

*Asymptotic behaviour for the  
Vlasov-Poisson System in the stellar  
dynamics case*

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A joint work in collaboration with  
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Będlewo, September 11, 2003

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - (\nabla_x \phi + \nabla_x \phi_0) \cdot \nabla_v f = 0 \\ \Delta_x \phi = 4\pi\gamma\rho, \quad \lim_{|x| \rightarrow \infty} \phi(t, x) = 0 \end{cases} \quad (VP)$$

in presence of an external potential  $\phi_0$ . If we look for stationary solutions taking the form

$$f(x, v) = g\left(\frac{1}{2}|v|^2 + \phi(x) + \phi_0(x) - \kappa\right)$$

Vlasov's equation is satisfied and the problem is reduced to solve the nonlinear Poisson equation

$$\Delta\phi = 4\pi\gamma G(\phi + \phi_0 - \kappa)$$

with

$$G(u) := \int_{\mathbb{R}^3} g\left(\frac{1}{2}|v|^2 + u\right) dv = \frac{4\pi}{3} \int_0^{+\infty} g(s + u) \sqrt{2s} ds$$

An example: let  $g(u) := (2\pi)^{-3/2}e^{-u}$  and  $M > 0$  fixed. The distribution function  $f$  is a gaussian and

$$\Delta\phi = 4\pi\gamma e^{\kappa} e^{-\phi-\phi_0} \quad \text{for} \quad 4\pi e^{\kappa} \int_{\mathbb{R}^3} e^{-\phi-\phi_0} dx = M$$

For  $\gamma = \pm 1$ ,  $\varphi = \pm\phi$ ,  $F(u) = e^{\mp u}$ ,  $e^{\phi_0} = \chi_{\Omega}$ , we have to solve

$$\Delta\varphi = M \frac{F(\varphi)}{\int_{\Omega} F(\varphi) dx}$$

Plasma case,  $\gamma = -1$ : if  $\mathcal{G}' = G$ , then  $\phi$  is a critical point of

$$\phi \mapsto \frac{1}{2} \int_{\mathbb{R}^3} |\nabla\phi|^2 dx + \int_{\mathbb{R}^3} \mathcal{G}(\phi + \phi_0 - \kappa[\phi + \phi_0]) dx - M \kappa[\phi + \phi_0]$$

which is convex if  $g$  is nonincreasing

The parameter  $\kappa$  is a functional of  $\phi$ , implicitly defined by

$$4\pi\gamma \int_{\mathbb{R}^3} G(\phi + \phi_0 - \kappa[\phi + \phi_0]) dx = M$$

where  $M > 0$  is the mass of the solution:  $M = \int_{\mathbb{R}^6} f(x, v) dx dv$   
 The solution is unique if one assumes that  $g \searrow$  ( $\mathcal{G}$  convex)

*Dual* approach: look for a critical point of the functional

$$f \mapsto \int_{\mathbb{R}^6} \left( \frac{1}{2} |v|^2 + \phi(x) + \phi_0(x) - \kappa \right) f(x, v) dx dv + \int_{\mathbb{R}^6} H(f(x, v)) dx dv$$

where  $H'(\cdot) = g^{(-1)}(-\cdot)$  and  $\kappa$  now appears as the Lagrange multiplier associated to the mass constraint.

$\gamma = -1$ : [Batt, Morrison, Rein], [Braasch, Rein, Vukadinović],  
 [Cáceres, Carrillo, J.D.]

$\gamma = +1$ : [Wolansky], [Rein], [Guo], [J.D., Sánchez, Soler], [Sánchez, Soler]

The three-dimensional *Vlasov-Poisson system* can be written in terms of a distribution function  $f : (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^+ \cup \{0\}$  and the corresponding mass density  $\rho(x, t) := \int_{\mathbb{R}^3} f(t, x, v) dv$  as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0 \\ f(t = 0, x, v) = f_0(x, v) \\ \Delta_x \phi = 4\pi\gamma\rho, \quad \lim_{|x| \rightarrow \infty} \phi(t, x) = 0 \end{cases} \quad (VP)$$

where  $\gamma = +1$  corresponds to the gravitational case and  $\gamma = -1$  to the plasma physical case.

