Entropy methods for linear and nonlinear parabolic equations

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I — Entropy methods for linear diffusions

The logarithmic Sobolev inequality

Convex Sobolev inequalities

- *logarithmic Sobolev inequality:* [Gross], [Weissler], [Coulhon],...
- *probability theory:* [Bakry], [Emery], [Ledoux], [Coulhon],...
- *linear diffusions (PDEs):* [Toscani], [Arnold, Markowich, Toscani, Unterreiter], [Otto, Kinderlehrer, Jordan], [Arnold, J.D.]

I-A. Intermediate asymptotics: heat equation

Heat equation: $$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n\,,\ t \in \mathbb{R}^+ \\ u(\cdot, t=0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0\, dx = 1 \end{cases} \qquad (1)$$

As $t \to +\infty$, $u(x,t) \sim \mathcal{U}(x,t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}$.

Optimal rate of convergence of $\|u(\cdot,t) - \mathcal{U}(\cdot,t)\|_{L^1(\mathbb{R}^n)}$?

The time dependent rescaling

$$u(x,t) = \frac{1}{R^n(t)}\, v\left(\xi = \frac{x}{R(t)}, \tau = \log R(t) + \tau(0)\right)$$

allows to transform this question into that of the convergence to the stationary solution $v_\infty(\xi) = (2\pi)^{n/2}\, e^{-|\xi|^2/2}$.

- Ansatz: $\frac{dR}{dt} = \frac{1}{R}$ $\quad R(0) = 1 \quad \tau(0) = 0$:

$$R(t) = \sqrt{1 + 2\,t}\,, \quad \tau(t) = \log R(t)$$

As a consequence: $v(\tau = 0) = u_0$.

- Fokker-Planck equation:

$$\begin{cases} v_\tau = \Delta v + \nabla(\xi\, v) & \xi \in \mathbb{R}^n\,,\ \tau \in \mathbb{R}^+ \\ v(\cdot, \tau = 0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0\, dx = 1 \end{cases} \tag{2}$$

Entropy (relative to the stationary solution v_∞):

$$\Sigma[v] := \int_{\mathbf{R}^n} v \log\left(\frac{v}{v_\infty}\right) dx = \int_{\mathbf{R}^n} \left(v \log v + \frac{1}{2}|x|^2 v\right) dx + Const$$

If v is a solution of (2), then (I is the Fisher information)

$$\frac{d}{d\tau}\Sigma[v(\cdot,\tau)] = -\int_{\mathbf{R}^n} v \left|\nabla \log\left(\frac{v}{v_\infty}\right)\right|^2 dx =: -I[v(\cdot,\tau)]$$

- Euclidean logarithmic Sobolev inequality: If $\|v\|_{L^1} = 1$, then

$$\int_{\mathbf{R}^n} v \log v \, dx + n\left(1 + \frac{1}{2}\log(2\pi)\right) \leq \frac{1}{2}\int_{\mathbf{R}^n} \frac{|\nabla v|^2}{v} dx$$

Equality: $v(\xi) = v_\infty(\xi) = (2\pi)^{-n/2} e^{-|\xi|^2/2}$

$$\Longrightarrow \Sigma[v(\cdot,\tau)] \leq \tfrac{1}{2} I[v(\cdot,\tau)]$$

$$\Sigma[v(\cdot,\tau)] \leq e^{-2\tau}\Sigma[u_0] = e^{-2\tau}\int_{\mathbf{R}^n} u_0 \log\left(\frac{u_0}{v_\infty}\right) dx$$

• Csiszár-Kullback inequality: Consider $v \geq 0$, $\bar{v} \geq 0$ such that $\int_{\mathbb{R}^n} v \, dx = \int_{\mathbb{R}^n} \bar{v} \, dx =: M > 0$

$$\int_{\mathbb{R}^n} v \log\left(\frac{v}{\bar{v}}\right) \, dx \geq \frac{1}{4\,M} \|v - \bar{v}\|^2_{L^1(\mathbb{R}^n)}$$

$$\Longrightarrow \|v - v_\infty\|^2_{L^1(\mathbb{R}^n)} \leq 4M \Sigma[u_0] \, e^{-2\tau}$$

$$\tau(t) = \log \sqrt{1 + 2t}$$

$$\|u(\cdot, t) - u_\infty(\cdot, t)\|^2_{L^1(\mathbb{R}^n)} \leq \frac{4}{1 + 2t} \Sigma[u_0]$$

$$u_\infty(x, t) = \frac{1}{R^n(t)} v_\infty\left(\frac{x}{R(t)}, \tau(t)\right)$$

Proof of the Csiszár-Kullback inequality: Taylor development at second order.

Euclidean logarithmic Sobolev inequality: other formulations

1) independent from the dimension [Gross, 75]

$$\int_{\mathbb{R}^n} w \log w \, d\mu(x) \le \frac{1}{2} \int_{\mathbb{R}^n} w \, |\nabla \log w|^2 \, d\mu(x)$$

with $w = \frac{v}{M\, v_\infty}$, $\|v\|_{L^1} = M$, $d\mu(x) = v_\infty(x)\, dx$.

2) invariant under scaling [Weissler, 78]

$$\int_{\mathbb{R}^n} w^2 \log w^2 \, dx \le \frac{n}{2} \log \left(\frac{2}{\pi\, n\, e} \int_{\mathbb{R}^n} |\nabla w|^2 \, dx \right)$$

for any $w \in H^1(\mathbb{R}^n)$ such that $\int w^2 \, dx = 1$

Proof: take $w = \sqrt{\frac{v}{M\,v_\infty}}$ and optimize on λ for $w_\lambda(x) = \lambda^{n/2} w(\lambda\,x)$

$$\int_{\mathbb{R}^n} |\nabla w_\lambda|^2\,dx - \int_{\mathbb{R}^n} w_\lambda^2 \log w_\lambda^2\,dx$$
$$= \lambda^2 \int_{\mathbb{R}^n} |\nabla w|^2\,dx - \int_{\mathbb{R}^n} w^2 \log w^2\,dx - n \log \lambda \int_{\mathbb{R}^n} w^2\,dx$$

Entropy-entropy production method

A method to prove the Euclidean logarithmic Sobolev inequality:

$$\frac{d}{d\tau}\left(I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)]\right) = -C \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \left| w_{ij} + a\,\frac{w_i w_j}{w} + b\,w\,\delta_{ij} \right|^2 dx < 0$$

for some $C > 0$, a, $b \in \mathbb{R}$. Here $w = \sqrt{v}$.

$I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)] \searrow I[v_\infty] - 2\Sigma[v_\infty] = 0$

$\Longrightarrow \quad \forall\, u_0\,, \quad I[u_0] - 2\Sigma[u_0] \geq I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)] \geq 0$ for $\tau > 0$

I-B. Entropy-entropy production method: improved convex Sobolev inequalities

Goal: large time behavior of parabolic equations:

$$\begin{cases} v_t = \mathrm{div}_x[D(x)\,(\nabla_x v + v\nabla_x A(x))] = \mathrm{div}[D(x)\,e^{-A}\nabla(ve^{A})] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0,\ x \in \mathbb{R}^n \\ v(x, t = 0) = v_0(x) \in L^1_+(\mathbb{R}^n) \end{cases} \tag{3}$$

$A(x) \ldots$ given 'potential'
$v_\infty(x) = e^{-A(x)} \in L^1 \ldots$ (unique) steady state
mass conservation: $\int_{\mathbb{R}^d} v(t)\,dx = \int_{\mathbb{R}^d} v_\infty\,dx = 1$

questions: exponential rate ? connection to logarithmic Sobolev inequalities ?

