Nonlinear diffusion equations as diffusion limits of kinetic equations

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Outline

- Methods based on self-similarity are now well established at the level of diffusion equations

- It is an important tool for the understanding of large time asymptotics

$\Longrightarrow$ Can we extend such methods to kinetic equations ?

- Self-similarity: a tool for understanding *dispersive* properties of kinetic equations

- Notion of *intermediate asymptotics* has to be replaced by *asymptotic stability*

How to relate kinetic descriptions with diffusive models ?
diffusion limits

Plan

I. Time-dependent rescalings and asymptotic stability
A joint work in collaboration with G. Rein

II. Measuring stability in kinetic equations [J.D., Sánchez, Soler], [Cáceres, Carrillo,J.D.]

III. Nonlinear diffusion equations as diffusion limits of kinetic equations
A joint work in collaboration with P. Markowich, D. Ölz, and C. Schmeiser. Other contributions by Poupaud, Golse, Goudon, Degond, Schmeiser, Perthame, Ben Abdallah, Chavanis et al., Laurençot, Lemou,...

I. Time-dependent rescalings and asymptotic stability

Consider the Vlasov-Poisson system

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0$$
$$-\Delta \phi = \int_{\mathbf{R}^d} f(t,x,v)\, dv$$

in the physical space: x, $v \in \mathbb{R}^3$, without confinement. Because of the repulsive mean field force, particles runaway at infinity and one expects to get dispersion estimates at least as $t \to \infty$

$$\frac{d}{dt}\left(\frac{1}{t}\int\int_{\mathbf{R}^3\times\mathbf{R}^3} |x - t\,v|^2 f(t,x,v)\, dx\, dv + \frac{1}{2}\, t \int_{\mathbf{R}^3} |\nabla \phi|^2\, dx\right)$$

$$= -\frac{1}{t^2}\int\int_{\mathbf{R}^3\times\mathbf{R}^3} |x - t\,v|^2 f(t,x,v)\, dx\, dv \leq 0$$

Since L^p norms of the distribution function $f(t\cdot,\cdot)$ are preserved:

$$\|f(t\cdot,\cdot)\|_{L^\infty(\mathbf{R}^3\times\mathbf{R}^3)} \leq \|f_0\|_{L^\infty(\mathbf{R}^3\times\mathbf{R}^3)} \quad \forall\, t>0$$

Dispersion can be measured by interpolation

$$0 \leq \rho(t,x) := \int_{\mathbf{R}^d} f(t,x,v)\, dv = \int_{|x-t\,v|\leq R} f\, dv + \int_{|x-t\,v|>R} f\, dv$$

$$0 \leq \rho(t,x) \leq 4\pi \left(\frac{R}{t}\right)^3 \|f_0\|_{L^\infty(\mathbf{R}^3\times\mathbf{R}^3)} + \frac{1}{R^2}\int_{\mathbf{R}^3} |x-t\,v|^2 f\, dv$$

Optimizing on $R=R(x,t)$ with x, t fixed, we get

$$0 \leq \rho(t,x) \leq C\, \|f_0\|^{2/5}_{L^\infty(\mathbf{R}^3\times\mathbf{R}^3)} \left(\int_{\mathbf{R}^3} |x-t\,v|^2 f\, dv\right)^{3/5} t^{-6/5}$$

$$\|\rho\|_{L^{5/3}(\mathbf{R}^3)} \leq C\, \|f_0\|^{2/5}_{L^\infty(\mathbf{R}^3\times\mathbf{R}^3)} \left(\int\!\!\int_{\mathbf{R}^3\times\mathbf{R}^3} |x-t\,v|^2 f\, dx\, dv\right)^{3/5} t^{-6/5}$$

Theorem 1. *[Perthame, Illner-Rein] Assume*

$$f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3),\ \int\int_{\mathbf{R}^3\times\mathbf{R}^3}(|x|^2 + |v|^2)\, f\ dx\, dv < \infty$$

Then

$$\|\rho(t,\cdot)\|_{L^{5/3}(\mathbf{R}^3)} = O(t^{-3/5})$$

Scalings

If (f, ϕ) is a solution to the Vlasov-Poisson system,

$$f_{\lambda,\mu}(t,x,v) := \lambda^{2-d} f\left(\lambda\, t, \mu\, x, \lambda^{-1}\, \mu\, v\right)$$

and the corresponding $\phi_{\lambda,\mu}$ given by the Poisson equation are also solutions to the Vlasov-Poisson system.

If we additionally require that the $L^1(\mathbb{R}^d \times \mathbb{R}^d)$ norm is preserved, we get: $\mu = \lambda^{2/d}$ and the corresponding distribution function is

$$f_{\lambda,\lambda^{2/d}}(t,x,v) := \lambda^{4-d} f\left(\lambda\, t, \lambda^{2/d}\, x, \lambda^{-1+2/d}\, v\right)$$

We immediately see that for $d = 3$, this scaling is not convenient since in the singular limit the initial data is not well defined as a measure.