Polytropic gas spheres ( $\gamma = +1$ ):

$$f(x, v) = (E_0 - |v|^2/2 - \phi(x))_+^\mu |x \times v|^{2k}$$

## Existence results

**Classical solutions:** K. Pfaffelmoser, Schaeffer, R. Glassey] Take  $f_0$  with compact support and control the growth of the size of the support:

$$Q(t) = 1 + \sup \left\{ |v| : \exists (t, x) \in (0, t) \times \mathbb{R}^3 \text{ s.t. } f(t, x, v) \neq 0 \right\}$$

Then the Cauchy problem for (VP) has a unique  $C^1$  solution and  $Q(t) \leq C_p(1+t)^p$  with  $p > \frac{33}{17}$

$\implies$  **Strong solutions:**  $f_0 \in L^1 \cap L^\infty$ , finite energy

**Weak solutions:** [Horst, Hunze]  $f_0 \in L^1 \cap L^p$ ,  $p > (12 + 3\sqrt{5})/11$ , finite energy

**Renormalized solutions:** [DiPerna, Lions]  $f_0 \in L^1$ ,  $f_0 \log f_0 \in L^1$ , finite energy

Other results using moments [Perthame, Lions], [Castella]

# 1. OPTIMAL BOUNDS FOR THE KINETIC AND POTENTIAL ENERGIES

Total energy associated to  $f$ :

$$E(f) := E_{KIN}(f) - \gamma E_{POT}(f)$$

Kinetic energy:

$$E_{KIN}(f) := \frac{1}{2} \int_{\mathbb{R}^6} |v|^2 f(t, x, v) dx dv$$

Potential energy :

$$E_{POT}(f) := \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx$$

The potential  $\phi$  associated to  $f$  is given by the Poisson equation:

$$\phi = -\frac{\gamma}{|\cdot|} * \int_{\mathbb{R}^3} f(\cdot, v) dv$$

For a smooth solution the total energy remains constant. Total mass is also preserved

$$\|f(t, \cdot, \cdot)\|_{L^1(\mathbb{R}^6)} = \int_{\mathbb{R}^6} f(t, x, v) dx dv$$

The transport also preserves uniform bounds :

$$\|f(t, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^6)} \leq \|f(0, \cdot, \cdot)\|_{L^\infty(\mathbb{R}^6)}.$$

Functional setting: for any  $M > 0$ , let

$$\Gamma_M = \{f \in L^1 \cap L^\infty(\mathbb{R}^6) : f(x, v) \geq 0, \|f\|_{L^1(\mathbb{R}^6)} = M, \|f\|_{L^\infty(\mathbb{R}^6)} \leq 1\}$$

and consider  $E_M := \inf \{E(f) : f \in \Gamma_M\}$

Let  $\gamma = 1$  (gravitational case). For any  $E \geq E_M$

$$K_\pm(E, M) := -2E_M \left( 1 - \frac{E}{2E_M} \pm \sqrt{1 - \frac{E}{E_M}} \right)$$

$$P_\pm(E, M) := -2E_M \left( 1 \pm \sqrt{1 - \frac{E}{E_M}} \right)$$



**Theorem 1**  $E_M$  is negative, bounded from below and for any  $f \in \Gamma_M$ , with  $E = E(f)$ , the following properties hold:

- (i)  $E_{KIN}(f) \in \left[ K_-(E, M), K_+(E, M) \right]$
- (ii)  $E_{POT}(f) \in \left[ \max\{0, P_-(E, M)\}, P_+(E, M) \right]$
- (iii)  $E_{POT}(f) \in \left[ 0, \sqrt{-4E_M E_{KIN}(f)} \right]$