Entropy-entropy production method

[Bakry, Emery, 84]

[Toscani '96], [Arnold, Markowich, Toscani, Unterreiter, 01]

Relative entropy of $v(x)$ w.r.t. $v_\infty(x)$:

$$\Sigma[v|v_\infty] := \int_{\mathbf{R}^d} \psi\left(\frac{v}{v_\infty}\right) v_\infty \, dx \geq 0$$

with

$$\begin{aligned} \psi(w) &\geq 0 \text{ for } w \geq 0, \quad \text{convex} \\ \psi(1) &= \psi'(1) = 0 \end{aligned}$$

Admissibility condtion: $(\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV}$

Examples:

$\psi_1 = w \ln w - w + 1$, $\Sigma_1(v|v_\infty) = \int v \ln\left(\frac{v}{v_\infty}\right) dx \ldots$ physical entropy

$\psi_p = w^p - p(w-1) - 1$, $1 < p \leq 2$, $\quad \Sigma_2(v|v_\infty) = \int_{\mathbf{R}^d} (v - v_\infty)^2 v_\infty^{-1} dx$

Exponential decay of entropy production

$$-I(v(t)|v_\infty) := \frac{d}{dt}\Sigma[v(t)|v_\infty] = -\int \psi''\left(\frac{v}{v_\infty}\right) |\underbrace{\nabla\left(\frac{v}{v_\infty}\right)}_{=:u}|^2 \, v_\infty \, dx \leq 0$$

Assume: $D \equiv 1$, $\underbrace{\frac{\partial^2 A}{\partial x^2}}_{\text{Hessian}} \geq \lambda_1 Id$, $\lambda_1 > 0$ ($A(x)\ldots$ unif. convex)

Entropy production rate:

$$-I' = 2\int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u \, v_\infty \, dx + \underbrace{2\int \mathrm{Tr}\,(XY)\, v_\infty \, dx}_{\geq 0}$$

$$\geq +2\lambda_1 I$$

Positivity of $\mathrm{Tr}\,(XY)$?

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} \geq 0$$

$$Y = \begin{pmatrix} \sum_{ij}(\frac{\partial u_i}{\partial x_j})^2 & u^T \cdot \frac{\partial u}{\partial x} \cdot u \\ u^T \cdot \frac{\partial u}{\partial x} \cdot u & |u|^4 \end{pmatrix} \geq 0$$

$$\Rightarrow \quad I(t) \leq e^{-2\lambda_1 t}\, I(t=0) \qquad t > 0$$

$$\forall v_0 \text{ with } I(v_0|v_\infty) < \infty$$

Exponential decay of relative entropy

$$\text{Known:} \quad I' \geq -2\lambda_1 \underbrace{I}_{=\Sigma'} \tag{4}$$

$$\int_t^\infty \ldots dt \qquad \Rightarrow \Sigma' = I \leq -2\lambda_1 \Sigma$$

Theorem 1 [Bakry, Emery], [Arnold, Markowich, Toscani, Unterreiter]

$\frac{\partial^2 A}{\partial x^2} \geq \lambda_1 Id$ (*"Bakry–Emery condition"*), $\Sigma[v_0|v_\infty]<\infty$

$$\Rightarrow \Sigma[v(t)|v_\infty] \leq \Sigma[v_0|v_\infty]\, e^{-2\lambda_1 t}, \qquad t > 0$$

$\|v(t) - v_\infty\|^2_{L^1} \leq C\,\Sigma[v(t)|v_\infty] \ldots$ Csiszár-Kullback

Convex Sobolev inequalities

Entropy–entropy production estimate (4) for $A(x) = -\ln v_\infty$ (uniformly convex):

$$\Sigma[v|v_\infty] \leq \frac{1}{2\lambda_1} |I(v|v_\infty)|$$

Example 1: logarithmic entropy $\psi_1(w) = w \ln w - w + 1$

$$\int v \ln\left(\frac{v}{v_\infty}\right) dx \leq \frac{1}{2\lambda_1} \int v \left|\nabla \ln\left(\frac{v}{v_\infty}\right)\right|^2 dx$$

$$\forall v,\ v_\infty \in L^1_+(\mathbb{R}^n),\ \int v\, dx = \int v_\infty\, dx = 1$$

logarithmic Sobolev inequality – "entropy version"

Logarithmic Sobolev inequality–dv_∞ measure version [Gross '75]

$$f^2 = \frac{v}{v_\infty} \Rightarrow \int f^2 \ln f^2 dv_\infty \le \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

$$\forall f \in L^2(\mathbb{R}^n, dv_\infty), \ \int f^2 dv_\infty = 1$$

Example 2: non-logarithmic entropies:

$\psi_p(w) = w^p - p(w-1) - 1, \qquad 1 < p \le 2$

$$(B_p) \ \ \frac{p}{p-1}\left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}}\, dv_\infty\right)^p\right] \le \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

from (4) with $\frac{v}{v_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}} dv_\infty}$ $\qquad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, v_\infty dx)$

Poincaré-type inequality [Beckner '89], $(B_p) \Rightarrow (B_2)$, $\quad 1 < p \le 2$

Refined convex Sobolev inequalities

Estimate of entropy production rate / entropy production:

$$I' = 2\int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u v_\infty dx + \underbrace{2\int \mathrm{Tr}\,(XY) v_\infty dx}_{\geq 0}$$

$$\geq -2\lambda_1 I$$

[Arnold, J.D.]: Observe that $\psi_p(w) = w^p - p(w-1) - 1$, $1 < p < 2$:

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} > 0$$

- Assume $\frac{\partial A^2}{\partial x^2} \geq \lambda_1 Id \Rightarrow \Sigma'' \geq -2\lambda_1 \Sigma' + \kappa \frac{|\Sigma'|^2}{1+\Sigma}$, $\qquad \kappa = \frac{2-p}{p} < 1$

$$\Rightarrow \boxed{k(\Sigma[v|v_\infty]) \leq \frac{1}{2\lambda_1}|\Sigma'|} = \frac{1}{2\lambda_1}\int \psi''\left(\frac{v}{v_\infty}\right)|\nabla \frac{v}{v_\infty}|^2 dv_\infty$$

Refined convex Sobolev inequality with $x \leq k(x) = \frac{1+x-(1+x)^\kappa}{1-\kappa}$

- Set $v/v_\infty = |f|^{\frac{2}{p}} / \int |f|^{\frac{2}{p}} dv_\infty \Rightarrow$

Theorem 2

$$\frac{1}{2}\left(\frac{p}{p-1}\right)^2 \left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^{2(p-1)} \left(\int f^2 dv_\infty\right)^{\frac{2-p}{p}}\right]$$
$$\leq \frac{2}{\lambda_1}\int |\nabla f|^2 dv_\infty \quad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, dv_\infty)$$

Refined Beckner inequality [Arnold, J.D. 2000, 2004]
$(rB_p) \Rightarrow (rB_2) = (B_2), \quad 1 < p \leq 2$

I-C. Applications...

- Homogeneous and non-homogenous collisional kinetic equations [L. Desvillettes, C. Villani, G. Toscani,...]
- Drift-diffusion-Poisson equations for semi-conductors [A. Arnold, P. Markowich, G. Toscani], [P. Biler, J.D., P. Markowich]
- The two-dimensional Keller-Segel model [B. Perthame, J.D.]
- Streater's models [P. Biler, J.D., M. Esteban, G. Karch]
- Heat equation with a source term [[J.D., G. Karch]
- The flashing ratchet [J.D., D. Kinderlehrer, M. Kowalczyk]
- Models for traffic flow [J.D., Reinhard Illner]
- Navier-Stokes in dimension 2 [T. Gallay, Wayne], [C. Villani], [J.D., A. Munnier]

... and questions under investigation

• Hierarchies of inequalities

• Derivation of entropy - entropy-production inequalities in non-standard frameworks:

- singular potentials: [JD, Nazaret, Otto]

- vanishing diffusion cooefficients: [Bartier, JD, Illner, Kowalczyk]

• Homogeneization and long time behaviour: [Allaire, JD, Kinderlehrer, Kowalczyk]

• Relaxation and diffusion properties on intermediate time scales, corrections to convex Sobolev inequalities: [Bartier, JD, Markowich]

I-D. An example of application: the flashing ratchet. Long time behavior and dynamical systems interpretation

[M. Chipot, D. Heath, D. Kinderlehrer, M. Kowalczyk, N. Walkington,...]

[J.D., David Kinderlehrer, Michał Kowalczyk]

Flashing ratchet: a simple model for a molecular motor (Brownian motors, molecular ratchets, or Brownian ratchets)

Diffusion tends to spread and dissipate density / transport concentrates density at specific sites determined by the energy landscape: unidirectional transport of mass.

