\mathbb{R}^2} n_0(x)\,dx < 8\,\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\,dx$ for any $t \geq 0$, see [W. Jäger, S. Luckhaus], [A. Blanchet, JD, B. Perthame]

$v = -\frac{1}{2\pi}\,\log|\cdot| * u$

Time-dependent rescaling

$$u(x,t) = \frac{1}{R^2(t)}\, n\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad v(x,t) = c\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1+2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\, \log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

[A. Blanchet, JD, B. Perthame] Convergence in self-similar variables

$$\lim_{t\to\infty} \|n(\cdot,\cdot+t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,\cdot+t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

means *intermediate asymptotics* in original variables:

$$\left\|u(x,t) - \tfrac{1}{R^2(t)}\, n_\infty\left(\tfrac{x}{R(t)}, \tau(t)\right)\right\|_{L^1(\mathbb{R}^2)} \searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

- Radial symmetry [Y. Naito]
- Uniqueness [P. Biler, G. Karch, P. Laurençot, T. Nadzieja]
- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon \in (0,1)$ [A. Blanchet, JD, B. Perthame]
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M:

$$\lim_{M \to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[D.D. Joseph, T.S. Lundgren] [JD, R. Stańczy]

The stationary solution when mass varies

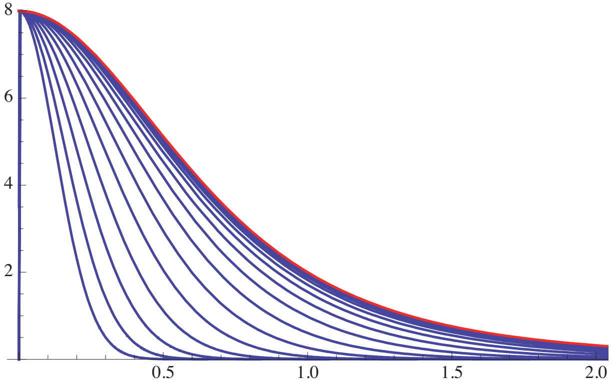


Figure: Representation of the solution appropriately scaled so that the 8π case appears as a limit (in red)

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla \Big[n \left(\log n - x + \nabla c \right) \Big]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \, dx + \int_{\mathbb{R}^2} \frac{1}{2} |x|^2 \, n \, dx - \frac{1}{2} \int_{\mathbb{R}^2} n \, c \, dx$$

satisfies

$$\frac{d}{dt} F[n(t,\cdot)] = - \int_{\mathbb{R}^2} n \left| \nabla \left(\log n \right) + x - \nabla c \right|^2 \, dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2 \, n(\lambda \, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2) \, dx + \int_{\mathbb{R}^2} \frac{\lambda^{-2}-1}{2} |x|^2 \, n \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) \, n(y) \, \log \frac{1}{\lambda} \, dx \, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi} \right)}_{>0 \text{ if } M < 8\pi} \log \lambda + \frac{\lambda^{-2} - 1}{2} \int_{\mathbb{R}^2} |x|^2 \, n \, dx$$

x

Keller-Segel with subcritical mass in self-similar variables

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

$$\lim_{t\to\infty} \|n(\cdot,\cdot+t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,\cdot+t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

$$n_\infty = M\,\frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

First result: small mass case

Theorem (A. Blanchet, JD, M. Escobedo, J. Fernández)

There exists a positive constant M^* *such that, for any initial data* $n_0 \in L^2(n_\infty^{-1}\,dx)$ *of mass* $M < M^*$ *satisfying the above assumptions, there is a unique solution* $n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau,\infty)\times\mathbb{R}^2)$ *for any* $\tau > 0$

Moreover, there are two positive constants, C *and* δ*, such that*

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \,\frac{dx}{n_\infty} \le C\, e^{-\,\delta\, t} \quad \forall\, t > 0$$

As a function of M*,* δ *is such that* $\lim_{M\to 0_+} \delta(M) = 1$

The condition $M \le 8\,\pi$ is necessary and sufficient for the global existence of the solutions, but there are two extra smallness conditions in our proof:

- Uniform estimate: the *method of the trap*
- *Spectral gap* of a linearised operator $\mathcal{L}$

