Mean-field attractive models: existence of stationary states and time-periodic solutions, orbital stability and symmetry issues

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NONLOCAL OPERATORS AND PARTIAL DIFFERENTIAL EQUATIONS (JUNE 26 – JULY 2, 2010)

Outline: Non-locality comes from a Poisson equation

- A first statement: there are non radially symmetric critical points
- Relative equilibria: examples and (partial) classification for systems of point particles. Time-periodic solutions
- Kinetic equations: extending the notion of relative equilibria to continuum mechanics
- Results for kinetic / diffusion equations
- The variational approach: heuristics
- The variational approach: a sketch of the proofs
- Flat systems: results and numerical computation
- Mean field gravitational models in quantum mechanics (with temperature)
- Concluding remarks: symmetry breaking and stability

Gravitational (non-relativistic) Vlasov-Poisson system in $\mathbb{R}^3$

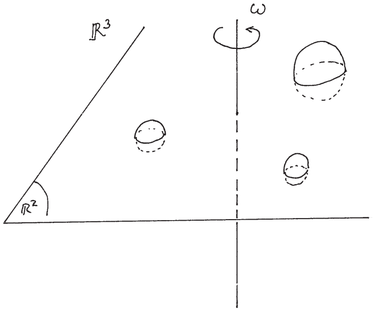
$$\begin{cases} \partial_t F + w \cdot \nabla_z F - \nabla_z \Phi \cdot \nabla_w F = 0 \\ \Delta \Phi = \int_{\mathbb{R}^3} F \, dw \end{cases} \tag{1}$$

Theorem

For any $N \geq 2$, any $p \in (1,5)$, any positive numbers $\lambda_1, \lambda_2, \ldots \lambda_N$ and any $\omega > 0$ small enough, there is a solution F^ω of (1) which is a relative equilibrium with angular velocity ω whose support has N disjoint connected components, each of them with mass m_i^ω such that

$$\lim_{\omega \to 0_+} m_i^\omega = \lambda_i^{(3-p)/2} \, m_* =: m_i$$

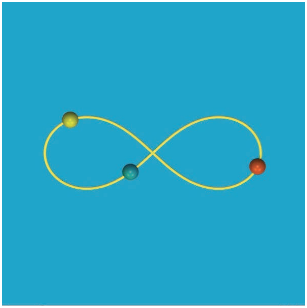
for some positive constant m_*. The center of mass $z_i^\omega(t)$ of each component is such that $\lim_{\omega \to 0_+} \omega^{2/3} \, z_i^\omega(t) =: z_i(t)$ is a relative equilibrium of the N-body Newton's equations with gravitational interaction

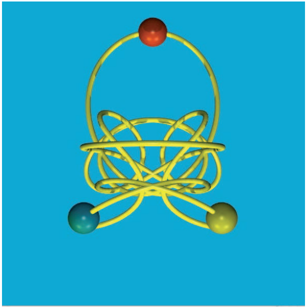

ω
$\mathbb{R}^3$
$\mathbb{R}^2$

Systems of discrete particles: the N-body problem in gravitation

... many solutions are known

- No stationary (time independent) solutions
- Periodic solutions in Hamiltonian dynamics: [Ekeland et al.]
- Choregraphies: [Chenciner et al.], [Terracini et al.]





Rysunek: Choregraphies, pictures taken from S. Terracini's web page

http://www.matapp.unimib.it/∼suster/files/index.html

Consider N point particles with masses m_i located at $z_i(t) \in \mathbb{R}^3$ subject to Newton's equations

$$m_i \frac{d^2 z_i}{dt^2} = \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{4\pi} \frac{z_j - z_i}{|z_j - z_i|^3} \tag{2}$$

Ansatz: the system is stationary in a reference frame rotating at constant angular velocity $\Omega = \omega\, \mathrm{e}_3$

Notation: $x' = (x^1, x^2, 0) = x - (x \cdot \mathrm{e}_3)\, \mathrm{e}_3$, a change of coordinates

$$x^3 = z^3 \ , \quad x^1 + i\, x^2 = e^{i \omega t}(z^1 + i\, z^2)$$

provides Newton's equations in a rotating frame

$$\frac{d^2 x_i}{dt^2} = \sum_{i \neq j = 1}^{N} \frac{m_j}{4\pi} \frac{x_j - x_i}{|x_j - x_i|^3} + \omega^2\, x_i' + 2\, \Omega \wedge \frac{dx_i}{dt}$$