Fokker-Planck type problem

$$\begin{array}{ll} u_t = (u_x + \psi_x u)_x & (x,t) \in \Omega \times (0,\infty) \\ u_x + \psi_x u = 0 & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & x \in \Omega \end{array} \tag{5}$$

$u_0 > 0$, $\int_\Omega u_0 = 1$, $\psi = \psi(x,t)$

Periodic state and asymptotic behaviour

Theorem 1 *Let $\psi \in L^\infty([0,T)\times\Omega)$ be a T-periodic potential and assume that there exists a finite partition of $[0,T)$ into intervals $[T_i,T_{i+1})$, $i=0,...,n$ with $T_0=0$, $T_n=T$ such that $\psi_{[T_i,T_{i+1})} \in L^\infty([T_i,T_{i+1}),W^{1,\infty}(\Omega))$. Then there exists a unique nonnegative T-periodic solution U to (5) such that $\int_\Omega U(x,t)\,dx = 1$ for any $t\in[0,T)$.*

Entropy and entropy production : $\sigma_q(u) = \begin{cases} \frac{u^q-1}{q-1} & \text{if } q>1, \\ u \ln u & \text{if } q=1. \end{cases}$

$$\Sigma_q[u|v] = \int_\Omega \left[\sigma_q\left(\frac{u}{v}\right) - \sigma_q'(1)\left(\frac{u}{v}-1\right)\right] v\,dx$$

$$I_q[u|v] = \int_\Omega \sigma_q''\left(\frac{u}{v}\right)\left|\nabla\left(\frac{u}{v}\right)\right|^2 v\,dx,$$

Theorem 2 *Let u_1, u_2 be any two solutions to (5).*

$$\Sigma_q[u_1(t)|u_2(t)] \le e^{-C_q t}\,\Sigma_q[u_1(0)|u_2(0)]$$

Proposition 3 *Ω is a bounded domain in $\mathbb{R}^d$ with C^1 boundary. Let u and v be two nonnegative functions in $L^1 \cap L^q(\Omega)$ if $q \in (1,2]$ and in $L^1(\Omega)$ with $u \log u$ and $u \log v$ in $L^1(\Omega)$ ($q = 1$).*

$$\Sigma_q[u|v] \ge 2^{-2/q} q \left[\max\left(\|u\|^{2-q}_{L^q(\Omega)}, \|v\|^{2-q}_{L^q(\Omega)}\right)\right]^{-1} \|u - v\|^2_{L^q(\Omega)}$$

Corollary 4 *Let $q \in [1,2]$. Any solution of (5) with initial data $u_0 \in L^1 \cap L^q(0,1)$ $u_0 \log u_0 \in L^1(0,1)$ if $q = 1$, converges to $\|u_0\|_{L^1} U(x,t)$, (periodic solution):*

$$\|u(x,t) - \|u_0\|_{L^1}\, U(x,t)\|_{L^q(0,1;dx)} \le K\, e^{-C_{q,\psi}\, t} \quad \forall\, t \ge 0$$

Let $u_\psi := \|u_0\|_{L^1} \frac{e^{-\psi}}{\int_\Omega e^{-\psi}\,dx}$.

$$\begin{aligned} \frac{d}{dt}\Sigma_1[u|u_\psi] &= \int_\Omega \left[1 + \log\left(\frac{u}{u_\psi}\right)\right] u_t\,dx - \int_\Omega \frac{u}{u_\psi}\, u_{\psi,t}\,dx \\ &= -I_1[u|u_\psi] - \int_\Omega \frac{u}{u_\psi}\, u_{\psi,t}\,dx \end{aligned}$$

Lemma 5 *Let $u \geq 0$ be a solution to (5) such that $\int_\Omega u\,dx = 1$. With the above notations, the following estimate holds:*

$$\frac{d}{dt}\Sigma_1[u|u_\psi] \leq -C_\psi\,\Sigma_1[u|u_\psi] + K_\psi.$$

Fixed-point for the map $\mathcal{T}(u(\cdot,0)) = u(\cdot,T)$ in

$\mathcal{Y} = \{u \in H^1(\Omega) \mid u \geq 0\,,\ \|u\|_{L^1(\Omega)} = 1\,,\ \Sigma_1[u|u_0(\cdot,0)] \leq K_\psi/C_\psi\}$.

Flashing potentials: same on each time interval.

Let $\mathcal{X}$ be the set of bounded nonnegative functions u in $L^1 \cap L^q(\Omega)$ (resp. in $L^1(\Omega)$ with $u \log u$ in in $L^1(\Omega)$) if $q \in (1,2]$ (resp. if $q = 1$) such that $\int_\Omega u\,dx = 1$.

Theorem 6 *Assume that* $v \in \mathcal{X}$ *with* $0 < m := \inf_\Omega v \leq v \leq \sup_\Omega v =: M < \infty$. *For any* $q \in [1,2]$

$$\mathcal{I} = \frac{q}{q-1} \inf_{\substack{u\in\mathcal{X} \\ u \neq v \, a.e.}} \frac{I_q[u|v]}{\Sigma_q[u|v]} \quad \textit{if } q > 1, \quad \mathcal{I} = \inf_{\substack{u\in\mathcal{X} \\ u \neq v \, a.e.}} \frac{I_1[u|v]}{\Sigma_1[u|v]} \quad \textit{if } q = 1 \tag{6}$$

can be estimated by

$$\mathcal{I} \geq 4\,\lambda_1(\Omega)\,\frac{m}{M} \tag{7}$$

where $\lambda_1(\Omega)$ *is Poincaré's constant of* Ω *(with weight* 1*).*

Relation between entropy and entropy production: exponential decay of the relative entropy.

DISSIPATION PRINCIPLE

[Jordan, Kinderlehrer, Otto], [Chipot, Kinderlehrer, Kowalczyk]

Wasserstein distance between Borel probability measures μ, μ^*:

$$d(\mu, \mu^*)^2 = \inf_{p \in \mathcal{P}(\mu,\mu^*)} \int_{\Omega\times\Omega} |x-\xi|^2 \, p(dx d\xi),$$

$\phi : \Omega \to \Omega$, $\phi(0) = 0$, $\phi(1) = 1$, strictly increasing continuous

$$\int_\Omega \zeta \, f \, d\xi = \int_\Omega \zeta(\phi(x)) \, f^*(x) \, dx, \qquad \text{for any } \zeta \in C^0(\Omega).$$

$f = F'$ is the push forward of $f^* = (F^*)'$, ϕ is the transfer function. In particular if $\zeta = \chi_{[0,x]}$, then

$$\int_0^{\phi(x)} f(\xi) \, d\xi = F(\phi(x)) = \int_0^x f^*(x') \, dx' = F^*(x) \Longrightarrow \phi = F^{-1} \circ F^* \, .$$

Wasserstein distance: $d(f, f^*)^2 = \int_\Omega |x - \phi(x)|^2 f^*(x) \, dx$.

[Benamou and Brenier]: convex duality. Differentiating with respect to t and x yields

$$f_\xi(\phi(x,t),t)\,\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x} + f_t(\phi(x,t),t)\,\frac{\partial\phi}{\partial x} + f(\phi(x,t),t)\,\frac{\partial^2\phi}{\partial x\partial t} = 0.$$

We implicitly define a velocity ν by $\nu(\phi,t) = \frac{\partial\phi}{\partial t}$. Using $\frac{\partial^2\phi}{\partial x\partial t} = \nu_\xi(\phi,t)\frac{\partial\phi}{\partial x}$ we find a continuity equation for $f(x,t)$:

$$f_t + (\nu f)_x = 0, \qquad \text{in } \Omega\times(0,\tau).$$

$$d(f^{**},f^*)^2 = \tau\min_\nu\int_0^\tau\int_\Omega \nu(x,t)^2 f(x,t)\,dxdt,$$

where the minimum is taken over all velocities ν such that

$$\begin{aligned} &f_t + (\nu f)_x = 0, && \text{in } \Omega\times(0,\tau),\\ &f(x,0) = f^*, \quad f(x,\tau) = f^{**}(x) && x\in\Omega. \end{aligned}$$

Free energy functional:

$$F(u) = \int_\Omega (\psi\, u + \sigma\, u \log u)\, dx.$$

[Kinderlehrer, Otto, Jordan]: Determine $u^{(k)}$ such that

$$\frac{1}{2} d(u^{(k-1)}, u^{(k)})^2 + \tau F(u^{(k)}) = \min_u \left[\frac{1}{2} d(u^{(k-1)}, u)^2 + \tau F(u) \right]. \tag{8}$$

Then $u_\tau(x,t) := u^{(k)}(x) \quad \text{if } t \in [k\,\tau, (k+1)\,\tau),\ x \in \Omega.$
(1) There exists a unique solution to the above scheme.
(2) As $\tau \to 0$, u_τ converges strongly in $L^1((0,t) \times \Omega)$ to the unique solution to (5).
Observe that in the limit $\tau \to 0$, $\nu(x,t) = -(\sigma \log u + \psi)_x$.
After [Kinderlehrer and Walkington], a new numerical scheme, based on the spatial discretization of

$$U_t = u(\log u + \psi)_x.$$

I-E. A model for traffic flow

[J.D., Reinhard Illner] $f = f(t,v)$ is an homogeneous distribution function, with velocities ranging in $(0,1)$:

$$f_t = (-B(t,v)\,f + D(t,v)\,f')'\,, \quad (t,v) \in \mathbb{R}^+ \times (0,1)$$

where $f_t = \partial f/\partial t$, $f' = \partial f/\partial v$. Let $C(t,v) := -\int_0^v \frac{B(t,w)}{D(t,w)}\,dw$

$$g(t,v) = \rho\,\frac{e^{-C(t,v)}}{\int_0^1 e^{-C(t,w)}\,dw} \quad \text{is a } \textit{local equilibrium}$$

Zero flux: $-B(t,v)\,g + D(t,v)\,g' = 0$ but $g_t \equiv 0$ is not granted.