First step: the trap

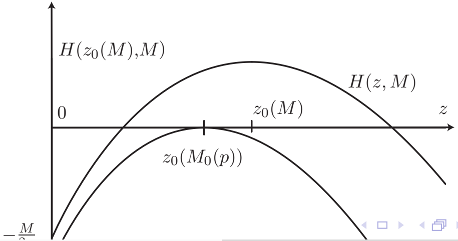
Lemma

For any $M < M_1$, there exists $C = C(M)$ such that, for any solution $u \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} \le C\, t^{-1} \quad \forall\, t > 0$$

The *method of the trap...* Duhamel's formula

$$H(t\,\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}\,, M) \le 0$$



Second step: L^p and H^1 estimates in the self-similar variables

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla\cdot(n\,(\nabla c - x)) & x\in\mathbb{R}^2\,,\ t>0\\ c = -\dfrac{1}{2\pi}\,\log|\cdot|*n & x\in\mathbb{R}^2\,,\ t>0\\ n(\cdot,t=0)=n_0\ge 0 & x\in\mathbb{R}^2 \end{cases}$$

With $K=K(x)=e^{|x|^2/2}$, rewrite the equation for n as

$$\frac{\partial n}{\partial t} - \frac{1}{K}\,\nabla\cdot(K\,\nabla n) = -\nabla c\cdot\nabla n + 2n + n^2$$

$$\frac12\,\frac{d}{dt}\int_{\mathbb{R}^2}|n|^2\,K\,dx + \int_{\mathbb{R}^2}|\nabla n|^2\,K\,dx$$
$$= -\int_{\mathbb{R}^2} n\nabla c\cdot\nabla n\,K\,dx + 2\int_{\mathbb{R}^2} n^2\,K\,dx + \int_{\mathbb{R}^2} n^3\,K\,dx$$
$$\le \varepsilon\int_{\mathbb{R}^2}|\nabla n|^2\,K\,dx + C$$

Third step: linearization and spectral gap

Proposition

For any $M \in (0, M_2)$*, for any* $f \in H^1(n_\infty\, dx)$ *such that*

$$\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0 \quad \Longrightarrow \quad \int_{\mathbb{R}^2} |\nabla f|^2\, n_\infty\, dx \geq \frac{1}{\Lambda(M)} \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

for some $\Lambda(M) > 0$ *and* $\lim_{M\to 0_+} \Lambda(M) = 1$

Corollary

If $M < M_2$*, then any solution* n *is bounded in*

$$L^\infty(\mathbb{R}^+, L^2(n_\infty^{-1}\, dx)) \cap L^\infty((\tau, \infty), H^1(n_\infty^{-1}\, dx))$$

for any $\tau > 0$

Fourth step: collecting the estimates

$$\begin{cases} \dfrac{\partial f}{\partial t} - \mathcal{L}(t,x,f,g) = -\dfrac{1}{n_\infty}\nabla\cdot\left[f\, n_\infty \nabla\left(g\, c_\infty\right)\right] & x\in\mathbb{R}^2\,,\ t>0 \\ -\Delta(c_\infty\, g) = f\, n_\infty & x\in\mathbb{R}^2\,,\ t>0 \end{cases}$$

Multiply by $f\, n_\infty$ and integrate by parts

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|f|^2\, n_\infty\, dx + \int_{\mathbb{R}^2}|\nabla f|^2\, n_\infty\, dx = \underbrace{\int_{\mathbb{R}^2}\nabla f\cdot\nabla\left(g\, c_\infty\right)\, n_\infty\, dx}_{=\mathrm{I}} + \underbrace{\int_{\mathbb{R}^2}\nabla f\cdot\nabla(g\, c_\infty)\, f\, n_\infty\, dx}_{=\mathrm{II}}$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|f|^2\, n_\infty\, dx \le -\left[1-\gamma(M)\right]\int_{\mathbb{R}^2}|\nabla f|^2\, n_\infty\, dx \le -\frac{\left[1-\gamma(M)\right]}{\Lambda(M)}\int_{\mathbb{R}^2}|f|^2\, n_\infty\, dx$$

3 – Keller-Segel model: inequalities and spectrum of the linearized operator

A parametrization of the solutions and the linearized operator

[J. Campos, JD]

$$-\Delta c = M \frac{e^{-\frac{1}{2}|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}|x|^2+c}\,dx}$$