Moreover, there exist functions which minimize the energy on  $\Gamma_M$ . They are stationary solutions to the VP system for which  $E = E_M$  and

$$K_{\pm}(E, M) = E_{KIN}(f) = \frac{1}{2} E_{POT}(f) = P_{\pm}(E, M)$$

## A potential energy estimate

**Lemma 1** *There exists a positive constant  $C$  such that  $\forall f \in L^1_{\neq} \cap L^\infty(\mathbb{R}^6)$  with  $|v|^2 f \in L^1(\mathbb{R}^6)$*

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 dx \leq C \|f\|_{L^1(\mathbb{R}^6)}^{7/6} \|f\|_{L^\infty(\mathbb{R}^6)}^{1/3} \left( \int_{\mathbb{R}^6} |v|^2 f(x, v) dx dv \right)^{1/2}$$

*Proof.* From the definition of  $\phi$  we have

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 dx = \int_{\mathbb{R}^3} (-\Delta \phi) \phi dx = 4\pi \int_{\mathbb{R}^6} \frac{\rho(y)\rho(x)}{|x-y|} dx dy$$

According to the **Hardy-Littlewood-Sobolev** inequalities,

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 dx \leq 4\pi \Sigma \|\rho\|_{L^{6/5}(\mathbb{R}^3)}^2$$

Because of **Hölder's** inequality,  $\|\rho\|_{L^{6/5}(\mathbb{R}^3)} \leq \|\rho\|_{L^1(\mathbb{R}^3)}^{7/12} \|\rho\|_{L^{5/3}(\mathbb{R}^3)}^{5/12}$

Interpolation of the  $L^{5/3}$ -norm of  $\rho$

$$\int_{\mathbb{R}^3} |\rho|^{5/3} dx \leq C \|f\|_{L^\infty(\mathbb{R}^6)}^{2/3} \int_{\mathbb{R}^6} |v|^2 f(x, v) dx dv$$

For any  $R > 0$ ,

$$\rho(x, t) := \int_{\mathbb{R}^3} f(t, x, v) dv = \int_{|v| \leq R} f(t, x, v) dv + \int_{v \geq R} f(t, x, v) dv$$

$$\rho(x, t) := \int_{\mathbb{R}^3} f(t, x, v) dv \leq \frac{4\pi}{3} R^3 \|f\|_{L^\infty(\mathbb{R}^6)} + \frac{1}{R^2} \int_{\mathbb{R}^3} |v|^2 f(t, x, v) dv$$

Optimize on  $R = R(t, x)$  for  $t, x$  fixed:  $R^5 = \frac{1}{2\pi} \frac{\int_{\mathbb{R}^3} |v|^2 f(t, x, v) dv}{\|f\|_{L^\infty(\mathbb{R}^6)}}$ .

$$\rho(x, t) \leq C \|f\|_{L^\infty(\mathbb{R}^6)}^{2/5} \left( \int_{\mathbb{R}^3} |v|^2 f(t, x, v) dv \right)^{3/5} \quad \square$$

An equivalent minimization problem

Let  $J_M = \inf \{J(f) : f \in \Gamma_M\}$ ,

$$J(f) = \frac{\frac{1}{2} \int_{\mathbb{R}^6} |v|^2 f \, dx \, dv}{\left(\frac{1}{8\pi} \int_{\mathbb{R}^6} |\nabla \phi|^2 \, dx\right)^2} \equiv \frac{E_{KIN}(f)}{(E_{POT}(f))^2}$$

**Lemma 2** *The minimization problems  $E(f) = E_M$  and  $J(f) = J_M$  over the set  $\Gamma_M$  are equivalent*

(i) *Their respective minima satisfy*

$$4 J_M E_M = -1$$

(ii) *If  $f_M \in \Gamma_M$  is a minimizer of the functional  $E$ , then it is also a minimizer of the functional  $J$ .*

If  $J(g_M) = J_M$  for some  $g_M \in \Gamma_M$ , then  $E(g_M^\sigma) = E_M$  where  $g_M^\sigma(x, v) := g_M(\sigma x, v/\sigma)$  and  $\sigma = \frac{E_{POT}(g_M)}{2E_{KIN}(g_M)}$