Relative entropy:

$$e[t,f] := \int_0^1 \left(f \log f - g \log g + C(t,v)(f-g)\right) dv$$
$$- \iint_{(0,1)\times(0,t)} C_t(s,v)(f-g)(s,v)\,dv\,ds$$

Then $\frac{d}{dt} e[t,f(t,.)] = -\int_0^1 D(t,v)\,f\,\left|\frac{f'}{f} - \frac{g'}{g}\right|^2\,dv$ ... but we dont have a lower bound for $e[t,f(t,.)]$.

Density: $\rho = \int_0^1 f(t,v)\, dv$ does not depend on t
Mean velocity: $u(t) = \frac{1}{\rho} \int_0^1 v\, f(t,v)\, dv$
Braking term:

$$B(t,v) = \begin{cases} -C_B\, |v-u(t)|^2 \rho \left(1 - \left|\frac{v-u(t)}{1-u(t)}\right|^\delta\right) & \text{if } v > u(t) \\ \\ C_A\, |v-u(t)|^2 (1-\rho) & \text{if } v \le u(t) \end{cases}$$

Diffusion term: $D(t,v) = \sigma\, m_1(\rho)\, m_2(u(t))\, |v-u(t)|^\gamma$

Proposition 7 [Illner-Klar-Materne02] *Any stationary solution is uniquely determined by ρ and its average velocity u. The set $(\rho, u[\rho])$ is in general multivalued. For any $\rho \in (0,1]$.*

Example. *The Maxwellian case.*

Convex entropies

Relative entropy of f *w.r.t.* g by $E[\,f\,|\,g\,] = \int_0^1 \Phi\left(\frac{f}{g}\right) g\,dv$

"Standard" example: $\Phi_\alpha(x) = (x^\alpha - x)/(\alpha - 1)$ for some $\alpha > 1$,
$\Phi(x) = x\log x$ if "$\alpha = 1$"

$$\begin{cases} f_t = \left[D(t,v)\,f\left(\frac{f'}{f} - \frac{g'}{g}\right)\right]' = \left[D(t,v)\,g\left(\frac{f}{g}\right)'\right]' & \forall\,(t,v) \in \mathbb{R}^+ \times (0,1) \\ \left(\frac{f}{g}\right)'(t,v) = 0 & \forall\, t \in \mathbb{R}^+,\ v = 0,\ 1 \end{cases}$$

$$g(t,v) := \kappa(t)\,e^{-C(t,v)} \text{ for some } \kappa(t) \neq 0.$$

$$\frac{d}{dt}E[\,f(t,\cdot)\,|\,g(t,\cdot)\,] = \int_0^1 \Phi'\left(\frac{f}{g}\right) f_t\,dv + \underbrace{\int_0^1 \left[\Phi\left(\frac{f}{g}\right) - \frac{f}{g}\,\Phi'(\frac{f}{g})\right] g_t\,dv}$$

$$= 0 \quad \text{if} \quad \dot{\kappa} = \kappa\,\frac{\int_0^1 \Psi(\frac{f}{g})\,g\,C_t(t,v)\,dv}{\int_0^1 \Psi(\frac{f}{g})\,g\,dv}, \quad \kappa(0) = 1$$

with $\Psi(x) := \Phi(x) - x\Phi'(x) < 0$

Convergence to a stationary solution

$$\limsup_{t\to+\infty} \kappa(t) < +\infty\ .$$

Theorem 8 *Let* $\Phi = \Phi_\alpha(x) = (x^\alpha - x)/(\alpha - 1)$, f *be a smooth global in time solution and assume that* $E[\,f\,|\,g\,]$ *is well defined and* C^1 *in* t. *If* $\exists\,\varepsilon \in (0, \frac{1}{2})$ *s.t.* $\varepsilon < u(t) = \frac{1}{\rho}\int_0^1 v\,f(t,v)\,dv < 1 - \varepsilon$ $\forall\ t > 0$, *then, as* $t \to +\infty$, $f(t,\cdot)$ *converges a.e. to a stationary solution* f_∞.

II — Porous media / fast diffusion equation and generalizations

[coll. Manuel del Pino (Universidad de Chile)] $\Longrightarrow$ Relate entropy and entropy-production by Gagliardo-Nirenberg inequalities

Other approaches:
1) "entropy – entropy-production method"
2) mass transportation techniques
3) hypercontractivity for appropriate semi-groups

• *nonlinear diffusions:* [Carrillo, Toscani], [Del Pino, J.D.], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vazquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub]

II-A. Porous media / Fast diffusion equation

[Del Pino, JD]

$$\begin{aligned} u_t = \Delta u^m \quad \text{in } \mathbb{R}^n \\ u_{|t=0} = u_0 \ge 0 \\ u_0(1+|x|^2) \in L^1\,, \quad u_0^m \in L^1 \end{aligned} \tag{9}$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0\,dx = 1$, the self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-n/(2-n(1-m))})$ as $t \to +\infty$, [Friedmann, Kamin, 1980]

$$\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-n/(2-n(1-m))})$$

Rescaling: Take $u(t,x) = R^{-n}(t)\, v\,(\tau(t), x/R(t))$ where

$$\dot{R} = R^{n(1-m)-1}\;, \quad R(0) = 1\;, \quad \tau = \log R$$

$$\boxed{v_\tau = \Delta v^m + \nabla \cdot (x\, v)\;, \quad v_{|\tau=0} = u_0}$$

[Ralston, Newman, 1984] Lyapunov functional: *Entropy*

$$\boxed{\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right)\, dx - \Sigma_0}$$

$$\frac{d}{d\tau}\Sigma[v] = -I[v]\;, \quad I[v] = \int v \left| \frac{\nabla v^m}{v} + x \right|^2\, dx$$

Stationary solution: C s.t. $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left(C + \frac{1-m}{2m}\,|x|^2\right)_+^{-1/(1-m)}$$

Fix Σ_0 so that $\Sigma[v_\infty] = 0$.

$$\Sigma[v] = \int \psi\left(\frac{v^m}{v_\infty^m}\right)\, v_\infty^{m-1}\, dx \quad \textit{with}\ \psi(t) = \frac{m t^{1/m} - 1}{1-m} + 1$$

Theorem 9 $m \in [\frac{n-1}{n}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\boxed{I[v] \geq 2\,\Sigma[v]}$$

An equivalent formulation

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) \, dx - \Sigma_0 \leq \frac{1}{2} \int v \left| \frac{\nabla v^m}{v} + x \right|^2 \, dx = \frac{1}{2} I[v]$$

$p = \frac{1}{2m-1}$, $v = w^{2p}$

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int |\nabla w|^2 dx + \left(\frac{1}{1-m} - n\right) \int |w|^{1+p} dx + K \geq 0$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$
$m = \frac{n-1}{n}$: Sobolev, $m \to 1$: logarithmic Sobolev

[Del Pino, J.D.], [Carrillo, Toscani], [Otto]

Optimal constants for Gagliardo-Nirenberg ineq.

[Del Pino, J.D.]

$$1 < p \leq \frac{n}{n-2} \text{ for } n \geq 3$$

$$\|w\|_{2p} \leq A \, \|\nabla w\|_2^\theta \, \|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi n}\right)^{\frac{\theta}{2}} \left(\frac{2y-n}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{n}{2})}\right)^{\frac{\theta}{n}}$$

$$\theta = \frac{n(p-1)}{p(n+2-(n-2)p)}, \; y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$. Uses [Serrin-Pucci], [Serrin-Tang].