Solve

$$-\phi'' - \frac{1}{r}\phi' = e^{-\frac{1}{2}r^2+\phi}\,, \quad r > 0$$

with initial conditions $\phi(0) = a$, $\phi'(0) = 0$ and get with $r = |x|$

$$M(a) := 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2}r^2+\phi_a}\,dx$$

$$n_a(x) = M(a) \frac{e^{-\frac{1}{2}r^2+\phi_a(r)}}{2\pi \int_{\mathbb{R}^2} r\,e^{-\frac{1}{2}r^2+\phi_a}\,dx} = e^{-\frac{1}{2}r^2+\phi_a(r)}$$

Mass

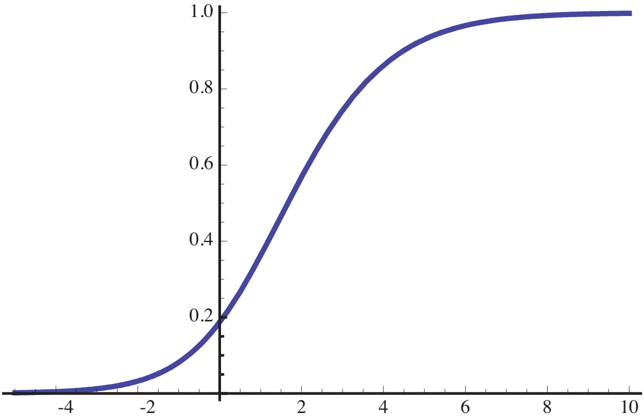


Figure: *The mass can be computed as* $M(a) = 2\pi \int_0^\infty n_a(r)\, r\, dr$. *Plot of* $a \mapsto M(a)/8\pi$

Bifurcation diagram

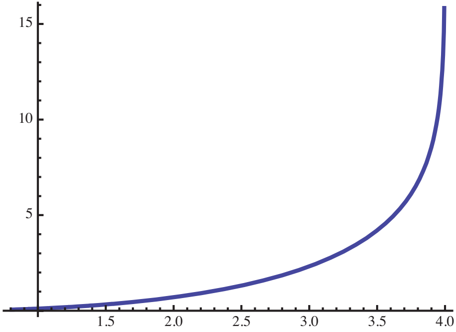


Figure: *The bifurcation diagram can be parametrized by* $a \mapsto \left(\frac{1}{2\pi} M(a), \|c_a\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\right)$ *with* $\|c_a\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = c_a(0) = a - b(a)$ *(cf. Keller-Segel system in a ball with no flux boundary conditions)*

Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

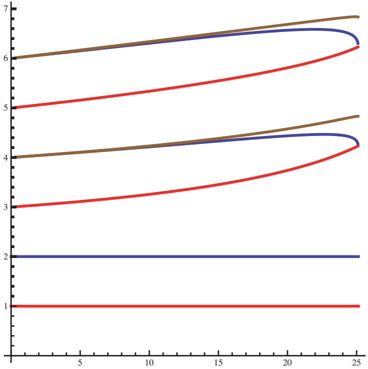


Figure: The lowest eigenvalues of $-\mathcal{L} = (-\Delta)^{-1}(n_a\, f)$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

[A. Blanchet, JD, M. Escobedo, J. Fernández], [J. Campos, JD], [V. Calvez, J.A. Carrillo], [J. Bedrossian, N. Masmoudi]

Spectral analysis in the functional framework determined by the relative entropy method

Simple eigenfunctions

Kernel Let $f_0 = \frac{\partial}{\partial M} c_\infty$ be the solution of

$$-\Delta f_0 = n_\infty f_0$$

and observe that $g_0 = f_0 / c_\infty$ is such that

$$\frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f_0 - c_\infty g_0) \big) =: \mathcal{L} f_0 = 0$$

Lowest non-zero eigenvalues $f_1 := \frac{1}{n_\infty} \frac{\partial n_\infty}{\partial x_1}$ associated with $g_1 = \frac{1}{c_\infty} \frac{\partial c_\infty}{\partial x_1}$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L} f_1 = f_1$

With $D := x \cdot \nabla$, let$f_2 = 1 + \frac{1}{2} D \log n_\infty = 1 + \frac{1}{2 n_\infty} D n_\infty$. Then

$$-\Delta (D c_\infty) + 2 \Delta c_\infty = D n_\infty = 2 (f_2 - 1) n_\infty$$

and so $g_2 := \frac{1}{c_\infty} (-\Delta)^{-1} (n_\infty f_2)$ is such that $-\mathcal{L} f_2 = 2 f_2$

Functional setting...