We look for stationary solutions in the rotating frame: relative equilibria

The configuration is *central* and planar: critical points of the function

$$\mathcal{V}_\omega(x_1', x_2', \ldots x_N') := -\frac{1}{8\pi} \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{|x_j' - x_i'|} - \frac{\omega^2}{2} \sum_{i=1}^{N} m_i\, |x_i'|^2$$

- All masses m_i are equal to some $m > 0$ and x'_i are located at the summits of a regular polygon, where $r = |x'_i|$ is adjusted so that

$$\frac{d}{dr}\left[\frac{a_N}{4\pi}\,\frac{m}{r} + \frac{1}{2}\,\omega^2\,r^2\right] = 0 \quad \text{with} \quad a_N := \frac{1}{\sqrt{2}}\sum_{j=1}^{N-1}\frac{1}{\sqrt{1-\cos\left(2\pi j/N\right)}}$$

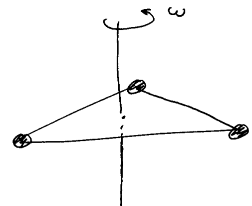
gives a *Lagrange solution* with $r = r(N,\omega) := \left(\frac{a_N\,m}{4\pi\,\omega^2}\right)^{1/3}$

[Perko-Walter]: all masses have to be equal if $N \geq 4$

Scale invariance:

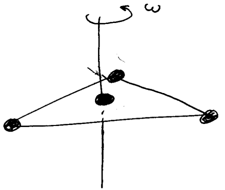
$$r(N, \varepsilon^{3/2}\,\omega) = \frac{1}{\varepsilon}\,r(N,\omega) \quad \forall\,\varepsilon > 0$$

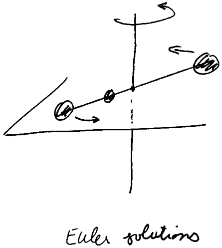
If $\nabla\mathcal{V}_\omega(x'_1, x'_2, \ldots x'_N) = 0$,
then $\nabla\mathcal{V}_{\varepsilon^{3/2}\,\omega}(\varepsilon^{-1}\,x'_1, \varepsilon^{-1}\,x'_2, \ldots$
the study of the critical points of $\mathcal{V}_\omega$ can be reduced to the case $\omega = 1$



Relative equilibria: other examples

- $N-1$ points particles of same mass are located at the summits of a regular centered polygon and one more point particle stands at the center (with not necessarily the same mass as the other ones). A solution is then found again by adjusting the size of the polygon
- The *Euler–Moulton solutions* are made of aligned points





Relative equilibria are critical points of the function $\mathcal{V}_\omega : (\mathbb{R}^2)^N \to \mathbb{R}$

$$\mathcal{V}_\omega(x_1', x_2', \dots x_N') := -\frac{1}{8\pi} \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{|x_j' - x_i'|} - \tfrac{1}{2}\,\omega^2 \sum_{i=1}^{N} m_i\, |x_i'|^2$$

Generic case: all masses are different

- $N = 2$:

$$|x_1 - x_2| = \left(\frac{M}{4\pi\,\omega^2}\right)^{1/3} \quad \text{and} \quad m_1\, x_1 + m_2\, x_2 = 0\,, \quad \text{with } M = m_1 + m_2$$

- $N = 3$:
  - *Lagrange solutions*: masses are located at the vertices of an equilateral triangle, and the distance between each point is $(M/(4\pi\,\omega^2))^{1/3}$ with $M = m_1 + m_2 + m_3$: two classes of solutions corresponding to the two orientations of the triangle when labeled by the masses
  - *Euler solutions* are made of aligned points and provide three classes of critical points, one for each ordering of the masses on the line

- $N \geq 4$: solutions made of aligned points are *Moulton's solutions*
- $N \geq 4$: Lagrange solutions (all particles are located at the vertices of a regular N-polygon) exists if and only if all masses are equal

- Standard variational setting [Smale]: for $m = (m_1, \dots m_N) \in \mathbb{R}_+^N$, consider the manifold $(q_1, \dots q_N) \in \mathbb{R}^{2N}$ such that

$$\sum_{i=1}^N m_i\, q_i = 0\,,\ \tfrac{1}{2}\sum_{i=1}^N m_i\, |q_i|^2 = 1\,,\ q_i \neq q_j \text{ if } i \neq j$$

quotiented by the equivalence classes associated to the invariances: rotations and scalings

$\dim(\mathcal{S}_m) = 2N - 3$, relative equilibria are critical points on $\mathcal{S}_m$ of the potential

$$U_m(q_1, \dots q_N) = -\frac{1}{8\pi} \sum_{i \neq j = 1}^N \frac{m_i\, m_j}{|q_j - q_i|}$$