*Proof.* The set  $\Gamma_M$  is stable under  $f \mapsto f^\sigma(x, v) = f(\sigma x, v/\sigma)$ ,  $\sigma > 0$ . Optimize on  $\sigma$ :

$$E(f^\sigma) = \sigma^2 E_{KIN}(f) - \sigma E_{POT}(f)$$

$$\sigma_{min} := \frac{E_{POT}(f)}{2 E_{KIN}(f)} E(f) \geq E(f^{\sigma_{min}}) = -\frac{1(E_{POT}(f))^2}{4 E_{KIN}(f)} = -\frac{1}{4J(f)}$$

Proof of Assertions of (i)-(iii) of Theorem ??.

$$E := E(f) = E_{KIN} - E_{POT} \quad \text{and} \quad \frac{E_{KIN}(f)}{(E_{POT}(f))^2} = J(f) \geq -\frac{1}{4 E_M}$$

$$-\frac{(E_{POT}(f))^2}{4 E_M} - E_{POT}(f) \leq E, \quad (E_{KIN} - E)^2 \leq -4 E_M E_{KIN}$$

Existence of minimizers ?

**Corollary 2** *Let  $f_M$  be a minimizing function for the functional  $E$  on  $\Gamma_M$ . Then*

$$E_{POT}(f_M) = 2 E_{KIN}(f_M) = -2 E_M$$

Note that  $E_M = -E_{KIN}(f_M) = -\frac{1}{2} E_{POT}(f_M) < 0$

This property is shared with any stationary solution.

## Nonincreasing symmetric rearrangements

The symmetric rearrangement  $A^*$  of the Borel set  $A \subset \mathbb{R}^n$  is the open ball in  $\mathbb{R}^n$  centered at the origin whose volume is that of  $A$ :

$$\chi_A^* := \chi_{A^*} = \begin{cases} 1 & \text{if } \frac{1}{n} |S^{n-1}| |x|^n \leq \|\chi_A\|_{L^1(\mathbb{R}^n)} \\ 0 & \text{otherwise} \end{cases}$$

Let  $h : \mathbb{R}^n \rightarrow \mathbb{C}$  be a Borel measurable function:

$$h^*(x) := \int_0^\infty \chi_{\{|h|>t\}}^*(x) dt$$

Partial symmetric nonincreasing rearrangement with respect to the  $x$  variable only:

$$g^{*x}(x, v) := \int_0^\infty \chi_{\{x \in \mathbb{R}^n : g(x, v) > t\}}^* dt$$

Elementary properties: use Fubini's theorem

$$\begin{aligned}
 \int_{\mathbb{R}^{2n}} g^{*x}(x, v) \, dx \, dv &= \int_{\mathbb{R}^{2n}} g(x, v) \, dx \, dv \\
 \|g^{*x}\|_{L^\infty(\mathbb{R}^{2n})} &= \|g\|_{L^\infty(\mathbb{R}^{2n})} \\
 \int_{\mathbb{R}^{2n}} |v|^2 g^{*x}(x, v) \, dx \, dv &= \int_{\mathbb{R}^{2n}} |v|^2 g(x, v) \, dx \, dv \\
 \int_{\mathbb{R}^n} g^{*x}(x, v) \, dv &= \int_{\mathbb{R}^n} g(x', v) \, dv \quad \text{if } |x| = |x'| \\
 \int_{\mathbb{R}^n} g^{*x}(|x|, v) \, dv &\geq \int_{\mathbb{R}^n} g^{*x}(|y|, v) \, dv \quad \text{if } |x| \leq |y| \\
 \int_{\mathbb{R}^n} \psi(|x|) g^{*x}(|x|, v) \, dv &< \int_{\mathbb{R}^n} \psi(|x|) g(x, v) \, dv \quad \text{for } \psi \nearrow
 \end{aligned}$$

**Theorem 3** [Riesz' theorem]

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(x - y) h(y) \, dx \, dy =: I(f, g, h) \leq I(f^*, g^*, h^*)$$

*If  $g$  is radially symmetric and strictly decreasing, equality holds only if  $\exists y \in \mathbb{R}^n$   $f(x) = f^*(x - y)$  and  $h(x) = h^*(x - y)$*