A special solution

The “monokinetic” distribution function:

$$f_\infty(t,x,v) := \rho(t,x)\,\delta(v-u(t,x))$$

is a solution of (VP) if

$$\rho(t,x) = \frac{1}{R^d(t)} \mathbb{1}_{B_{R(t)}}(x)\ , \qquad u(t,x) = \frac{1}{R(t)} \frac{dR}{dt}\, x$$

are solutions to the pressureless Euler-Poisson system (EP)

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho\, u) = 0 \\ \partial_t u + (u \cdot \nabla_x) u = -\nabla_x \phi \\ -\Delta \phi = \rho \end{cases}$$

$$\Longleftrightarrow \frac{dR}{dt} = R^{1-d}$$

Time-dependent rescalings

[Rein,J.D.] We define a change of variables which leaves (VP) as invariant as possible

$$(t, x, v) \mapsto (\tau, \xi, \eta)$$

$$dt = A^2(t) d\tau \ , \quad x = R(t)\xi$$

Assuming that $t \mapsto x(t)$ and $\tau \mapsto \xi(\tau)$ satisfy $\frac{dx}{dt} = v$ and $\frac{d\xi}{d\tau} = \eta$ $\Longleftrightarrow$ *Galilean invariance*, the new velocity variable η has to satisfy

$$v = \frac{dx}{dt} = \dot{R}(t)\xi + R(t)\frac{d\xi}{d\tau}\frac{d\tau}{dt} = \dot{R}(t)\xi + \frac{R(t)}{A^2(t)}\eta$$

Rescaled distribution function:

$$f(t,x,v) = G(t)F(\tau,\xi,\eta)$$

$$d\tau = A^{-2}(t)dt,\ \xi = R^{-1}(t)x,\ \eta = \frac{A^2(t)}{R(t)}\left(v - \frac{\dot{R}(t)}{R(t)}x\right).$$

If ν and W are defined as the rescaled spatial density and the rescaled potential respectively, then

$$\nu(\tau,\xi) = \int_{\mathbf{R}^d} F(\tau,\xi,\eta)\, d\eta = \tfrac{A^{2d}}{R^d G}\rho(t,x),$$

$$W(\tau,\xi) = \tfrac{A^{2d}}{R^{d+2}G}\phi(t,x), \quad \nabla_\xi W(\tau,\xi) = \tfrac{A^{2d}}{R^{d+1}G}\nabla_x\phi(t,x),$$

Rescaled Vlasov equation

$$\partial_\tau F + \eta \cdot \nabla_\xi F \;+\; 2A^2\left(\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right)\eta \cdot \nabla_\eta F$$
$$-\; \ddot{R}\frac{A^4}{R}\xi \cdot \nabla_\eta F - R^d G A^{4-2d}\nabla_\xi W \cdot \nabla_\eta F + A^2\frac{\dot{G}}{G}F = 0$$

L^1 norm is preserved under the change of variables

$$\frac{\dot{A}}{A} - \frac{\dot{R}}{R} = \frac{1}{2d}\frac{\dot{G}}{G} \quad \Longrightarrow \quad G = \left(\frac{A}{R}\right)^{2d}$$

No time-dependent factor in front of the nonlinear term:
$A = R^{d/4}$, $G = R^{\frac{d-4}{2}d}$ and R has to solve

$$\ddot{R} + R^{1-d} = 0$$

Rescaled Vlasov-Poisson system (RVP):

$$\partial_\tau F + \eta \cdot \nabla_\xi F + \nabla_\eta \left[\left(\xi - \nabla_\xi W + \tfrac{d-4}{2} R^{\frac{d}{2}-1} \dot{R} \eta \right) F \right] = 0$$
$$-\Delta W = \nu(\tau, \xi) = \int_{\mathbf{R}^d} F(\tau, \xi, \eta)\, d\eta$$

The relation between the old and the new variables is

$$dt = R^{d/2} d\tau, \quad d\tau = R^{-d/2} dt$$
$$x = R\,\xi, \quad \xi = R^{-1} x$$
$$v = \dot{R}\xi + R^{1-\frac{d}{2}} \eta, \quad \eta = R^{\frac{d}{2}-1} \left(v - \frac{\dot{R}}{R} x \right)$$

and the rescaled functions are given by

$$F(\tau, \xi, \eta) = R^{\frac{4-d}{2} d} f(t, x, v), \quad \nu(\tau, \xi) = R^d \rho(t, x)$$
$$W(\tau, \xi) = R^{d-2} \phi(t, x), \quad \nabla_\xi W(\tau, \xi) = R^{d-1} \nabla_x \phi(t, x)$$

$F^M_\infty(\xi,\eta) = \nu^M_\infty(\xi)\delta(\eta)$ where δ is the usual Dirac distribution is a steady state, with

$$\nabla_\xi W^M_\infty(\xi) = \begin{cases} \xi & \text{if } |\xi| \leq (M/|S^{d-1}|)^{1/d} \\ \xi/|\xi|^d & \text{if } |\xi| > (M/|S^{d-1}|)^{1/d} \end{cases}$$

and associated spatial density

$$\nu^M_\infty(\xi) = d \cdot \mathbb{1}_{B^d((M/|S^{d-1}|)^{1/d})}$$

By the inverse rescaling transformation, we get

$$f^M_\infty(t,x,v) = \frac{d}{R(t)^d} \mathbb{1}_{B^d(R(t)(M/|S^{d-1}|)^{1/d})}(x)\, \delta\left(v - \frac{\dot{R}(t)}{R(t)} x\right)$$

and $\rho^M_\infty(t,x) = \frac{d}{R(t)^d} \mathbb{1}_{B^d(R(t)(M/|S^{d-1}|)^{1/d})}(x)$, $u^M_\infty(t,x) = \frac{\dot{R}(t)}{R(t)} x$
This defines a weak solution of (VP) or (EP)