For $N \geq 4$, various classification results have been achieved by [Palmore]

- For $N \geq 3$, the index of a relative equilibrium is always greater or equal than $N-2$. This bound is achieved by Moulton's solutions
- For $N \geq 3$, there are at least $\mu_i(N) := \binom{N}{i}(N-1-i)\,(N-2)\,!$ distinct relative equilibria in $\mathcal{S}_m$ of index $2N-4-i$ if U_m is a Morse function. As a consequence, there are at least

$$\sum_{i=0}^{N-2} \mu_i(N) = [2^{N-1}(N-2)+1]\,(N-2)\,!$$

  distinct relative equilibria in $\mathcal{S}_m$ if U_m is a Morse function
- For every $N \geq 3$ and for almost all masses $m \in \mathbb{R}^N_+$, U_m is a Morse function
- There are only finitely many classes of relative equilibria for every $N \geq 3$ and for almost all masses $m \in \mathbb{R}^N_+$
- If $N \geq 4$, the set of masses for which there exist degenerate classes of relative equilibria has positive k-dimensional Hausdorff measure if $0 \leq k \leq N-1$

The kinetic problem

Gravitational Vlasov-Poisson system (with centrifugal and Coriolis forces):

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f - \omega^2 x' \cdot \nabla_v f + 2\,\Omega \wedge v \cdot \nabla_v f = 0$$

$$\Delta \phi = \rho := \int_{\mathbb{R}^3} f \; dv$$

Boundary conditions: $\phi = -\frac{1}{4\pi\,|\cdot|} * \rho$

Change of coordinates: $f(t,x,v) = F(t,z,w)$, $\phi(t,x) = \Phi(t,z)$

$$x = \exp(\omega\, t\, A)\, z\,, \quad v = \Omega \wedge x + \exp(\omega\, t\, A)\, w \quad \text{with} \quad A := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For some arbitrary convex function β, critical points of the *free energy*

$$\mathcal{F}[f] := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \beta(f)\; dx\, dv + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(|v|^2 - \omega^2\, |x'|^2\right) f \; dx\, dv - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 \; dx$$

give stationary solutions under the constraint $\iint_{\mathbb{R}^3 \times \mathbb{R}^3} f \; dx\, dv = M$

For $\omega \neq 0$: no minimizers

[Binney-Tremaine] A typical example of such a function is

$$\beta(f) = \kappa\, f^q$$

A critical point of $\mathcal{F}$ such that $\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\;dx\,dv = M$ is given by

$$f(x,v) = \gamma\left(\lambda + \frac{1}{2}\,|v|^2 + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right)$$

where $\gamma(s) = (\beta')^{-1}(-s)$: $\gamma(s) = (-s)_+^{1/(q-1)}$
The problem is reduced to solve a non-linear Poisson equation

$$\Delta\phi = g\big(\lambda + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\big)\,\chi_{\mathrm{supp}(\rho)}$$

$$g(\mu) := \int_{\mathbb{R}^3} \gamma\big(\mu + \frac{1}{2}\,|v|^2\big)\;dv$$

Variational approach:

$$\mathcal{J}[\phi] := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\;dx + \int_{\mathbb{R}^3} G\left(\lambda + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right)\,dx - \int_{\mathbb{R}^3} \lambda\,\rho\;dx$$

where $\lambda = \lambda[x,\phi]$ is now a functional which is constant on each connected component K_i of the support of $\rho(x)$

The total mass is $M = \sum_{i=1}^{N} m_i$

$$\int_{K_i} g\left(\lambda_i + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right)\,dx = m_i$$

$$\mathcal{J}[\phi] = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx + \sum_{i=1}^{N}\left[\int_{K_i} G\left(\lambda_i + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2\right)\,dx - m_i\,\lambda_i\right]$$

Heuristics. The various components K_i are far away from each other so that the dynamics of their center of mass is described by the N-body point particles system, at first order. On each component K_i, the solution is a perturbation of an isolated minimizer of $\mathcal{F}$ (without angular rotation) under the constraint that the mass is equal to m_i. Alternatively, we consider a critical point of $\mathcal{J}$ obtained as the perturbation of a superposition of single components critical points of $\mathcal{J}$ of mass m_i, which are supported in a neighborhood of K_i, for all $i = 1, 2, \ldots N$, provided the centers of mass x_i of each of the components are close enough of a critical point of $\mathcal{V}_\omega$, with $\omega > 0$, small