$1 < p = \frac{1}{2m-1} \leq \frac{n}{n-2} \iff$ Fast diffusion case: $\frac{n-1}{n} \leq m < 1$

$0 < p < 1 \iff$ Porous medium case: $m > 1$

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau}$ + Csiszár-Kullback inequalities

$\Rightarrow$ Intermediate asymptotics [Del Pino, J.D.]

(i) $\frac{n-1}{n} < m < 1$ if $n \geq 3$

$$\limsup_{t \to +\infty} t^{\frac{1-n(1-m)}{2-n(1-m)}} \|u^m - u_\infty^m\|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t \to +\infty} t^{\frac{1+n(m-1)}{2+n(m-1)}} \|\, [u - u_\infty]\; u_\infty^{m-1}\, \|_{L^1} < +\infty$$

Generalization

Intermediate asymptotics for:

$$u_t = \Delta_p u^m$$

Convergence to a stationary solution for:

$$v_t = \Delta_p v^m + \nabla(x\,v)$$

Let $q = 1 + m - (p-1)^{-1}$. Whether q is bigger or smaller than 1 determines two different regimes like for $p = 1$.

$q < 1 \iff$ Fast diffusion case

$q > 1 \iff$ Porous medium case

For $q > 0$, define the *entropy* by

$$\Sigma[v] = \int \Big[\sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty)\Big]\;dx$$

$\sigma(s) = \frac{s^q - 1}{q-1}$ if $q \neq 1$
$\sigma(s) = s\log s$ if $q = 1$ ($p \neq 2$: see below)

Nonhomogeneous version – Gagliardo-Nirenberg ineq.

$b = \frac{p\,(p-1)}{p^2-p-1}$, $a = b\,q$, $v = w^b$. For $p \neq 2$, let

$$\mathcal{F}[v] = \int v^{-\frac{1}{p-1}} |\nabla v|^p \, dx - \frac{1}{q} \left(\frac{n}{1-\kappa_p} + \frac{p}{p-2} \right) \int v^q \, dx$$

$\kappa_p = \frac{1}{p}\,(p-1)^{\frac{p-1}{p}}$. Based on [Serrin, Tang] (uniqueness result)

Corollary 3 $n \geq 2$, $(2n+1)/(n+1) \leq p < n$. $\forall\ v$ s.t. $\|v\|_{L^1} = \|v_\infty\|_{L^1}$

$$\mathcal{F}[v] \geq \mathcal{F}[v_\infty]$$

$$\|w\|_b \leq \mathcal{S}\, \|\nabla w\|_p^\theta\, \|w\|_a^{1-\theta} \quad \text{if } a > p$$

$$\|w\|_a \leq \mathcal{S}\, \|\nabla w\|_p^\theta\, \|w\|_b^{1-\theta} \quad \text{if } a < p$$

[Del Pino, J.D.] Intermediate asymptotics of $u_t = \Delta_p u^m$

Theorem 10 $n \geq 2$, $1 < p < n$, $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{p}{p-1}$ *and* $q = 1 + m - \frac{1}{p-1}$

(i) $\quad \|u(t,\cdot) - u_\infty(t,\cdot)\|_q \leq K\, R^{-(\frac{\alpha}{2}+n(1-\frac{1}{q}))}$

(ii) $\quad \|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}}$

(i): $\frac{1}{p-1} \leq m \leq \frac{p}{p-1}$ *(ii)*: $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{1}{p-1}$

$\alpha = (1 - \frac{1}{p}(p-1)^{\frac{p-1}{p}})\,\frac{p}{p-1}$, $R = (1+\gamma\, t)^{1/\gamma}$, $\gamma = (mn+1)(p-1) - (n-1)$

$u_\infty(t,x) = \frac{1}{R^n}\, v_\infty(\log R, \frac{x}{R})$

$v_\infty(x) = (C - \frac{p-1}{mp}\,(q-1)\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ *if* $m \neq \frac{1}{p-1}$

$v_\infty(x) = C\, e^{-(p-1)^2|x|^{p/(p-1)}/p}$ *if* $m = (p-1)^{-1}$.

Use $v_t = \Delta_p v^m + \nabla\cdot(x\, v)$

$w = v^{(mp+q-(m+1))/p}$, $a = b\, q = p\,\frac{m(p-1)+p-2}{mp(p-1)-1}$.

II-B. The $W^{1,p}$ logarithmic Sobolev inequality and consequences

[Del Pino, JD]

Optimal constants for Gagliardo-Nirenberg ineq.

[Del Pino, J.D.]

Theorem 11 $1 < p < n$, $1 < a \le \frac{p(n-1)}{n-p}$, $b = p\,\frac{a-1}{p-1}$

$$\|w\|_b \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_a^{1-\theta} \quad \text{if } a > p$$
$$\|w\|_a \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_b^{1-\theta} \quad \text{if } a < p$$

Equality if $w(x) = A\,(1 + B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$

$a > p$: $\theta = \frac{(q-p)\,n}{(q-1)\,(n\,p-(n-p)\,q)}$

$a < p$: $\theta = \frac{(p-q)\,n}{q\,(n\,(p-q)+p\,(q-1))}$

The optimal L^p-Euclidean logarithmic Sobolev inequality (an optimal under scalings form) [Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

Theorem 12 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx \right]$$

$$\mathcal{L}_p = \tfrac{p}{n} \left(\tfrac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\,\frac{p-1}{p}+1)} \right]^{\frac{p}{n}}$$

Equality: $u(x) = \left(\pi^{\frac{n}{2}} (\frac{\sigma}{p})^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)} \right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}$

$p = 2$: Gross' logaritmic Sobolev inequality [Gross, 75], [Weissler, 78]

$p = 1$: [Ledoux 96], [Beckner, 99]

For some purposes, it is sometimes more convenient to use this inequality in a non homogeneous form, which is based upon the fact that

$$\inf_{\mu>0} \left[\frac{n}{p} \log\left(\frac{n}{p\mu}\right) + \mu \frac{\|\nabla w\|_p^p}{\|w\|_p^p}\right] = n \log\left(\frac{\|\nabla w\|_p}{\|w\|_p}\right) + \frac{n}{p} \, .$$

Corollary 13 *For any* $w \in W^{1,p}(\mathbb{R}^n)$, $w \neq 0$, *for any* $\mu > 0$,

$$p \int |w|^p \log\left(\frac{|w|}{\|w\|_p}\right) dx + \frac{n}{p} \log\left(\frac{p\,\mu\,e}{n\,\mathcal{L}_p}\right) \int |w|^p \, dx \leq \mu \int |\nabla w|^p \, dx \, .$$

II-C. Consequences for $u_t = \Delta_p u^{1/(p-1)}$

[Del Pino, JD, Gentil]

- Existence
- Uniqueness
- Hypercontractivity, Ultracontractivity
- Large deviations

EXISTENCE

Consider the Cauchy problem

$$\begin{cases} u_t = \Delta_p(u^{1/(p-1)}) & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(\cdot, t=0) = f \geq 0 \end{cases} \qquad (10)$$

$\Delta_p u^m = \mathrm{div}\left(|\nabla u^m|^{p-2} \nabla u^m\right)$ is 1-homogeneous $\Longleftrightarrow m = 1/(p-1)$.

Notations: $\|u\|_q = (\int_{\mathbb{R}^n} |u|^q \, dx)^{1/q}$, $q \neq 0$. $p^* = p/(p-1)$, $p > 1$.

Theorem 14 *Let $p > 1$, $f \in L^1(\mathbb{R}^n)$ s.t. $|x|^{p^*} f$, $f \log f \in L^1(\mathbb{R}^n)$. Then there exists a unique weak nonnegative solution $u \in C(\mathbb{R}_t^+, L^1)$ of (10) with initial data f, such that $u^{1/p} \in L^1_{\mathrm{loc}}(\mathbb{R}_t^+, W^{1,p}_{\mathrm{loc}})$.*

[Alt-Luckhaus, 83] [Tsutsumi, 88] [Saa, 91] [Chen, 00] [Agueh, 02]
[Bernis, 88], [Ishige, 96]
Crucial remark: [Benguria, 79], [Benguria, Brezis, Lieb, 81], [Diaz,Saa, 87]
The functional $u \mapsto \int |\nabla u^\alpha|^p \, dx$ is convex for any $p > 1$, $\alpha \in [\frac{1}{p}, 1]$.