The behavior of $R(t)$ depends on the dimension. For $d \neq 2$,

$$\frac{1}{2}\left(\frac{dR}{dt}(t)\right)^2 - \frac{1}{2}\left(\frac{dR}{dt}(0)\right)^2 = \frac{1}{2-d}R^{2-d}(t) - R^{2-d}(0) = 0$$

and, for $d = 2$,

$$\frac{1}{2}\left(\frac{dR}{dt}(t)\right)^2 - \frac{1}{2}\left(\frac{dR}{dt}(0)\right)^2 = \log\left(\frac{R(t)}{R(0)}\right)$$

If we choose $\frac{dR}{dt}(0) = 0$ and $R(0) = 1$, initial data for the rescaled problem is the same as for the original system

As $t \to \infty$, we get the following equivalences

$$\begin{aligned} R(t) &\sim t^2 && \text{if} && d = 1 \\ R(t) &\sim t\sqrt{\log t} && \text{if} && d = 2 \\ R(t) &\sim t && \text{if} && d \geq 3 \end{aligned}$$

In terms of the rescaled time variable, this means

$$\begin{aligned} \tau(t) &\sim \log t && \text{if} && d = 1 \\ \tau(t) &\sim \sqrt{\log t} && \text{if} && d = 2 \\ \tau(t) &\to \tau_\infty < \infty && \text{if} && d \geq 3 \end{aligned}$$

The friction coefficient

$$\beta := \frac{d-4}{2} R^{\frac{d}{2}-1} \dot{R}$$

is not constant, but converges as $t \to \infty$ to a positive constant, at least for $d < 4$. In the case $d = 4$, one recovers that the Vlasov-Poisson system is conformally invariant

Decay of the energy of the rescaled system

$$\frac{d}{d\tau} \left[\int\int_{\mathbf{R}^d \times \mathbf{R}^d} \left(\frac{1}{2} |\eta|^2 + \frac{1}{2} |\xi|^2 + \frac{1}{2} W \right) F(\tau, \xi, \eta)\, d\xi\, d\eta \right]$$
$$= -\beta(\tau) \int\int_{\mathbf{R}^d \times \mathbf{R}^d} |\eta|^2 F(\tau, \xi, \eta)\, d\xi\, d\eta \leq 0$$

II. Measuring stability in kinetic equations

Consider the (VP) system in the gravitational case

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - (\nabla_x \phi + \nabla_x \phi_0) \cdot \nabla_v f = 0 \\ \\ \Delta_x \phi = 4\pi\rho\,, \quad \lim_{|x|\to\infty} \phi(t,x) = 0 \end{cases} \qquad (VP)$$

in presence of an external potential ϕ_0. If we look for stationary solutions taking the form

$$f(x,v) = \gamma\left(\frac{1}{2}\,|v|^2 + \phi(x) + \phi_0(x) - \kappa\right)$$

Vlasov's equation is satisfied and the problem is reduced to solve the nonlinear Poisson equation

$$\Delta\phi = 4\pi\, G(\phi + \phi_0 - \kappa)$$

with

$$G(u) := \int_{\mathbb{R}^3} \gamma\left(\frac{1}{2}\,|v|^2 + u\right)\, dv = \frac{4\pi}{3} \int_0^{+\infty} \gamma(s+u)\,\sqrt{2\,s}\, ds$$

A POTENTIAL ENERGY ESTIMATE

Lemma 2. *There exists a positive constant C such that* $\forall\, f \in \mathrm{L}^1_+ \cap \mathrm{L}^\infty(\mathbb{R}^6)$ *with* $|v|^2 f \in L^1(\mathbb{R}^6)$

$$\int_{\mathbf{R}^3} |\nabla\phi|^2\, dx \le C\, \|f\|^{7/6}_{\mathrm{L}^1(\mathbf{R}^6)}\, \|f\|^{1/3}_{\mathrm{L}^\infty(\mathbf{R}^6)} \left(\int_{\mathbf{R}^6} |v|^2 f(x,v)\, dx\, dv\right)^{1/2}$$

$$\int_{\mathbf{R}^3} |\nabla\phi|^2\, dx = \int_{\mathbf{R}^3} (-\Delta\phi)\, \phi\, dx = 4\,\pi \int_{\mathbf{R}^6} \frac{\rho(y)\rho(x)}{|x-y|}\, dx\, dy$$

According to the Hardy-Littlewood-Sobolev inequalities,

$$\int_{\mathbf{R}^3} |\nabla\phi|^2\, dx \le 4\pi\, \Sigma\, \|\rho\|^2_{\mathrm{L}^{\frac{6}{5}}(\mathbf{R}^3)}$$

Because of Hölder's inequality, $\|\rho\|_{\mathrm{L}^{6/5}(\mathbf{R}^3)} \le \|\rho\|^{7/12}_{\mathrm{L}^1(\mathbf{R}^3)} \|\rho\|^{5/12}_{\mathrm{L}^{5/3}(\mathbf{R}^3)}$

Use interpolation inequalities to bound $\|\rho\|_{\mathrm{L}^{5/3}(\mathbf{R}^3)}$