$\omega = 0$: [Guo et al.], [Lemou-Méhats-Raphaël], [Rein et al.], [Sánchez et al.], [Soler et al.], [Schaeffer], [Wolansky], [JD-Fernández]

The spatial density $\rho^\omega := \int_{\mathbb{R}^3} f^\omega \, dv$ has exactly N disjoint connected components K_i^ω and

$$m_i^\omega = \int_{K_i^\omega} \rho^\omega \, dx \,, \quad z_i^\omega(t) = \exp\left(-\omega \, t \, A\right) \, x_i^\omega \quad \text{where} \quad x_i^\omega := \frac{1}{m_i^\omega} \int_{K_i^\omega} x \, \rho^\omega \, dx$$

$$\rho_i(x) := \frac{1}{\lambda_i^p} \, \rho^\omega \left(\lambda_i^{(1-p)/2}(x + x_i^\omega)\right) \, \chi_{K_i^\omega} \left(\lambda_i^{(1-p)/2}(x + x_i^\omega)\right)$$

converges to a density function $\rho_* = (w-1)_+^p$ given by

$$-\Delta w = (w-1)_+^p \quad \text{in } \mathbb{R}^3 \,, \qquad \lim_{|x| \to \infty} w(x) = 0$$

Theorem

For any $N \geq 2$, any $p \in (1,5)$, any positive numbers $\lambda_1, \lambda_2, \ldots \lambda_N$ and any $\omega > 0$ small enough, there is a relative equilibrium solution F^ω s.t.

$$\lim_{\omega \to 0_+} m_i^\omega = \lambda_i^{(3-p)/2} \, m_* =: m_i$$

for $m_* = \int_{\mathbb{R}^3} \rho_* \, dx$. The center of mass $z_i^\omega(t)$ of each component is such that $\lim_{\omega \to 0_+} \omega^{2/3} \, z_i^\omega(t) =: z_i(t)$ is a relative equilibrium of the N-body problem (Newton's equations)

Let $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$, fix $\lambda_1, \dots \lambda_N$ and $\omega > 0$, small: the problem is

$$-\Delta u = \sum_{i=1}^{N} \rho_i \quad \text{in } \mathbb{R}^3 , \quad \rho_i := \left(u - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^p \chi_i \tag{3}$$

where χ_i denotes the characteristic function of K_i
Boundary condition $\lim_{|x|\to\infty} u(x) = 0$

Mass and center of mass associated to each component by

$$m_i := \int_{\mathbb{R}^3} \rho_i \, dx \quad \text{and} \quad x_i := \frac{1}{m_i} \int_{\mathbb{R}^3} x\, \rho_i \, dx$$

We shall say that two solutions u_1 and u_2 are equivalent if there is a rotation $R \in \mathrm{SO}(2) \times \{\mathrm{Id}\}$, *i.e.* a rotation in the plane orthogonal to the direction of rotation, such that $u_2(x) = u_1(R\,x)$ for any $x \in \mathbb{R}^3$. We shall say that u_1 and u_2 are distinct if they are not equivalent

Theorem

- *For ω small enough, and for almost every positive $(\lambda_1, \dots \lambda_N) \in (0,\infty)^N$, (3) has at least $[2^{N-1}(N-2)+1]\,(N-2)\,!$ distinct solutions which continuoulsy depend on ω*
- *If u^ω is such a solution, there are points $\xi_1^\omega, \dots \xi_N^\omega \in \mathbb{R}^3$ such that as $\omega \to 0$, $|\xi_j^\omega - \xi_i^\omega| \to \infty$ for any $j \neq i$ and $u^\omega(\cdot + \xi_i^\omega)$ locally converges to the unique radial nonnegative solution of*

$$-\Delta w = (w - \lambda_i)_+^p \quad in\ \mathbb{R}^3$$

- *For $\omega > 0$ small enough, the support of ρ^ω has N connected components*

$$\lim_{\omega \to 0} m_i^\omega = \lambda_i^{(3-p)/2}\, m_* =: m_i \quad and \quad \lim_{\omega \to 0} \omega^{2/3}\, \xi_i^\omega := \xi_i$$

and $(\xi_1, \dots \xi_N)$ is a relative equilibrium with masses $(m_i)_{1 \le i \le N}$