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality [A. Blanchet, JD, B. Perthame]

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)\,(c - c_\infty)\,dx \geq 0$$

achieves its minimum for $n = n_\infty$

$$\mathsf{Q}_1[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} F[n_\infty(1 + \varepsilon f)] \geq 0$$

if $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$. Notice that f_0 generates the kernel of Q_1

Lemma (J. Campos, JD)

Poincaré type inequality For any $f \in H^1(\mathbb{R}^2, n_\infty\, dx)$ such that $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$, we have

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f\, n_\infty)|^2\, n_\infty\, dx = \int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, n_\infty\, dx \leq \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

... and eigenvalues

With g such that $-\Delta(g\,c_\infty) = f\,n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1\,f_2\,n_\infty\,dx - \int_{\mathbb{R}^2} f_1\,n_\infty\,(g_2\,c_\infty)\,dx$$

on the orthogonal space to f_0 in $L^2(n_\infty\,dx)$

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\,c_\infty)|^2\,n_\infty\,dx \quad \text{with} \quad g = -\frac{1}{c_\infty}\,\frac{1}{2\pi}\,\log|\cdot| * (f\,n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L}\,f \rangle = \mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma (J. Campos, JD)

$\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is 1

Linearized Keller-Segel theory

$$\mathcal{L}\, f = \frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f - c_\infty\, g) \big)$$

Corollary (J. Campos, JD)

$$\langle f, f \rangle \leq \langle \mathcal{L}\, f, f \rangle$$

The linearized problem takes the form

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f$$

where $\mathcal{L}$ is a self-adjoint operator on the orthogonal of f_0 equipped with $\langle \cdot, \cdot \rangle$. A solution of

$$\frac{d}{dt} \langle f, f \rangle = -\, 2\, \langle \mathcal{L}\, f, f \rangle$$

has therefore exponential decay

More functional inequalities

A subcritical logarithmic HLS inequality

Recall that

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality [A. Blanchet, JD, B. Perthame]

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)\,(c - c_\infty)\, dx \geq 0$$

achieves its minimum for $n = n_\infty$

Lemma (J. Campos, JD)

Poincaré type inequality For any $f \in H^1(\mathbb{R}^2, n_\infty\, dx)$ *such that* $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$, *we have*

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f\, n_\infty)|^2\, n_\infty\, dx = \int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, n_\infty\, dx \leq \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

... Legendre duality

A new Onofri type inequality

Theorem (J. Campos, JD)

For any $M \in (0, 8\pi)$, if $n_\infty = M \frac{e^{c_\infty - \frac{1}{2}|x|^2}}{\int_{\mathbb{R}^2} e^{c_\infty - \frac{1}{2}|x|^2}\,dx}$ with $c_\infty = (-\Delta)^{-1} n_\infty$, $d\mu_M = \frac{1}{M} n_\infty\,dx$, we have the inequality

$$\log\left(\int_{\mathbb{R}^2} e^\phi\,d\mu_M\right) - \int_{\mathbb{R}^2} \phi\,d\mu_M \le \frac{1}{2\,M}\int_{\mathbb{R}^2} |\nabla\phi|^2\,dx \quad \forall\,\phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$$

Corollary (J. Campos, JD)

The following Poincaré inequality holds

$$\int_{\mathbb{R}^2} \left|\psi - \overline{\psi}\right|^2 n_\infty\,dx \le \int_{\mathbb{R}^2} |\nabla\psi|^2\,dx \quad \text{where} \quad \overline{\psi} = \int_{\mathbb{R}^2} \psi\,d\mu_M$$

An improved interpolation inequality (coercivity estimate)

Lemma (J. Campos, JD)

For any $f \in L^2(\mathbb{R}^2, n_\infty\, dx)$ such that $\int_{\mathbb{R}^2} f\, f_0\, n_\infty\, dx = 0$ holds, we have

$$-\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\, n_\infty(x)\, \log|x-y|\, f(y)\, n_\infty(y)\, dx\, dy \le (1-\varepsilon)\int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

for some $\varepsilon > 0$, where $g\, c_\infty = G_2 * (f\, n_\infty)$ and, if $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$ holds,

$$\int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, dx \le (1-\varepsilon)\int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