An equivalent minimization problem

$\Gamma_M = \{f \in \mathsf{L}^1 \cap \mathsf{L}^\infty(\mathbb{R}^6) \ : \ f(x,v) \geq 0\,,\ \|f\|_{\mathsf{L}^1(\mathbb{R}^6)} = M\,,\ \|f\|_{\mathsf{L}^\infty(\mathbb{R}^6)} \leq 1\}$
Let $J_M = \inf\{J(f) \ : \ f \in \Gamma_M\}$,

$$J(f) = \frac{\frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f \, dx\, dv}{\left(\frac{1}{8\pi}\int_{\mathbb{R}^6} |\nabla\phi|^2 \, dx\right)^2} \equiv \frac{E_{KIN}(f)}{(E_{POT}(f))^2}$$

Lemma 3. *The minimization problems $E(f) = E_M$ and $J(f) = J_M$ over the set Γ_M are equivalent*
(i) *Their respective minima satisfy*

$$4\, J_M\, E_M = -1$$

(ii) *If $f_M \in \Gamma_M$ is a minimizer of the functional E, then it is also a minimizer of the functional J*

Theorem 1 (J.D., Sánchez, Soler). *Let f_M be a minimizing function for the functional E on Γ_M, with radial mass density. Then*

$$f_M(x,v) = \begin{cases} 1 \quad \textit{if} \quad \frac{1}{2}|v|^2 + \phi_{f_M}(x) < \frac{7}{3}\frac{E(f_M)}{M} \\ \\ 0 \quad \textit{otherwise} \end{cases}$$

where ϕ_{f_M} is the unique radial solution on $\mathbb{R}^3$ of

$$\Delta \phi_{f_M} = \frac{1}{3}(4\pi)^2 \left[2\left(\frac{7}{3}\frac{E_M}{M} - \phi_{f_M} \right)_+ \right]^{3/2}$$

It is the unique minimizer with radial mass density and it is also a steady-state solution to the VP system. Moreover, if f is another minimizing function, then with $\bar{x} := \frac{1}{M}\int_{\mathbb{R}^6} x\, f(x,v)\, dx\, dv$

$$f(x,v) = f_M(x - \bar{x}, v) \quad \forall\, (x,v) \in \mathbb{R}^6$$

Nonlinear stability for the evolution problem

[Guo, Rein, Wolansky, Sánchez, Soler, Schaeffer, Lemou-Méhats-Raphaël]
Consider for any g, $h \in \Gamma_M$ the distance d defined by

$$d(g,h) = E(g) - E(h) + \frac{1}{4\pi} \|\nabla\phi_g - \nabla\phi_h\|^2_{L^2(\mathbb{R}^3)}$$

Theorem 2. *For every $\epsilon > 0$, there exists a $\delta > 0$ such that, if f is a solution of the VP system with an initial condition $f_0 \in \Gamma_M$, then*

$$d(f_0, f_M) \leq \delta \quad \Longrightarrow \quad d(f^*(t), f_M) \leq \epsilon \quad \forall\, t \geq 0$$

The result is easily achieved by contradiction since $E(f^*(t)) - E(f_M) \leq E(f_0) - E(f_M) \searrow 0$ implies $\|\nabla\phi_{f^*(t)} - \nabla\phi_{f_M}\|_{L^2(\mathbb{R}^3)} \searrow 0$ $\square$

III. Nonlinear diffusion equations as diffusion limits of kinetic equations

Model

$$\epsilon^2 \partial_t f + \epsilon v \cdot \nabla_x f - \epsilon \nabla_x V(x) \cdot \nabla_v f = Q[f] \qquad (1)$$

$$Q[f] := G_f - f$$

$$G_f := \gamma\left(\frac{1}{2}|v|^2 + V(x) - \mu_{\rho_f}(x,t)\right)$$

Local Fermi level: μ_{ρ_f} is implicitly determined by the condition

$$\int_{\mathbb{R}^3} G_f \, dv = \rho_f := \int_{\mathbb{R}^3} f \, dv$$

The collision operator can be rewritten as

$$\mu_{\rho_f}(x,t) = V(x) + \bar{\mu}(\rho_f(x,t)) \quad \text{and} \quad \int_{\mathbb{R}^3} \gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}(\rho_f)\right) dv = \rho_f$$

$$Q[f] = G_f - f\,, \quad G_f = \gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}(\rho_f)\right)$$

Assumptions

The energy profile $\gamma : (E_1, E_2) \to \mathbb{R}_+$ is a nonincreasing, non-negative $\mathcal{C}^1$ function, $-\infty \le E_1 < E_2 \le \infty$, $\lim_{E \to E_2} \gamma(E) = 0$. If $E_2 < \infty$, we extend γ to $[E_2, \infty)$ by 0 and assume that there are constants $k > 0$ and $C > 0$ such that

$$\gamma(E) \le C(E_2 - E)^k \quad \text{on} \quad (\hat{E}, E_2)$$

If $E_2 = \infty$ we require

$$\gamma(E) = O(E^{-5/2}) \quad \text{as} \quad E \to \infty$$

to ensure existence of second velocity moments

$+$ technical assumptions on γ close to E_2

Formal asymptotic as $\epsilon \to 0$

$$f = \sum_{i=0}^{\infty} f^i \epsilon^i \quad \rho^i := \int_{\mathbb{R}^3} f^i \, dv \quad \rho = \sum_{i=1}^{\infty} \rho^i \epsilon^i$$