The scaling invariance is recovered only in the limit $\omega \to 0_+$

$$-\Delta w = (w-1)_+^p \quad \text{in } \mathbb{R}^3 \tag{4}$$

Lemma

Under the condition $\lim_{|x|\to\infty} w(x) = 0$, Equation (4) has a unique solution, up to translations, which is positive and radially symmetric. It is a non-degenerate critical point of $\frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2\,dx + \frac{1}{p+1}\int_{\mathbb{R}^3} (w-1)_+^{p+1}\,dx$

$$w_i(x) := w^{\lambda_i}(x-\xi_i)\,, \quad W_\xi := \sum_{i=1}^{N} w_i$$

Compatibility condition: for a large, fixed $\mu > 0$, and all small $\omega > 0$,

$$|\xi_i| < \mu\,\omega^{-\frac{2}{3}}\,, \quad |\xi_i - \xi_j| > \mu^{-1}\omega^{-\frac{2}{3}} \tag{5}$$

Ansatz: we look for a solution of (3) of the form

$$u = W_\xi + \phi$$

with $\operatorname{supp}(w^{\lambda_i} - \lambda_i)_+ \subset B(0,R)$ for all $i = 1, \dots N$ for $R > 0$ large

$$\Delta\phi + \sum_{i=1}^{N} p\left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p-1}\chi_i\,\phi = -\mathrm{E} - \mathrm{N}[\phi]$$

We want to solve

$$\Delta\phi + \sum_{i=1}^{N} p\left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p-1}\chi_i\,\phi = -\mathsf{E} - \mathsf{N}[\phi]$$

where

$$\mathsf{E} := \Delta W_\xi + \sum_{i=1}^{N}\left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p}\,\chi_i$$

$$\mathsf{N}[\phi] := \sum_{i=1}^{N}\Big[\left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2 + \phi\right)_+^{p} - \left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p} - p\left(W_\xi - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p-1}\phi\Big]\chi_i$$

The variational scheme

- A linear theory
- The projected nonlinear problem (Lagrange multipliers)
- The variational reduction
- A variational approach in finite dimension

[Floer-Weinstein 1986] + many others...

$$\|\phi\|_* = \sup_{x\in\mathbb{R}^3} \left(\sum_{i=1}^N |x-\xi_i| + 1 \right) |\phi(x)| \,, \quad \|h\|_{**} = \sup_{x\in\mathbb{R}^3} \left(\sum_{i=1}^N |x-\xi_i|^4 + 1 \right) |h(x)|$$

Consider the *projected problem*

$$\mathrm{L}[\phi] = h + \sum_{i=1}^N \sum_{j=1}^3 c_{ij}\, Z_{ij}\, \chi_i$$

where $Z_{ij} := \partial_{x_j} w_i$, subject to orthogonality conditions

$$\int_{\mathbb{R}^3} \phi\, Z_{ij}\, \chi_i\, dx = 0 \quad \forall\, i\,,\ j = 1, 2 \ldots N$$

Lemma

Assume that (5) holds. Given h with $\|h\|_{**} < +\infty$, there is a unique solution $\phi =: \mathrm{T}[h]$ and there exists a positive constant C, which is independent of ξ such that, for $\omega > 0$ small enough,

$$\|\phi\|_* \leq C\, \|h\|_{**}$$

Find ϕ with $\|\phi\|_* < +\infty$, solution of

$$\mathrm{L}[\phi] = -\mathrm{E} - \mathrm{N}[\phi] + \sum_{i=1}^{N} \sum_{j=1}^{3} c_{ij}\, Z_{ij} \chi_i$$

such that $\phi(x) \to 0 \quad$ as $|x| \to +\infty$ and

$$\int_{\mathbb{R}^3} \phi\, Z_{ij}\, \chi_i\, dx = 0 \quad \text{for all } i\,,\ j\,.$$

To do this analysis we have to measure the size of the error E. We recall that

$$\mathrm{E} = \sum_{i=1}^{N} \Big[(w_i + \sum_{j \neq i} w_j - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2)_+^p - (w_i - \lambda_i)_+^p \Big]\, \chi_i$$

$$= \sum_{i=1}^{N} p\,(w_i - \lambda_i + t\,(\sum_{j \neq i} w_j + \tfrac{1}{2}\,\omega^2\,|x'|^2))_+^{p-1} (\sum_{j \neq i} w_j + \tfrac{1}{2}\,\omega^2\,|x'|^2)\, \chi_i$$

for some $t \in (0, 1)$

$$|\mathsf{E}| \le C \sum_{i=1}^{N} \Big[\sum_{j\neq i} \frac{1}{|\xi_i - \xi_j|} + \frac{1}{2}\omega^2 |\xi_i|^2 \Big] \chi_i \ \le C\omega^{\frac{2}{3}} \sum_{i=1}^{N} \chi_i$$