## Spherical symmetry and regularity of the potential

**Lemma 3** *Let  $M > 0$ . There exists a minimizing sequence  $(f_n)_{n \in \mathbf{N}} \in \Gamma_M^{\mathbf{N}}$  of the functional  $E$  such that  $\rho_n(x) = \int_{\mathbb{R}^3} f_n(x, v) dv$  is a radial nonincreasing function.*

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx &= 4\pi \int_{(\mathbb{R}^3)^4} \frac{f(x, v) f(x', v')}{|x - x'|} dx dx' dv dv' \\ &\leq 4\pi \int_{(\mathbb{R}^3)^4} \frac{f^{*x}(x, v) f^{*x}(x', v')}{|x - x'|} dx dx' dv dv' \end{aligned}$$

**Lemma 4** *Let  $\rho \in L^1(\mathbb{R}^3) \cap L^{5/3}(\mathbb{R}^3)$  be a spherically symmetric function. Then  $\phi \in W_{\text{loc}}^{2,5/3}(\mathbb{R}^3)$  and  $\exists \eta > 0, \forall R > 0$*

$$\int_{|x| < R} |\nabla \phi|^{2+\eta} dx \leq C(M, R) \left( \int_{|x| < R} \rho^{5/3} dx + 1 \right)$$

## A priori estimates, scalings and tools of the concentration-compactness

Poisson equation

$$\Delta \phi = 4 \pi \rho, \quad \lim_{|x| \rightarrow +\infty} \phi(x) = 0$$

$\phi$  is radial, nondecreasing and strictly increasing in the interior of the support of  $\rho$  if  $\rho \in L^1_+$  is radial. Let  $\int_{\mathbf{R}^3} \rho(x) dx =: M > 0$

(i) For any  $r > 0$ ,

$$\phi'(r) \leq \frac{M}{r^2} \quad \text{and} \quad -\frac{M}{r} \leq \phi(r) \leq 0$$

(ii) For any  $R > 0$ ,

$$\int_{|x| \geq R} |\nabla \phi(x)|^2 dx \leq 4 \pi \frac{M^2}{R}$$

## Lemma 5 [Scaling]

$$E_M = M^{7/3} E_1$$

**Lemma 6 [Splitting]** Let  $f \in \Gamma_M$  be such that the mass density  $\rho(x) = \int_{\mathbb{R}^3} f(x, v) dv$  is spherically symmetric. Given  $R > 0$ , if  $M - \lambda = \int_{|x| < R} \int_{\mathbb{R}^3} f(x, v) dv dx$  for some  $\lambda \in [0, M]$ , then

$$E(f) - E_M \geq - \left( \frac{7}{3} \frac{E_M}{M^2} + \frac{1}{4\pi R} \right) (M - \lambda) \lambda$$

*Proof.* Let  $\phi = \phi_1 + \phi_2$ , with

$$\Delta\phi_1(x) = \int_{\mathbb{R}^3} \chi_{B_R}(x) f(x, v) dv, \quad \Delta\phi_2(x) = \int_{\mathbb{R}^3} (1 - \chi_{B_R}(x)) f(x, v) dv$$

$$\begin{aligned}
E(f) &= E_{KIN}(\chi_{B_R} f) + E_{KIN}((1 - \chi_{B_R})f) \\
&\quad - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_1|^2 dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla \phi_2|^2 dx - \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla \phi_1 \cdot \nabla \phi_2 dx \\
&= E(\chi_{B_R} f) + E((1 - \chi_{B_R})f) - \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla \phi_1 \cdot \nabla \phi_2 dx \\
&\geq E_{M-\lambda} + E_\lambda + \frac{1}{4\pi} \int_{\mathbb{R}^3} \phi_2 \Delta \phi_1 dx \\
&\geq \left[ \left(1 - \frac{\lambda}{M}\right)^{\frac{7}{3}} + \left(\frac{\lambda}{M}\right)^{\frac{7}{3}} - 1 \right] E_M + \frac{1}{4\pi} \int_{\mathbb{R}^3} \phi_2 \Delta \phi_1 dx
\end{aligned}$$