Uniqueness

Consider two solutions u_1 and u_2 of (10).

$$\frac{d}{dt}\int u_1 \log\left(\frac{u_1}{u_2}\right) dx$$
$$= \int \left(1 + \log\left(\frac{u_1}{u_2}\right)\right) (u_1)_t \, dx - \int \left(\frac{u_1}{u_2}\right) (u_2)_t \, dx$$
$$= -(p-1)^{-(p-1)} \int u_1 \left[\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right] \cdot \left[\left|\frac{\nabla u_1}{u_1}\right|^{p-2} \frac{\nabla u_1}{u_1} - \left|\frac{\nabla u_2}{u_2}\right|^{p-2} \frac{\nabla u_2}{u_2}\right] dx \,.$$

It is then straightforward to check that two solutions with same initial data f have to be equal since

$$\frac{1}{4\,\|f\|_1} \|u_1(\cdot,t) - u_2(\cdot,t)\|_1^2 \le \int u_1(\cdot,t) \log\left(\frac{u_1(\cdot,t)}{u_2(\cdot,t)}\right) dx \le \int f \log\left(\frac{f}{f}\right) dx = 0$$

by the Csiszár-Kullback inequality.

Hyper- and Ultra-contractivity

Understanding the regularizing properties of

$$u_t = \Delta_p u^{1/(p-1)}$$

Theorem 15 *Let α, $\beta \in [1, +\infty]$ with $\beta \geq \alpha$. Under the same assumptions as in the existence Theorem, if moreover $f \in L^\alpha(\mathbb{R}^n)$, any solution with initial data f satisfies the estimate*

$$\|u(\cdot,t)\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta)\, t^{-\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}} \quad \forall\, t > 0$$

with $A(n,p,\alpha,\beta) = (\mathcal{C}_1\,(\beta-\alpha))^{\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}}\,\mathcal{C}_2^{\frac{n}{p}}$, $\mathcal{C}_1 = n\,\mathcal{L}_p\,e^{p-1}\,\frac{(p-1)^{p-1}}{p^{p+1}}$,

$\mathcal{C}_2 = \frac{(\beta-1)^{\frac{1-\beta}{\beta}}}{(\alpha-1)^{\frac{1-\alpha}{\alpha}}}\,\frac{\beta^{\frac{1-p}{\beta}-\frac{1}{\alpha}+1}}{\alpha^{\frac{1-p}{\alpha}-\frac{1}{\beta}+1}}$.

Case $p=2$: $\mathcal{L}_2 = \frac{2}{\pi\,n\,e}$, [Gross 75]

As a special case of Theorem 15, we obtain an *ultracontractivity* result in the limit case corresponding to $\alpha = 1$ and $\beta = \infty$.

Corollary 16 *Consider a solution u with a nonnegative initial data $f \in L^1(\mathbb{R}^n)$. Then for any $t > 0$*

$$\|u(\cdot,t)\|_\infty \leq \|f\|_1 \left(\frac{\mathcal{C}_1}{t}\right)^{\frac{n}{p}} .$$

Case $p = 2$, [Varopoulos 85]

Proof. Take a nonnegative function $u \in L^q(\mathbb{R}^n)$ with $u^q \log u$ in $L^1(\mathbb{R}^n)$. It is straightforward that

$$\frac{d}{dq} \int u^q \, dx = \int u^q \log u \, dx \; . \tag{11}$$

Consider now a solution $u_t = \Delta_p u^{1/(p-1)}$. For a given $q \in [1, +\infty)$,

$$\frac{d}{dt} \int u^q \, dx = -\frac{q(q-1)}{(p-1)^{p-1}} \int u^{q-p} |\nabla u|^p \, dx \; . \tag{12}$$

Assume that q depends on t and let $F(t) = \|u(\cdot, t)\|_{q(t)}$. Let $' = \frac{d}{dt}$. A combination of (11) and (12) gives

$$\frac{F'}{F} = \frac{q'}{q^2} \left[\int \frac{u^q}{F^q} \log \left(\frac{u^q}{F^q} \right) dx - \frac{q^2(q-1)}{q'(p-1)^{p-1}} \frac{1}{F^q} \int u^{q-p} |\nabla u|^p \, dx \right] \; .$$

Since $\int u^{q-p}|\nabla u|^p\,dx = (\frac{p}{q})^p \int |\nabla u^{q/p}|^p\,dx$, Corollary 13 applied with $w = u^{q/p}$,

$$\mu = \frac{(q-1)\,p^p}{q'\,q^{p-2}\,(p-1)^{p-1}}$$

gives for any $t \geq 0$

$$F(t) \leq F(0)\,e^{A(t)} \qquad \text{with } A(t) = \frac{n}{p}\int_0^t \frac{q'}{q^2}\log\left(\mathcal{K}_p\,\frac{q^{p-2}\,q'}{q-1}\right)ds$$

$$\text{and } \mathcal{K}_p = \frac{n\,\mathcal{L}_p}{e}\,\frac{(p-1)^{p-1}}{p^{p+1}}\,.$$

Now let us minimize $A(t)$: the optimal function $t \mapsto q(t)$ solves the ODE

$$q''\,q = 2\,{q'}^2 \Longleftrightarrow q(t) = \frac{1}{a\,t+b}\,.$$

Take $q_0 = \alpha$, $q(t) = \beta$ allows to compute $at = \frac{\alpha-\beta}{\alpha\beta}$ and $b = \frac{1}{\alpha}$. $\square$

Conclusion

The three following identities are equivalent:

(i) For any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p \, dx = 1$,

$$\int |w|^p \log |w| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla w|^p \, dx \right]$$

(ii) Let P_t^p be the semigroup associated $u_t = \Delta_p(u^{1/(p-1)})$:

$$\|P_t^p f\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta) \, t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$

(iii) Let Q_t^p be the semigroup associated to $v_t + \frac{1}{p} |\nabla v|^p = 0$:

$$\|e^{Q_t^p g}\|_\beta \leq \|e^g\|_\alpha \, B(n,p,\alpha,\beta) \, t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$

The Prékopa-Leindler inequality implies (iii).

III — L^1 *intermediate asymptotics*

for scalar conservation laws

Joint work with Miguel ESCOBEDO

Let $q > 1$ and consider a nonnegative entropy solution of

$$\begin{cases} U_\tau + (U^q)_\xi = 0\,, \quad \xi \in \mathbb{R}\,, \quad \tau > 0 \\ \\ U(\tau = 0, \cdot) = U_0 \end{cases} \tag{13}$$

Question: what is the asymptotic behavior as $t \to +\infty$?

P. Lax (1957): $\|U(\tau,\cdot) - W_\infty(\tau,\cdot)\|_1 = O(\tau^{-1/2}) \quad$ as $\quad \tau \to \infty$ if

$$U_\tau + f(U)_\xi = 0$$

with $f \in C^2$ near the origin + additional conditions.

T.-P. Liu & M. Pierre: $\lim_{\tau\to\infty} \tau^{\frac{1}{q}(1-\frac{1}{p})} \|U(\tau) - U_\infty(\tau)\|_p = 0$
where U_∞ is the self-similar solution
Y.-J. Kim (2001): $q \in (1,2)$, intermediate asymptotics in L^1
M. Escobedo, J.D. (2003) $q \in (1,2)$, intermediate asymptotics in L^1 + additional estimates

Theorem 17 *Let U be a global, piecewise C^1 entropy solution of (13) corresponding to a nonnegative initial data U_0 in $L^1 \cap L^\infty(\mathbb{R})$ which is compactly supported in $(\xi_0, +\infty)$ for some $\xi_0 \in \mathbb{R}$ and such that*

$$\liminf_{\xi \to (\xi_0)_+} \frac{U_0(\xi)}{|\xi - \xi_0|^{1/(q-1)}} > 0$$

Then, for any $\alpha \in (0, \frac{q}{q-1})$ and $\epsilon > 0$,

$$\limsup_{\tau \to +\infty} \tau^{\alpha - \epsilon} \int_{\mathbb{R}} |U(\tau, \xi) - U_\infty(\tau, \xi - \xi_0)| \frac{d\xi}{|\xi - \xi_0|^\alpha} = 0$$

Self-similar solution: $U_\infty(\tau, \xi) = (|\xi|/q\,\tau)^{1/(q-1)} \chi_{\xi \le c(\tau)}$

Corollary 18 *For any $\beta < 1$, there exists a constant C_β such that*

$$\|U(\tau, \cdot) - U_\infty(\tau, \xi - \xi_0)\|_1 \le C_\beta \, \tau^{-\beta}$$