Thus $\|\mathsf{E}\|_{**} \le C\omega^{\frac{2}{3}}$. Moreover, for $\|\phi\|_* \le 1$,

$$|\mathsf{N}[\phi]| \le C \sum_{i=1}^{N} |\phi|^{\gamma} \chi_i \ , \quad \gamma = \min\{p, 2\}$$

$\|\mathsf{N}[\phi]\|_{**} \le C\|\phi\|_*^{\gamma}$ and $\|N(\phi_1) - N(\phi_2)\|_{**} \le o(1)\|\phi_1 - \phi_2\|_*^{\gamma}$
We look for a fixed point $\phi = \mathcal{A}[\phi] := -\mathsf{T}\big[\mathsf{E} + \mathsf{N}[\phi]\big]$ on the region

$$\mathcal{B} = \Big\{ \phi \ : \ \|\phi\|_* \le K\omega^{\frac{2}{3}} \Big\}$$

Lemma

$\exists\, !\phi_\xi = \phi(\xi_1, \dots \xi_k)$ *which depends continuously on its parameters for the* $\|\ \|_*$*-norm and* $\|\phi_\xi\|_* \le C\omega^{\frac{2}{3}}$*,*

$$\phi_{e^{\theta A}\xi} = \phi_\xi(e^{-\theta A}\cdot) \quad \textit{and} \quad c_{i\cdot}(e^{\theta A}\xi) = e^{-\theta A} c_{i\cdot}$$

Lemma

We have that $c_{ij} = 0$ for all i , j if and only if the k-tuple $(\xi_1, \dots \xi_N)$ is a critical point of the functional

$$(\xi_1, \dots \xi_N) \mapsto \Lambda(\xi_1, \dots \xi_N) := J(W_\xi + \phi_\xi)$$

A Taylor expansion

$$J(W_\xi) = J(W_\xi + \phi_\xi) - DJ(W_\xi + \phi_\xi)[\phi_\xi] + \frac{1}{2}\int_0^1 D^2 J(W_\xi + (1-t)\phi_\xi)[\phi_\xi]^2 \, dt$$

$$D^2 J(W_\xi + (1-t)\phi_\xi)[\phi_\xi]^2 = O(\omega^{\frac{4}{3}})$$

$$\Lambda(\xi) = \sum_{i=1}^N \lambda_i^{5-p} e_* \, - \, \Big[\frac{1}{2} \sum_{i \neq j} \frac{m_i \, m_j}{|\xi_i - \xi_j|} + \tfrac{1}{2} \, \omega^2 \sum_{i=1}^N m_i \, |\xi_i|^2 \Big] \, + \, O(\omega^{\frac{4}{3}})$$

In the region

$$\mathfrak{B} = \left\{ (\xi_1, \dots \xi_k) \ : \ |\xi_i - \xi_j| > \rho\, \omega^{-\frac{2}{3}}, \quad |\xi_i| < \rho^{-1}\, \omega^{-\frac{2}{3}} \text{ for all } i\,,\ j \right\}$$

where $\rho > 0$ is chosen small enough and fixed, we have that

$$\sup_{\mathfrak{B}} \Lambda > \sup_{\partial \mathfrak{B}} \Lambda$$

so that this functional has a local maximum somewhere in $\mathfrak{B}$. Hence a critical point of this functional does exist in $\mathfrak{B}$ □

Lemma

For any $\lambda_i > 0$, ξ_i, we have found a critical point of Λ, i.e. a solution of

$$\mathsf{L}[\phi] = -\mathsf{E} - \mathsf{N}[\phi]$$

$$U_m(q_1, \dots q_N) = -\frac{1}{8\pi} \sum_{i \neq j = 1}^{N} \frac{m_i \, m_j}{|q_j - q_i|}$$

is a Morse function on $\mathcal{S}_m$. Take a local system of coordinates $(\eta_1, ..., \eta_{2N-4})$ on a neighborhood of a critical point $\bar{q}$ and for $\alpha > 0$, let

$$\xi(\alpha, p, \eta) = (\alpha(q_1(\eta) + p), \dots \alpha(q_N(\eta) + p))$$

$$\Phi(\alpha, p, \eta) = \omega^{-\frac{2}{3}} \Lambda(\xi(\omega^{-\frac{2}{3}} \alpha, p, q(\eta)))$$

$$\nabla \Phi(\alpha, p, \eta) = \nabla \Big(-\frac{1}{2} \sum_{i \neq j = 1}^{N} \frac{m_i \, m_j}{\alpha |q_j(\eta) - q_i(\eta)|} - \frac{1}{2} \alpha^2 \sum_{i=1}^{N} m_i \, |q_i(\eta)|^2 + \frac{1}{2} \alpha^2 |p|^2 \sum_{i=1}^{N} m_i \Big) + O(\omega^{\frac{2}{3}})$$