$$1) \forall x \in [0, 1], (1 - x)^{7/3} + x^{7/3} - 1 \leq -\frac{7}{3} x (1 - x)$$

$$2) \left| \frac{1}{4\pi} \int_{\mathbb{R}^3} \phi_2 \Delta \phi_1 dx \right| \leq \|\phi_2\|_{L^\infty(\mathbb{R}^3)} (M - \lambda)$$

$$\text{where } \|\phi_2\|_{L^\infty(\mathbb{R}^3)} = |\phi_2(0)| = |\phi_2(R)| \leq \frac{\lambda}{4\pi R} \quad \square$$

**Lemma 7** [No vanishing] Let  $R_0 > \frac{3M^2}{28\pi|E_M|}$ ,  $(f_n)_{n \in \mathbf{N}} \in \Gamma_M^{\mathbf{N}}$  a minimizing sequence as in Lemma ???. Then

$$\limsup_{n \rightarrow \infty} \int_{|x| \geq R_0} \int_{\mathbb{R}^3} f_n \, dv \, dx = 0$$

*Proof.* By contradiction:  $\lim_{n \rightarrow \infty} \int_{|x| \geq R_0} \int_{\mathbb{R}^3} f_n \, dv \, dx = \lambda > 0$ .

$$\exists R_n > R_0 \quad \frac{\lambda}{2} = \int_{|x| \geq R_n} \int_{\mathbb{R}^3} f_n \, dv \, dx$$

Apply now Lemma ??? to each  $f_n$  with  $R = R_n$ :

$$\begin{aligned} E(f_n) - E_M &\geq - \left( \frac{7E_M}{3M^2} + \frac{1}{4\pi R_n} \right) \left( M - \frac{\lambda}{2} \right) \frac{\lambda}{2} \\ &\geq - \left( \frac{7E_M}{3M^2} + \frac{1}{4\pi R_0} \right) \left( M - \frac{\lambda}{2} \right) \frac{\lambda}{2} > 0 \end{aligned}$$

□

## Convergence of a minimizing sequence

**Proposition 8** Let  $(f_n)_{n \in \mathbf{N}} \in \Gamma_M^{\mathbf{N}}$  be a minimizing sequence for the functional  $E$ , with radial nonincreasing mass densities. Up to a subsequence, the sequence converges to a minimizer  $f_M \in \Gamma_M$  s.t.  $E_M = E(f_M)$ ,  $\text{supp}(f_M) \subset B_{R_0} \times \mathbb{R}^3$  where  $R_0 = \frac{3M^2}{28\pi|E_M|}$

*Proof.* Lemma ??  $\Rightarrow \lim_{n \rightarrow \infty} \int_{B_{R_0}} \int_{\mathbb{R}^3} f_n \, dv \, dx = M$ . Dunford-Pettis:

- (i) [boundedness]  $(f_n)_{n \in \mathbf{N}}$  is bounded in  $L^1(\mathbb{R}^6)$
- (ii) [no concentration] for any measurable set  $A$

$$\int_A f_n \, dx \, dv \leq \|f_n\|_{L^\infty(\mathbb{R}^6)} |A| \leq |A|$$

- (iii) [no vanishing] for any  $K_1, K_2$ , either  $K_1 \geq R_0$  or

$$\int_{|x| > K_1} \int_{|v| > K_2} f_n \, dx \, dv \leq \int_{\mathbb{R}^3} \int_{|v| > K_2} f_n \, dx \, dv \leq \frac{1}{K_2^2} E_{KIN}(f_n) \quad \square$$