Let $G^i := \mathrm{sign}(\rho^i)\, \gamma(|v|^2/2 - \bar{\mu}(|\rho^i|))$, $G_f \approx \sum_{i=1}^{\infty} G^i \epsilon^i$

$$\epsilon^0 : \qquad G^0(x,v,t) = G_{f^0} = \gamma(|v|^2/2 + V(x) - \mu^0(x,t)) = f^0$$

$$\epsilon^1 : \qquad v \cdot \nabla_x f^0 - \nabla_x V \cdot \nabla_v f^0 = G^1 - f^1$$

$$\epsilon^2 : \qquad \partial_t f^0 + v \cdot \nabla_x f^1 - \nabla_x V \cdot \nabla_v f^1 = G^2 - f^2$$

$$\partial_t \int_{\mathbb{R}^3} f^0 \, dv + \nabla_x \cdot \int_{\mathbb{R}^3} v f^1 dv = O(\epsilon)$$

$$f^1 = v \cdot \nabla_x \mu^0 \, \gamma' \left(\frac{1}{2} v^2 + V(x) - \mu^0(x,t) \right) + G^1 \qquad \int_{\mathbb{R}^3} v f^1 \, dv = -\rho^0 \nabla_x \mu^0$$

Collecting these estimates, we get, for $\rho^0(x,t) = \int_{\mathbb{R}^3} f^0(x,v,t)\,dv$,

$$\partial_t \rho^0 = \nabla \cdot (\rho^0 \nabla \mu^0)$$

Use: $\mu^0 = \bar{\mu}(\rho^0) + V$ to recover the expected drift-diffusion equation :

$$\partial_t \rho^0 = \nabla \cdot (\rho^0 (\nabla \bar{\mu}(\rho^0) + \nabla V(x))) \qquad (2)$$

MOTIVATION

- collisions : short time scale
- Gibbs states are usually better known than collision kernels
- Gibbs states $\Longleftrightarrow$ generalized entropies
- nonlinear diffusion equations are difficult to justify directly
- global Gibbs states are the same at the kinetic / diffusion levels

[Ben Abdallah, J.D.], [Chavanis, Laurençot, Lemou], [Degond, Ringhofer]

Example 1. Power law case : $\gamma(E) := DE^{-k}$, $D > 0$, $k > 5/2$ (existence of second velocity moments)

$$\bar{\mu}(\rho) = -\left(\frac{\rho}{D\beta(k)}\right)^{\frac{1}{\frac{3}{2}-k}}, \quad \text{where} \quad \beta(k) := 4\pi\sqrt{2}\int_0^\infty \frac{\sqrt{s}}{(s+1)^k}\, ds$$

Fast diffusion equations :

$$\partial_t \rho = \nabla \cdot \left(\Theta \nabla (\rho^{\frac{k-5/2}{k-3/2}}) + \rho \nabla V\right), \quad \text{where} \quad \Theta := \frac{1}{k-\frac{5}{2}}\left(\frac{1}{D\beta(k)}\right)^{\frac{1}{\frac{3}{2}-k}}$$

Outside of a finite ball, the potential grows faster than a power

$$V(x) \geq C|x|^q, \quad \text{a.e. for} \quad |x| > R \quad \text{with} \quad q > \frac{3}{k-5/2}\,.$$

Example 2. Maxwell distribution : $\gamma(E) = \exp(-E)$

$$\bar{\mu}(\rho) = \log \rho - \frac{3}{2} \log(2\pi)$$

Linear drift-diffusion equation : $\nu(\rho) = \rho$

$$\partial_t \rho = \nabla \cdot \Big(\nabla \rho + \rho \nabla V \Big)$$

"the linear case"

Growth assumption on the potential

$$V(x) \geq q \log(|x|), \quad \text{a.e. for} \quad |x| > R \quad \text{with} \quad q > 3$$

Example 3. Let γ be a cut-off power with positive exponent :

$$\gamma(E) = (E_2 - E)_+^k := \begin{cases} D(E_2 - E)^k & \text{for} \quad E < E_2, \quad D > 0, \quad k > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{\mu}(\rho) = \left(\frac{\rho}{D\alpha(k)}\right)^{\frac{1}{k+\frac{3}{2}}} - E_2\,, \quad \text{where} \quad \alpha(k) = 4\pi\sqrt{2}\int_0^1 \sqrt{u}(1-u)^k\,du$$

Porous medium equations : $\nu(\rho) = \Theta\rho^{\frac{2k+5}{2k+3}}$

$$\partial_t \rho = \nabla \cdot \left(\Theta\nabla\left(\rho^{\frac{k+5/2}{k+3/2}}\right) + \rho\nabla V\right), \quad \text{where} \quad \Theta := \frac{1}{k+\frac{5}{2}}\left(\frac{1}{D\alpha(k)}\right)^{\frac{1}{k+\frac{3}{2}}}$$

Growth condition on the potential : if μ^* is the upper bound for the Fermi energy

$$\Big(E_2 + \mu^* - V(x)\Big)_+ = O\left(\frac{1}{|x|^q}\right) \quad \text{a.e. as} \quad |x| \to \infty, \quad q > \frac{3}{k+\frac{3}{2}}$$