$(\bar{\lambda}^{\frac{1}{3}}, 0, \bar{\eta})$ is nondegenerate, so the local degree $\deg(\nabla \tilde{\Phi}, \mathcal{U}, 0)$ is well defined and nonzero: there exists a critical point (α^*, p^*, η^*) as $\omega \to 0$

A flat (non-local) model: theory and numerical results

[JD-Fernández] Written in cartesian coordinates, the equation is

$$\partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2\,\omega\, v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f = 0$$
$$\phi = -\frac{1}{4\pi\,|x|} * \int_{\mathbb{R}^2} f\, dv$$

where $a \wedge b := a^\perp \cdot b = a_1\, b_2 - a_2\, b_1$ and $x,\ v \in \mathbb{R}^2$

Definition

A localized minimizer is a critical point ρ of $\mathcal{G}_\omega$ which is compactly supported in a ball $B(0, R - \varepsilon)$ for some $R > 0$ and $\varepsilon \in (0, R)$, and which is a minimizer of $\mathcal{G}_\omega$ restricted to the set

$$\left\{ \rho \in L^1_+(\mathbb{R}^2)\ :\ \mathrm{supp}(\rho) \subset B(0, R) \text{ and } \int_{\mathbb{R}^2} \rho\, dx = M \right\}$$

Theorem

For any $M > 0$, there exists $\omega_*(M) = \omega_* > 0$ and $\omega^*(M) = \omega^* > 0$ with $\omega_* \le \omega^*$ such that

(i) *If $\omega \in [-\omega_*, \omega_*]$, the reduced free energy functional*

$$\mathcal{G}_\omega[\rho] := \frac{\kappa}{m-1}\int_{\mathbb{R}^2} \rho^m \, dx - \frac{\omega^2}{2}\int_{\mathbb{R}^2} |x|^2 \rho \, dx - \frac{1}{8\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\rho(x)\,\rho(y)}{|x-y|}\, dx\, dy$$

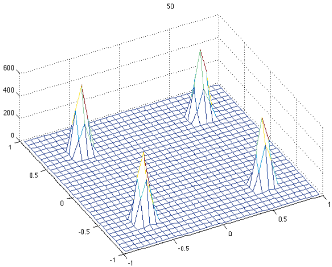
admits a localized minimizer

(ii) *If $|\omega| > \omega^*$, $\mathcal{G}_\omega[\rho]$ admits no localized minimizer*

More detailed results in the radial case. How does symmetry breaking occur ?

Goal: investigate the energy landscape [JD-Fernández-Salomon]

- Local minimizers (under appropriate constraints) have a very small basin of attraction
- Compact support has to be enforced at each step
- Iteration method inspired by mean-field models in quantum mechanics work, with similar difficulties: the Cancès-LeBris method of relaxation has to be introduced to achieve convergence
- how branches of solutions vary in terms of ω is open



Mean field gravitational models in quantum mechanics (with temperature) – Boson stars

A physical state of the system is represented by a density matrix operator: ρ is a positive self-adjoint trace class operator on $L^2(\mathbb{R}^3;\mathbb{C})$

$$\rho = \sum_{j\in\mathbb{N}} \lambda_j |\psi_j\rangle\langle\psi_j|$$

$\lambda_j \in \ell^1$ are the *occupation probabilities*, $(\psi_j)_{j\in\mathbb{N}}$ is a complete orthonormal basis of $L^2(\mathbb{R}^3)$. Particle density is given by

$$n_\rho(x) = \sum_{j\in\mathbb{N}} \lambda_j \, |\psi_j(x)|^2 \in L^1_+(\mathbb{R}^3) \;, \quad \int_{\mathbb{R}^3} n_\rho(x)\,\mathrm{d}x = M = \sum_{j\in\mathbb{N}} \lambda_j$$

Hartree energy of the system is given by

$$\mathcal{E}_H[\rho] := \mathcal{E}_{\mathrm{kin}}[\rho] - \mathcal{E}_{\mathrm{pot}}[\rho] = \mathrm{tr}(-\Delta\rho) - \frac{1}{2}\,\mathrm{tr}(V_\rho \; \rho) \;, \quad V_\rho = n_\rho * \frac{1}{|\cdot|}$$

$$\mathcal{E}_H[\rho] = \sum_{j\in\mathbb{N}} \lambda_j \, |\nabla\psi_j(x)|^2 - \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{n_\rho(x)\, n_\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$