Uniform estimates: *Graph convergence*

Theorem 19 *Under the same assumptions as above,*

$$\lim_{\tau\to+\infty} \sup_{\xi\in\mathrm{supp}(U(\tau,\cdot))} \tau^{1/q}\,|U(\tau,\xi)-U_\infty(\tau,\cdot-\xi_0)| = 0$$

$$\lim_{\tau\to+\infty} (1+q\,\tau)^{-1/q}\max\left[\mathrm{supp}(U(\tau,\cdot))\right] = Const(q,U_0)$$

Notions of solution, time-dependent rescaling, shocks

Let U be a nonnegative piecewise C^1 entropy solution of (13), whose points of discontinuity are given by the curves $\xi_1(\tau) < \xi_2(\tau) < \cdots < \xi_n(\tau)$. Then the rescaled function

$$u(t,x) = e^t\,U\left((e^{qt}-1)/q, e^t x\right)$$

is a piecewise C^1 function, whose points of discontinuity are given by the curves $s_i(t) \equiv e^{-t}\xi_i((e^{qt}-1)/q)$
Rankine-Hugoniot condition

$$s_i'(t) = \frac{(u_i^+)^q - s_i(t)\,u_i^+ - (u_i^-)^q + s_i(t)\,u_i^-}{u_i^+ - u_i^-}$$

Out of the curves $x = s_i(t)$ the function u is a classical solution of

$$u_t = (x\,u - u^q)_x \tag{14}$$

and across these curves it satisfies

$$u_i^- := \lim_{\substack{x\to s_i(t)\\ x<s_i(t)}} u(t,x) > \lim_{\substack{x\to s_i(t)\\ x>s_i(t)}} u(t,x) := u_i^+$$

Moreover u and U have the same initial data $U_0 := U(0,\cdot) = u(0,\cdot)$. Finally, if $U_0 \in L^1(\mathbb{R})$, then $||u(t)||_1 = ||U_0||_1$, for all $t > 0$.

Entropy

For every $c > 0$, let u^c_∞ be the *stationary solution* of (14) :

$$u^c_\infty(x) = \begin{cases} x^{1/(q-1)} & 0 \le x \le c \\ \\ 0 & \text{if } x < 0 \text{ or } x > c \end{cases}$$

Relative entropy Σ of the solution u with respect to u^c_∞: For any positive constant c, let

$$\Sigma(t) = \int_0^c \mu(x) \left| u(t,x) - u^c_\infty(x) \right| dx$$

Define $f(v) = v - v^q$ for $v > 0$

$$\frac{d\Sigma}{dt} \le \int_0^c \mu'(u^c_\infty)^q \left| f\left(\frac{u}{u^c_\infty}\right) \right| dx \le 0$$

Assume for simplicity that $u(t,.)$ has exactly one shock. Let $v^{\pm} = u^{\pm}/u^c_\infty$ at $x = s(t) \in (0,c)$: $v^- > v^+$ and

$$s'(t) = -(u^{c'}_\infty)^{q-1} \frac{f(v^+) - f(v^-)}{v^+ - v^-}$$

$$\frac{d\Sigma}{dt} = \int_0^c \mu \, u_t \left[1\!\!1_{u > u^c_\infty} - 1\!\!1_{u < u^c_\infty} \right] dx$$

$$+ \quad [\mu(s)|u - u^c_\infty(s)| \cdot s'(t)\,]_{u=u^+}^{u=u^-}$$

$$\frac{d\Sigma}{dt} \leq \int_0^c \mu' \,(u^c_\infty)^q \left| f\left(\frac{u}{u^c_\infty}\right)\right| dx + \mu(s)\,(u^c_\infty(s))^q \, \Psi(v^-, v^+)$$

$+$ boundary terms

$$\Psi(v^-,v^+) := \left[f(v^+) - f(v^-)\right] \cdot \frac{|v^+-1|-|v^--1|}{v^+-v^-} + |f(v^+)| - |f(v^-)|$$

- $1 \le v^+ \le v^-$: $f(v^-) \le f(v^+) \le 0$ and $\Psi(v^-,v^+) = 0$

- $v^+ < 1 \le v^-$: $f(v^-) \le 0 < f(v^+)$

$$\begin{aligned} \frac{1}{2}\Psi(v^-,v^+) = & \ \frac{v^--1}{v^--v^+} f(v^+) + \frac{1-v^+}{v^--v^+} f(v^-) \\ \le & \ \ f\left(\frac{v^--1}{v^--v^+} v^+ + \frac{1-v^+}{v^--v^+} v^-\right) = f(1) = 0 \end{aligned}$$

- $v^+ < v^- \le 1$: $f(v^-) \ge 0$ and $f(v^+) \ge 0$ and $\Psi(v^-,v^+) = 0$

$$\frac{d\Sigma_\alpha}{dt} \leq -\alpha \int_0^c x^{-\alpha-1+\frac{q}{q-1}} \left| f\left(\frac{u}{u_\infty^c}\right)\right| dx + \text{boundary terms}$$

Taylor expansion:

$$f\left(\frac{u}{u_\infty^c}\right) = (1-q)\left(\frac{u}{u_\infty^c}-1\right) + q(1-q)\left(\frac{u}{u_\infty^c}-1\right)^2 \int_0^1 (1-\theta)\left(\theta\frac{u}{u_\infty^c}+1-\theta\right)^{q-2} d\theta$$

$$\int_0^c x^{-\alpha+\frac{1}{q-1}} \left(\frac{u}{u_\infty^c}-1\right)^2 dx \leq \underbrace{\left\|\frac{u}{u_\infty^c}-1\right\|_{L^\infty(0,c(t))}}_{\to 0} \int_0^c \mu \left|u - u_\infty^c\right| dx$$

is neglectible compared to $\Sigma_\alpha(t)$ as $t \to +\infty$.

$$\boxed{\frac{d\Sigma_\alpha}{dt} + (q-1)\,\alpha\,\Sigma_\alpha(t) = o(\Sigma_\alpha(t))}$$

IV — Fourth order operators

$$\begin{aligned} & u_t + (u(\log u)_{xx})_{xx} = 0 \\ & u(\cdot, 0) = u_0 \quad \text{in } S^1 \end{aligned} \tag{15}$$

Joint work with Ansgar JÜNGEL and Ivan GENTIL,
in progress

[Jüngel et al.]
[Cáceres, Carrillo, Toscani]

$$u_t + (u(\log u)_{xx})_{xx} = 0, \quad u(\cdot, 0) = u_0 \quad \text{in } S^1$$

There are several Lyapunov functionals:

$$\frac{d}{dt}\int_{S^1} u(\log u - 1)dx + \int_{S^1} u(\log u)_{xx}^2 dx = 0$$

$$\frac{d}{dt}\int_{S^1} (u - \log u)dx + \int_{S^1} (\log u)_{xx}^2 dx = 0$$

Existence of periodic solutions

Theorem 4 *Let* $u_0 : S^1 \to \mathbb{R}$ *be a measurable function such that* $\int(u_0 - \log u_0)dx < \infty$. *Then there exists a global weak solution* u *of (15) satisfying*

$$u \in L^q_{\text{loc}}(0,\infty; W^{1,p}(S^1)) \cap W^{1,1}_{\text{loc}}(0,\infty; H^{-2}(S^1)),$$
$$u \geq 0 \quad \text{in } S^1 \times (0,\infty), \quad \log u \in L^2_{\text{loc}}(0,\infty; H^2(S^1)),$$

where $p \in (1, 4/3)$, $q = 5p/(4p-2) \in (2, 5/2)$, *and for all* $T > 0$ *and all smooth test functions* ϕ

$$\int_0^T \langle u_t, \phi \rangle_{(H^2)^*, H^2} dt + \int_0^T \int_{S^1} u(\log u)_{xx} \phi_{xx} dx dt = 0.$$

Optimal logarithmic Sobolev inequality on S^1

Theorem 5 *Let* $H = \{u \in H^1(S^1) : u_x \not\equiv 0 \text{ a.e.}\}$ *and* $\|u\|_2^2 = \int_{S^1} u^2 dx/L$. *Then*

$$\inf_{u \in H} \frac{\int_{S^1} u_x^2 dx}{\int_{S^1} u^2 \log(u^2/\|u\|_2^2) dx} = \frac{\pi^2}{2L^2}.$$

Lower bound: Expand the quotient for $u = 1+\varepsilon v$ with $\int_{S^1} v dx = 0$ in powers of ε and use the Poincaré inequality.
Upper bound: entropy - entropy-production method:

$$v_t = v_{xx} \quad \text{in } S^1 \times (0,\infty), \quad v(\cdot,0) = u^2 \quad \text{in } S^1$$