Example 4. Fermi-Dirac distribution : $\gamma(E) = \frac{1}{\exp(E)+\alpha}$

$$(\bar{\mu}^{-1})(\theta) = \frac{4\pi\sqrt{2}}{\alpha}\int_0^\infty \frac{\sqrt{p}\,dp}{\exp(p-\theta-\log\alpha)+1} = -\frac{(2\pi)^{\frac{3}{2}}}{\alpha}\mathrm{Li}_{3/2}\big(-\alpha\exp(\theta)\big)$$

$$\bar{\mu}(\rho) = \log\left(-\frac{1}{\alpha}\big(\mathrm{Li}_{3/2}^{-1}\big)\big(-\frac{\alpha\rho}{(2\pi)^{3/2}}\big)\right), \quad \mathrm{Li}_n(z) := \sum_{k=1}^{\infty}\frac{z^k}{k^n}$$

Macroscopic equation : $\partial_t\rho = \nabla\cdot\Big(\big(D(\rho)\nabla\rho + \rho\nabla V\big)\Big)$

$$D(\rho) = \nu'(\rho) = \rho\,\bar{\mu}'(\rho) = \frac{-\alpha}{(2\pi)^{3/2}}\frac{\rho}{\mathrm{Li}_{1/2}\Big(\big(\mathrm{Li}_{3/2}^{-1}\big)\big(\frac{-\alpha\rho}{(2\pi)^{3/2}}\big)\Big)}$$

Moreover the expansion of $D(\rho)$ at $\rho = 0$ gives

$$D(\rho) = 1 + \frac{\sqrt{2}}{4}\frac{\alpha\rho}{(2\pi)^{3/2}} + \left(\frac{3}{8} - \frac{2\sqrt{3}}{9}\right)\frac{\alpha^2\rho^2}{(2\pi)^3} + O(\rho^3)$$

Example 5. Bose-Einstein distribution : $\gamma(E) = \frac{1}{\exp(E)-\alpha}$

$$(\bar{\mu}^{-1})(\theta) = \frac{4\pi\sqrt{2}}{\alpha}\int_0^\infty \frac{\sqrt{p}\,dp}{\exp(p-\theta-\log\alpha)-1} = \frac{(2\pi)^{\frac{3}{2}}}{\alpha}\mathrm{Li}_{3/2}\Big(\alpha\exp(\theta)\Big)$$

$$\bar{\mu}(\rho) = \log\left(\frac{1}{\alpha}\Big(\mathrm{Li}_{3/2}^{-1}\Big)\Big(\frac{\alpha\rho}{(2\pi)^{3/2}}\Big)\right)$$

Macroscopic equation : $\partial_t\rho = \nabla\cdot\Big((D(\rho)\nabla\rho + \rho\nabla V)\Big)$

$$D(\rho) = \nu'(\rho) = \rho\bar{\mu}'(\rho) = \frac{\alpha}{(2\pi)^{3/2}}\frac{\rho}{\mathrm{Li}_{1/2}\Big((\mathrm{Li}_{3/2}^{-1})(\frac{\alpha\rho}{(2\pi)^{3/2}})\Big)}$$

Observe that $\lim_{\rho\to\overline{\rho}}\nu'(\rho) = 0$ and $\lim_{\rho\to 0}\nu'(\rho) = 1$

Maximal density $\bar{\rho}$: $\overline{\rho} = \frac{(2\pi)^{3/2}\zeta(\frac{3}{2})}{\alpha} \approx \frac{41{,}144}{\alpha}$

Local Gibbs state and diffusion coefficient : *Modelization*

$$Q[f] = G_f - f\,, \quad G_f = \gamma\Big(\frac{1}{2}|v|^2 - \bar{\mu}(\rho_f)\Big)$$

$\bar{\mu}^{-1} : (-E_2, -E_1) \to (0, \infty)$ is such that

$$(\bar{\mu}^{-1})(\theta) = 4\pi\sqrt{2}\int_0^\infty \gamma(p-\theta)\sqrt{p}\,dp$$

We extend $\bar{\mu}^{-1}$ by the value 0 on $(-\infty, -E_2)$. Differentiation with respect to θ leads to the Abelian equation

$$\frac{(\bar{\mu}^{-1})'(\theta)}{2\pi\sqrt{2}} = \int_{-\infty}^{\theta} \frac{\gamma(-q)}{\sqrt{\theta - q}}\,dq$$

and gives an explicit expression of γ in terms of $\bar{\mu}^{-1}$

$$\gamma(E) = \frac{1}{\sqrt{2}\;2\pi^2}\frac{d^2}{dE^2}\int_{-\infty}^{-E} \frac{(\bar{\mu}^{-1})(\theta)}{\sqrt{-E-\theta}}\,d\theta$$

Initial data

We assume that there is a constant *Fermi level* μ^* such that

$$0 \leq f_I(x,v) \leq f^*(x,v) := \gamma\Big(\frac{1}{2}|v|^2 + V(x) - \mu^*\Big) \quad \forall (x,v) \in \mathbb{R}^6$$

Maximal macroscopic density :

$$\bar{\rho} := \lim_{\theta \to -E_1^+} \int_{\mathbb{R}^3} \gamma\Big(\frac{1}{2}|v|^2 - \theta\Big)\, dv$$

If $\bar{\rho} < \infty$ we assume

$$\bar{\mu}^{-1}(\mu^*) \leq \bar{\rho}$$

Potential

$\nabla_x V \in W^{1,\infty}\mathbb{R}^3)$ and V is bounded from below

$$\inf_{x\in\mathbb{R}^3} V(x) = V_{\text{min}} = 0$$

Confinement condition : with $f^*(x,v) := \gamma\Big(\frac{1}{2}|v|^2 + V(x) - \mu^*\Big)$

$$f^* \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \quad \text{and} \quad \iint_{\mathbb{R}^6}\Big(\frac{1}{2}|v|^2 + V(x)\Big)\, f^*(x,v)\, dv\, dx < \infty$$

Observe that this implies $\|f^*\|_{L^1} \geq \|f_I\|_{L^1} = M$

A *compatibility assumption :* Given a Gibbs state (a function γ), impose some minimal growth conditions on V.