The *free energy functional* given by

$$\mathcal{F}_T[\rho] := \mathcal{E}_H[\rho] - T\,\mathcal{S}[\rho]$$

$T \geq 0$ denotes the temperature, $\beta(s) \in \mathbb{R}_+$ is a convex *entropy generating function*, and $\mathcal{S}[\rho]$ is the *entropy functional*

$$\mathcal{S}[\rho] := -\operatorname{tr}\beta(\rho) = -\sum_{j\in\mathbb{N}} \beta(\lambda_j)$$

Define $i_{M,T} := \inf_{\operatorname{tr}\rho=M} \mathcal{F}_T[\rho]$. A useful *a priori* estimate

$$\mathcal{E}_{\mathrm{pot}}[\rho] \leq C\, \|n_\rho\|_{L^1}^{3/2} \operatorname{tr}(-\Delta\rho)^{1/2}$$

$$\mathcal{F}_T[\rho] \geq \operatorname{tr}(-\Delta\rho) - C\,M^{3/2}\, \operatorname{tr}(-\Delta\rho)^{1/2} \geq -\frac{1}{4}\, C^2\, M^3$$

Binding inequality: if there are minimizers for $i_{M_1,T}$ and $i_{M_2,T}$, then

$$i_{M_1+M_2,T} < i_{M_1,T} + i_{M_2,T}$$

$$\mathfrak{H}_M = \left\{\rho \in \mathfrak{S}_1 \ : \ \rho^* = \rho \geq 0, \ \sqrt{-\Delta}\,\rho\,\sqrt{-\Delta} \in \mathfrak{S}_1, \ \mathrm{tr}\,\rho = M\right\}$$

$$i_{M,T} = \inf_{\rho \in \mathfrak{H}_M} \mathcal{F}_T[\rho]\,, \quad T^*(M) := \sup\{T > 0 \ : \ i_{M,T} < 0\}$$

β is strictly convex, nonnegative with $\beta(0) = \beta'(0) = 0$ and of class C^1 on its domain: $\beta(s) = s^p$, $1 < p < 3$, for instance

Theorem

[Aki, JD, Sparber] *Let $M > 0$: $T^* = T^*(M) > 0$, possibly infinite, is the maximal temperature*

(i) *For all $T < T^*$, there exists $\rho \in \mathfrak{H}_M$ such that $\mathcal{F}_T[\rho] = i_{M,T}$*

(ii) *The set of all minimizers is orbitally stable under the dynamics*

(iii) *Minimizer $\rho \in \mathfrak{M}_M$ are pure states (density matrix operators of rank one) if and only if $T \in [0, T_c]$, for some critical temperature $T_c > 0$*

Comments

- Maximal temperature: if $p \in (1, 7/5)$ and $\beta(s) = s^p$, then the maximal temperature, $T^*(M)$, is finite
- Interpolation and Lieb-Thirring inequalities: proving the boundedness from below of the free energy in presence of a general potential amounts to establish interpolation inequalities of Gagliardo-Nirenberg type, which are themselves equivalent to Lieb-Thirring inequalities: [JD, Felmer, Loss, Paturel]
- Application in chemistry: [JD, Felmer, Mayorga], [JD, Felmer, Lewin]
- Relative equilibria ? Likely to occur
- Boson stars: a crude model. Physics has to be better understood (temperature)
- Relativistic models (take $\sqrt{1-\Delta}$ instead of $-\Delta$ in the kinetic term) and critical mass issues (Chandrasekhar): [Lenzmann, Lewin]

Not yet the end...

Conclusion: symmetry breaking and stability

- In nonlinear PDEs, symmetry breaking usually occurs because of a competition between the nonlinearity and an external potential
- A classical example is the (PDE) Hénon problem in elliptic PDEs
- The case covered by the theorem of [Gidas-Ni-Nirenberg] is the trivial one: the nonlinearity and an external potential cooperate
- Symmetry breaking is usually achieved by eigenvalue considerations
- Here we have an example based on multiscale analysis (new)
- Main issue (especially in gravitation) is dynamical / orbital stability
  Constrained (localized) minimization and mass transport methods [McCann] but new ideas are required. Non-locality !
- Building examples of periodic solutions (choregraphies ?) would be an intermediate step

Thank you for your attention !