4 – Keller-Segel model, a functional analysis approach

Exponential convergence for any mass $M \leq 8\pi$

If $n_{0,*}(\sigma)$ stands for the symmetrized function associated to n_0, assume that for any $s \geq 0$

$$(H) \quad \exists\, \varepsilon \in (0, 8\,\pi - M) \quad \text{such that} \quad \int_0^s n_{0,*}(\sigma)\, d\sigma \leq \int_{B\left(0,\sqrt{s/\pi}\right)} n_{\infty,M+\varepsilon}(x)\, dx$$

Theorem (J. Campos, JD)

Under the above assumption, if $n_0 \in L^2_+(n_\infty^{-1}\, dx)$ and $M := \int_{\mathbb{R}^2} n_0\, dx < 8\,\pi$, then any solution with initial datum n_0 is such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \frac{dx}{n_\infty(x)} \leq C\, e^{-\,2\,t} \quad \forall\, t \geq 0$$

for some positive constant C, where n_∞ is the unique stationary solution with mass M

Sketch of the proof

- [J. Campos, JD] Uniform convergence of $n(t,\cdot)$ to n_∞ can be established for any $M \in (0, 8\pi)$ by an adaptation of the symmetrization techniques of [J.I. Díaz, T. Nagai, J.M. Rakotoson]
- Uniform estimates (with no rates) easily result
- Estimates based on Duhammel's formula inspired by [M. Escobedo, E. Zuazua] allow to prove uniform convergence
- Spectral estimates can be incorporated to the relative entropy approach
- Exponential convergence of the relative entropy follows

Step 1: symmetrization (1/2)

To any measurable function $u : \mathbb{R}^2 \mapsto [0, +\infty)$,
we associate the distribution function defined by $\mu(t, \tau) := |\{u > \tau\}|$
and its decreasing rearrangement given by

$$u_* : [0, +\infty) \rightarrow [0, +\infty] \,, \quad s \mapsto u_*(s) = \inf\{\tau \geq 0 \ : \ \mu(t, \tau) \leq s\} \,.$$

1. For every measurable function $F : \mathbb{R}^+ \mapsto \mathbb{R}^+$, we have

$$\int_{\mathbb{R}^2} F(u)\, dx = \int_{\mathbb{R}^+} F(u_*)\, ds$$

2. If $u \in W^{1,q}(0, T; L^p(\mathbb{R}^N))$ is a nonnegative function, with $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then $u_* \in W^{1,q}(0, T; L^p(0, \infty))$ and the formula

$$\int_0^{\mu(t,\tau)} \frac{\partial u_*}{\partial t}(t, \sigma)\, d\sigma = \int_{\{u(t,\cdot) > \tau\}} \frac{\partial u}{\partial t}(t, x)\, dx$$

holds for almost every $t \in (0, T)$ [J.I. Díaz, T. Nagai, J.M. Rakotoson]

Step 1: symmetrization (2/2)

Lemma

If n is a solution, then the function

$$k(t,s) := \int_0^s n_*(t,\sigma)\,d\sigma$$

satisfies $k \in L^\infty([0,+\infty)\times(0,+\infty)) \cap H^1\left([0,+\infty); W^{1,p}_{\rm loc}(0,+\infty)\right) \cap L^2\left([0,+\infty); W^{2,p}_{\rm loc}(0,+\infty)\right)$ *and*

$$\begin{cases} \frac{\partial k}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 k}{\partial s^2} - (k + 2\,s)\,\frac{\partial k}{\partial s} \le 0 & a.e.\ \ in\ (0,+\infty)\times(0,+\infty) \\ k(t,0) = 0\ , \quad k(t,+\infty) = \int_{\mathbb{R}^2} n_0\,dx & for\ t\in(0,+\infty) \\ k(0,s) = \int_0^s (n_0)_*\,d\sigma & for\ s\ge 0 \end{cases}$$

Step 2: Uniform estimates

Proposition (J.I. Díaz, T. Nagai, J.M. Rakotoson)

Let f, g be two continuous functions on $Q = \mathbb{R}^+ \times (0, +\infty)$ such that ...