Existence and uniqueness

Proposition 4. *For any $p \in (1, \infty)$, Eq. (1) has a unique weak solution in*

$$\mathcal{V} := \{f \in \mathcal{C}(0,\infty; (L^1 \cap L^p)(\mathbb{R}^6)) : 0 \le f \le f^*, \ \forall t > 0 \ a.e.\}$$

Proof: Cf. [Poupaud-Schmeiser, 1991] : define the map $f \mapsto \Gamma[f] = g$

$$\begin{aligned} &\epsilon^2 \partial_t g + \epsilon\, v \cdot \nabla_x g - \epsilon\, \nabla_x V \cdot \nabla_v g = G_f - g \\ &g(t=0) = f_I \end{aligned}$$

Γ maps $\mathcal{V}$ into itself and is a contraction for sufficiently small time intervals.

FREE ENERGY

$$\mathcal{F}[f] := \iint_{\mathbb{R}^6} \left[\left(\frac{1}{2}|v|^2 + V \right) f + \beta_\gamma[f] \right] dv\ dx$$

$$\beta_\gamma[f] := \int_0^f -\gamma^{-1}(s)\ ds$$

$-\gamma^{-1}$ is monotonically increasing $\Longrightarrow$ β_γ is a convex function

Microscopic energy associated to a distribution function f :

$$E_f(x,v,t) := \frac{1}{2}|v|^2 + V(x) - \mu_{\rho_f}(x,t) = \frac{1}{2}|v|^2 - \bar{\mu}\Big(\rho_f(x,t)\Big) = -\gamma^{-1}[G_f]$$

$$\epsilon^2 \frac{d}{dt}\mathcal{F}\Big(f(.,.,t)\Big) = \iint_{\mathbb{R}^6} \Big(G_f - f\Big)\Big(\gamma^{-1}[G_f] - \gamma^{-1}[f]\Big)\ dv\ dx := -D[f] \le 0$$

Consequences

$$\epsilon^2\Big[\mathcal{F}\big(f(,.,.,t)\big)-\mathcal{F}\big(f_I\big)\Big]=-\int_0^T D[f](t)\,dt$$

$$\mathcal{F}_{\mathsf{loc}}[f](x,.,t):=\int_{\mathbb{R}^3}\left[\left(\frac{1}{2}|v|^2+V(x)-\mu_f(x,t)\right)f(x,v,t)+\beta_\gamma\big(f(x,v,t)\big)\right]dv$$

is convex. Minimum if and only if $f=G_f$

$$0=\frac{1}{2}|v|^2+V(x)-\mu_f(x,t)+\beta'_\gamma[f]=\frac{1}{2}|v|^2+V(x)-\mu_f(x,t)-\gamma^{-1}[f]$$

$$\mathcal{F}_{\mathsf{loc}}[f](x,t)\geq\mathcal{F}_{\mathsf{loc}}[G_f](x,t)\quad\text{and}\quad\mathcal{F}\geq\mathcal{F}[G_f]\geq\mathcal{F}[g]$$

$$\mathcal{F}[g]=\iint_{\mathbb{R}^6}\gamma\left(\frac{1}{2}|v|^2+V-\mu\right)\left(\mu-\frac{|v|^2}{3}\right)dv\,dx=\int_{\mathbb{R}^3}\big(\mu\rho_g-\nu(\rho_g)\big)\,dx$$

with $\quad\rho_g=\bar{\mu}^{-1}(\mu-V)\quad\nu(\rho_g):=\dfrac{1}{3}\displaystyle\int_{\mathbb{R}^3}|v|^2\gamma\left(\frac{1}{2}|v|^2-\bar{\mu}(\rho_g)\right)dv$

Consider the partition of the support of f^* according to

$$\Omega_+ := \left\{(x,v,t) \in \text{supp}\, f^* \subset \mathbb{R}^6 \times (0,T) : E_f = \frac{1}{2}|v|^2 - \bar{\mu}(\rho_f(x,t)) < E_2\right\}$$

$$\Omega_0 := \left\{(x,v,t) \in \text{supp}\, f^* \subset \mathbb{R}^6 \times (0,T) : E_f = \frac{1}{2}|v|^2 - \bar{\mu}(\rho_f(x,t)) \geq E_2\right\}$$

The family of solutions f^ϵ, up to the extraction of a subsequence, converges to its local Gibs state $G_{f^\epsilon} = \gamma(E_{f^\epsilon})$ a.e. on Ω_+ and it converges to 0 a.e. on Ω_0.

$$\Omega_+^{x,t} := \{v \in \mathbb{R}^3 : (x,v,t) \in \Omega_+\} \quad \text{and} \quad \Omega_0^{x,t} := \{v \in \mathbb{R}^3 : (x,v,t) \in \Omega_0\}$$

Notice that $\Omega_+^{x,t} = \mathbb{R}^3$ if $E_2 = \infty$

Lemma 5. *For any nonnegative function $f \le f^*$ there exists a constant, which does not depend on x and t, such that*

$$\int_{\Omega_+^{x,t}} v_i^{2m} \frac{G_f - f}{\gamma^{-1}[f] - E_f}\, dv \le \mathcal{M}\,,$$

for any $m = 1, 2$, $i = 1, 2, 3$.