Then

$$\frac{d}{dt}\left(\int_{S^1} (\sqrt{v}_x)^2 dx - \frac{\pi^2}{2L^2}\int_{S^1} v \log v dx\right) \le -\frac{2}{3}\int_{S^1} \frac{(\sqrt{v}_x)^4}{v} dx \le 0$$

Corollary 6 *Let* $\mathcal{H} = \{u \in H^2(S^1) : u_x \not\equiv 0 \text{ a.e.}\}$. *Then*

$$\inf_{u \in \mathcal{H}} \frac{\int_{S^1} u_{xx}^2 dx}{\int_{S^1} u^2 \log(u^2/\|u\|_2^2) dx} = \frac{\pi^2}{2L^4}.$$

- Asymptotic behavior

$$\frac{d}{dt} \int_{S^1} u \log\left(\frac{u}{\bar{u}}\right) dx \le -\frac{2L^4}{\pi^2} \int_{S^1} u \log\left(\frac{u}{\bar{u}}\right) dx$$

- Hyper-contractivity: in progress

V — Inequalities and transport

[Cordero-Erausquin, Gangbo, Houdré, Nazaret, Villani],
[Agueh, Ghoussoub, Kang]

- Sobolev inequality: $\|f\|_{L^{2^*}} \leq S \, \|\nabla f\|_{L^2}$
- (Standard) logarithmic Sobolev inequality
- Logarithmic Sobolev inequality in $W^{1,p}(\mathbb{R}^N)$

Sobolev inequalities

$$\|f\|_{L^{2^*}} \leq S\,\|\nabla f\|_{L^2}$$

$N \geq 3$. Optimal function: $f(x) = (\sigma + |x|^2)^{-(N-2)/2}$.

A proof based on mass transportation:

$$\inf\left\{\frac{1}{2\lambda^2}\int_{\mathbf{R}^N} |\nabla f|^2\,dx \;:\; \int_{\mathbf{R}^N} |f|^{2^*}\,dx = 1\right\}$$

$$= \tfrac{n(n-2)}{2(n-1)}\,\sup\left\{\textstyle\int_{\mathbf{R}^N} |g|^{2^*(1-\frac{1}{n})}dy - \frac{\lambda^2}{2}\int_{\mathbf{R}^N} |y|^2\,|g|^{2^*}dy \;:\; \int_{\mathbf{R}^N} |g|^{2^*}dy = 1\right\}$$

Mass transportation: basic results

μ and ν two Borel probability measures on $\mathbb{R}^N$. $T : \mathbb{R}^N \to \mathbb{R}^N$
$T\#\mu = \nu \Longleftrightarrow \nu(A) = \mu(T^{-1}(A))$ for any Borel measurable set A.

Theorem 20 (Brenier, McCann) *$\exists\, T = \nabla\phi$ such that $T\#\mu = \nu$ and ϕ is convex.*

$\mu = F(x)\,dx$, $\nu = G(x)\,dx$, $\int_{\mathbf{R}^N} F(x)\,dx = \int_{\mathbf{R}^N} G(y)\,dy = 1$

$$\forall\, b \in C(\mathbb{R}^N, \mathbb{R}^+) \quad \int_{\mathbf{R}^N} b(y)G(y)\,dy = \int_{\mathbf{R}^N} b(\nabla\phi(x))F(x)\,dx$$

Under technical assumptions: $\phi \in C^2$, supp(F) or supp(G) is convex... [Caffarelli] ϕ solves the Monge-Ampère equation

$$G(\nabla\phi)\,\det \mathrm{Hess}(\phi) = F$$

A proof of the Sobolev inequality

$$G(\nabla\phi)^{-\frac{1}{n}} = (\det \mathrm{Hess}(\phi))^{\frac{1}{n}} F^{-\frac{1}{n}} \le \frac{1}{n} \Delta\phi \, F^{-\frac{1}{n}}$$

$$\int G(y)^{1-\frac{1}{n}} dy \le \frac{1}{n} \int G(\nabla\phi(x))^{1-\frac{1}{n}} (\det \mathrm{Hess}(\phi))^{\frac{1}{n}} \Delta\phi \, dx$$

$$= \frac{1}{n} \int F^{1-\frac{1}{n}} \Delta\phi \, dx = -\frac{1}{n} \int \nabla(F^{1-\frac{1}{n}}) \cdot \nabla\phi \, dx$$

by the arithmetic-geometric inequality. $F = |f|^{2^*}$, $G = |g|^{2^*}$

$$\int |g|^{2^*(1-\frac{1}{n})} dy \le -\frac{2(n-1)}{n(n-2)} \int (f^{\frac{n}{n-2}}) \nabla f \cdot \nabla\phi \, dx$$

$$\frac{n(n-2)}{2(n-1)} \int |g|^{2^*(1-\frac{1}{n})} dy \le \frac{2}{\lambda^2} \int |\nabla f|^2 dx + \frac{\lambda^2}{2} \int |f|^{2^*} |\nabla\phi|^2 dx$$

by Young's inequality. Use: $\int F \, |\nabla\phi|^2 \, dx = \int G|y|^2 dy$

A proof of the standard logarithmic Sobolev inequality

$G(y) = e^{-|y|^2/2}$, $F(x) = f(x)\, e^{-|x|^2/2}$, $\nabla\phi \# F\, dx = G\, dy$.

$$e^{-|\nabla\phi|^2/2} \det \mathrm{Hess}(\phi) = f(x)\, e^{-|x|^2/2}$$

$\theta(x) = \phi(x) - \frac{1}{2}|x|^2$

$$f(x)\, e^{-|x|^2/2} = \det\left(\mathrm{Id} + \mathrm{Hess}(\theta)\right) e^{-|x+\nabla\theta(x)|^2/2}$$

$$\begin{aligned} \log f - |x|^2/2 &= -|x+\nabla\theta(x)|^2/2 + \log\Big[\det\left(\mathrm{Id} + \mathrm{Hess}(\theta)\right)\Big] \\ &\leq -|x+\nabla\theta(x)|^2/2 + \Delta\theta \end{aligned}$$

(use $\log(1+t) \leq t$). Let $d\mu(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx$.

$$\log f \leq -\frac{1}{2}|\nabla\theta|^2 - x\cdot\nabla\theta + \Delta\theta$$

$$\int f \log f\, d\mu \leq -\frac{1}{2}\int \left|\sqrt{f}\,\nabla\theta + \frac{\nabla f}{\sqrt{f}}\right|^2 d\mu + \frac{1}{2}\int \frac{|\nabla f|^2}{f}\, d\mu \leq \frac{1}{2}\int \frac{|\nabla f|^2}{f}\, d\mu$$

Logarithmic Sobolev inequality in $W^{1,p}(\mathbb{R}^N)$

$G(y) = c_{p,n}\, e^{-\frac{p}{p-1}\,|y|^{p/(p-1)}} =: f_\infty(y)$, $F(x) = f(x)\, c_{p,n}\, e^{-\frac{p}{p-1}\,|x|^{p/(p-1)}}$
$\nabla\phi \# F\, dx = G\, dy$, $d\mu(x) = f_\infty^p(x)\, dx$

$$f(x)\, e^{-\frac{p}{p-1}\,|x|^{p/(p-1)}} = \det\left(\mathrm{Hess}(\phi)\right) e^{-\frac{p}{p-1}\,|x+\nabla\theta(x)|^{p/(p-1)}}$$

$$f^p(x) = f_\infty^p(\nabla\phi)\det\left(\mathrm{Id} + \mathrm{Hess}(\phi)\right)$$

$$\int f^p \log f^p\, d\mu = \int f^p \log f_\infty^p\, d\mu + \int (\Delta\phi - n) f^p d\mu$$

$$\int \Delta\phi\, f^p d\mu = -p\int f^{p-1}\nabla f\cdot\nabla\phi\, d\mu \le \frac{\lambda^{-q}}{q}\int |f|^p\,|\nabla\phi|^{p/(p-1)} + \frac{\lambda^p}{p}\int |\nabla f|^p d\mu$$

using Young's inequality: $X = f^{p-1}\nabla\phi$, $Y = \nabla f$

$$\int X\cdot Y\, d\mu \le \frac{\lambda^{-q}}{q}\,\|X\|_q^q + \frac{\lambda^p}{p}\,\|Y\|_p^p$$