$$\begin{cases} \frac{\partial f}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 f}{\partial s^2} - (f + 2\,s)\,\frac{\partial f}{\partial s} \leq \frac{\partial g}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 g}{\partial s^2} - (g + 2\,s)\,\frac{\partial g}{\partial s} \;\; a.e. \;\; in \;\; Q \\ f(t,0) = 0 = g(t,0) \quad and \quad f(t,+\infty) \leq g(t,+\infty) \; for \; any \; t \in (0,+\infty) \\ f(0,s) \leq g(0,s) \; for \; s \geq 0 \,, and \; g(t,s) \geq 0 \; in \; Q \end{cases}$$

then $f \leq g$ on Q

Corollary

Assume that $n_0 \in L^2_+(n_\infty^{-1}\,dx)$ satisfies (H) and $M := \int_{\mathbb{R}^2} n_0\,dx < 8\,\pi$. Then there exist positive constants $C_1 = C_1(M,p)$ and $C_2 = C_2(M,p)$ such that

$$\|n\|_{L^p(\mathbb{R}^2)} \leq C_1 \quad and \quad \|\nabla c\|_{L^\infty(\mathbb{R}^2)} \leq C_2$$

Step 3: Estimates based on Duhammel's formula

Consider the kernel associated to the Fokker-Planck equation

$$K(t,x,y) := \frac{1}{2\pi\,(1-e^{-2\,t})}\,e^{-\frac{1}{2}\,\frac{|x-e^{-t}y|^2}{1-e^{-2\,t}}} \quad x\in\mathbb{R}^2\,,\quad y\in\mathbb{R}^2\,,\quad t>0$$

If n is a solution, then

$$n(t,x) = \int_{\mathbb{R}^2} K(t,x,y)\,n_0(y)\,dy + \int_0^t \int_{\mathbb{R}^2} \nabla_x K(t-s,x,y)\cdot n(s,y)\,\nabla c(s,y)\,dy\,ds$$

Corollary

Assume that n is a solution. Then

$$\lim_{t\to\infty} \|n(t,\cdot)-n_\infty\|_{\mathrm{L}^p(\mathbb{R}^d)} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|\nabla c(t,\cdot)-\nabla c_\infty\|_{\mathrm{L}^q(\mathbb{R}^d)} = 0$$

for any $p\in[1,\infty]$ and any $q\in[2,\infty]$

Step 4: Spectral estimates can be incorporated

With $\mathsf{Q}_1[f] = \langle f, f \rangle$ and $\mathsf{Q}_2[f] = \langle f, \mathcal{L} f \rangle$

1. For any function f in the orthogonal of the kernel of $\mathcal{L}$, we have

$$\mathsf{Q}_1[f] \le \mathsf{Q}_2[f]$$

2. For any radial function $f \in \mathcal{D}(\mathsf{L}_2)$, we have

$$2\,\mathsf{Q}_1[f] \le \mathsf{Q}_2[f]$$

Step 5: Exponential convergence of the relative entropy

$$\frac{\partial f}{\partial t} = \mathcal{L} f - \frac{1}{n_\infty} \nabla \left[n_\infty \, f \, \nabla (g \, c_\infty) \right]$$

$$\frac{d}{dt} \mathsf{Q}_1[f(t,\cdot)] = -\,2\,\mathsf{Q}_2[f(t,\cdot)] + \int_{\mathbb{R}^2} \nabla(f - g\,c_\infty)\, f\, n_\infty \cdot \nabla(g\,c_\infty)\,dx$$

$$\frac{d}{dt} \mathsf{Q}_1[f(t,\cdot)] \le -\,2\,\mathsf{Q}_2[f(t,\cdot)] + \delta(t,\varepsilon)\,\sqrt{\mathsf{Q}_1[f(t,\cdot)]\,\mathsf{Q}_2[f(t,\cdot)]}$$

$$\mathsf{Q}_1[f(t,\cdot)] \le \mathcal{Q} \quad \forall\, t \ge 0$$

$$\frac{d}{dt} \mathsf{Q}_1[f(t,\cdot)] \le -\,\mathsf{Q}_1[f(t,\cdot)] \left[2 - \,\delta(t,\varepsilon) \left(\left(\mathsf{Q}_1[f(t,\cdot)]\right)^{\frac{1-\varepsilon}{2-\varepsilon}} + \mathsf{Q}_1[f(t,\cdot)]\right)^{\frac{1}{2+\varepsilon}} \right) \right]$$

As a consequence, we finally get that

$$\limsup_{t\to\infty} e^{2\,t}\, \mathsf{Q}_1[f(t,\cdot)] < \infty$$

Some key ideas

1. Lyapunov / Entropy functionals and functional inequalities
2. Linearization and best constants
3. Functional framework for linearized operators can be deduced from the entropy functional

Thank you for your attention !