Scaled perturbations of the first and second moments

$$j^\epsilon := \int_{\mathbb{R}^3} v \, \frac{f^\epsilon - G_{f^\epsilon}}{\epsilon} \, dv \quad \text{and} \quad \kappa^\epsilon := \int_{\mathbb{R}^3} v \otimes v \, \frac{f^\epsilon - G_{f^\epsilon}}{\epsilon} \, dv$$

Lemma 6. *For any bounded, open set $U \subset \mathbb{R}^3 \times [0, T)$, there are two constants $\mathcal{M}_U^1$ and $\mathcal{M}_U^2$, which do not depend on ϵ, such that*

$$\|j^\epsilon\|_{L^2_{x,t}(U)} \le \mathcal{M}_U^1 \quad \textit{and} \quad \|\kappa^\epsilon\|_{L^2_{x,t}(U)} \le \mathcal{M}_U^2 \quad \textit{as} \quad \epsilon \to 0\,.$$

If $g(x, v, t) := \gamma(|v|^2/2 - \bar{\mu}(\rho(x, t)))$, then

$$\int_{\mathbb{R}^3} v \otimes v \, g \, dv = \nu(\rho) \, \mathrm{Id}^{3\times 3} \quad \text{where} \quad \nu(\rho) := \int_0^\rho \sigma \bar{\mu}'(\sigma) \, d\sigma$$

Proposition 7. *$\rho^\epsilon \to \rho^0$ in L^p_{loc} strongly for all $p \in (1, \infty)$.*

Div-Curl Lemma as in [Goudon-Poupaud, 2001]. Integrate (1) with respect to dv and $v\,dv$

$$\begin{cases} \partial_t \rho^\epsilon + \nabla_x \cdot j^\epsilon = 0 \\ \epsilon^2 \partial_t j^\epsilon + \nabla_x \cdot \int_{\mathbb{R}^3} v \otimes v f^\epsilon \, dv = -j^\epsilon - \rho^\epsilon \nabla_x V \end{cases}$$

Split the second moments of f

$$\int_{\mathbb{R}^3} v \otimes v f^\epsilon \, dv = \int_{\mathbb{R}^3} v \otimes v G^{f^\epsilon} \, dv + \int_{\mathbb{R}^3} v \otimes v (f^\epsilon - G^{f^\epsilon}) \, dv = \nu(\rho^\epsilon) I^{3\times 3} + \epsilon \kappa^\epsilon$$

$$\begin{cases} \partial_t \rho^\epsilon + \nabla_x \cdot j^\epsilon = 0 \\ \nabla_x \nu(\rho^\epsilon) = -j^\epsilon - \rho^\epsilon \nabla_x V - \epsilon \nabla_x \cdot \kappa^\epsilon - \epsilon^2 \partial_t j^\epsilon \end{cases}$$

Apply the Div-Curl Lemma to

$$U^\epsilon := (\rho^\epsilon, j^\epsilon), \quad V^\epsilon := (\nu(\rho^\epsilon), 0, 0, 0)$$

With $(\text{curl}\, w)_{ij} = w^i_{x_j} - w^j_{x_i}$ and

$$\begin{cases} \text{div}_{t,x} U^\epsilon = 0 \,, \\ (\text{curl}_{t,x} V^\epsilon)_{1,2\ldots 4} = -j^\epsilon - \rho^\epsilon \nabla_x V - \epsilon \nabla_x \cdot \kappa^\epsilon - \epsilon^2 \partial_t j^\epsilon \end{cases}$$

we obtain the convergence of $U^{\epsilon_i} \cdot V^{\epsilon_i} = \rho^{\epsilon_i} \nu(\rho^{\epsilon_i})$

As in [Marcati-Milani, 1990], we deduce using Young measures that the convergence of ρ^{ϵ_i} is strong. The strict convexity assumption is replaced by the strict monotonicity of the function ν in $\rho\, \nu(\rho)$.

Theorem 8. *For any* $\varepsilon > 0$*, the equation has a unique weak solution* $f^\varepsilon \in C(0,\infty; L^1 \cap L^p(\mathbb{R}^6))$ *for all* $p < \infty$*. As* $\varepsilon \to 0$*,* f^ε *weakly converges to a local Gibbs state* f^0 *given by*

$$f^0(x,v,t) = \gamma\left(\frac{1}{2}|v|^2 + V(x) - \bar{\mu}(\rho(x,t))\right) \quad \forall\,(x,v,t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+$$

where ρ *is a solution of the nonlinear diffusion equation*

$$\partial_t \rho = \nabla_x \cdot (\nabla_x \nu(\rho) + \rho\, \nabla_x V(x))$$

with initial data $\rho(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v)\,dv$

$$\nu(\rho) = \int_0^\rho s\,\bar{\mu}'(s)\,ds$$

Moreover, $\int_{\mathbb{R}^3} f^\varepsilon\,dv$ *strongly converges to* ρ *in* L^p_{loc} *as* $\varepsilon \to 0$