## 2. SOLUTIONS OF THE VP SYSTEM WITH MINIMAL ENERGY

**Theorem 4** *Let  $f_M$  be a minimizing function for the functional  $E$  on  $\Gamma_M$ , with radial mass density. Then*

$$f_M(x, v) = \begin{cases} 1 & \text{if } \frac{1}{2}|v|^2 + \phi_{f_M}(x) < \frac{7E(f_M)}{3M} \\ 0 & \text{otherwise} \end{cases}$$

where  $\phi_{f_M}$  is the unique radial solution on  $\mathbb{R}^3$  of

$$\Delta \phi_{f_M} = \frac{1}{3} (4\pi)^2 \left[ 2 \left( \frac{7E_M}{3M} - \phi_{f_M} \right)_+ \right]^{3/2}$$

*It is the unique minimizer with radial mass density and it is also a steady-state solution to the VP system. Moreover, if  $f$  is another minimizing function, then with  $\bar{x} := \frac{1}{M} \int_{\mathbb{R}^6} x f(x, v) dx dv$*

$$f(x, v) = f_M(x - \bar{x}, v) \quad \forall (x, v) \in \mathbb{R}^6$$

## Explicit form of the minimizers

### Lemma 9

$$f_M(x, v) = \begin{cases} 1 & \text{for } (x, v) \text{ such that } |v| \leq \left(\frac{3}{4\pi} \rho_M(x)\right)^{1/3} \text{ a.e.} \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* 1)  $f_M \|_{L^\infty(\mathbb{R}^6)} = 1$  otherwise let:

$$\bar{f}(x, v) = \kappa f(\kappa^{2/3}x, \kappa^{-1/3}v)$$

$$\|\bar{f}\|_{L^1(\mathbb{R}^6)} = \|f\|_{L^1(\mathbb{R}^6)}, \quad \|\bar{f}\|_{L^\infty(\mathbb{R}^6)} = \kappa \|f\|_{L^\infty(\mathbb{R}^6)}, \quad E(\bar{f}) = \kappa^{2/3} E(f)$$

for  $\kappa = \|f_M\|_{L^\infty(\mathbb{R}^6)}^{-1} > 1$ , a contradiction



2) Let  $\epsilon \in (0, 1)$  and  $g(x, v) \in L^1(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$  be a test function such that  $g \geq 0$  a.e. in  $\mathbb{R}^6 \setminus \text{supp}(f_M)$ , with compact support contained inside

$$(\text{supp}(f_M) \setminus S_\epsilon)^c \equiv (\mathbb{R}^6 \setminus \text{supp}(f_M)) \cup S_\epsilon$$

where  $S_\epsilon = \{(x, v) \in \mathbb{R}^6 : \epsilon \leq f_M(x, v) \leq 1 - \epsilon\}$ . Let

$$g(t) := M \frac{tg + f_M}{\|tg + f_M\|_{L^1(\mathbb{R}^6)}}, \quad 0 < t < T := M\epsilon \left( M\|g\|_{L^\infty} + \|g\|_{L^1} \right)^{-1}$$

$$\begin{aligned} 0 &\leq -T\|g\|_{L^\infty(\mathbb{R}^6)} + \epsilon \leq tg + f_M && \text{in } S_\epsilon \\ 0 &\leq f_M = tg + f_M && \text{in } \text{supp}(f_M) \setminus S_\epsilon \\ 0 &\leq tg = tg + f_M && \text{in } \mathbb{R}^6 \setminus \text{supp}(f_M) \end{aligned}$$

$$\left. \begin{aligned} \|g(t)\|_{L^\infty(S_\epsilon)} &\leq M \frac{T\|g\|_{L^\infty(\mathbb{R}^6)} + 1 - \epsilon}{M - T\|g\|_{L^1(\mathbb{R}^6)}} = 1 \\ \|g(t)\|_{L^1(\mathbb{R}^6)} &= M \end{aligned} \right\} \Rightarrow g(t) \in \Gamma_M$$

Taylor expansion at  $t = 0_+$

$$g(t) - f_M = t g'(0) + \frac{t^2}{2} g''(\theta) = t \left( g - \frac{1}{M} \left[ \int_{\mathbb{R}^6} g \, dx \, dv \right] f_M \right) + \frac{t^2}{2} g''(\theta)$$

$$E(g(t)) - E_M = t \int_{\mathbb{R}^6} \left( \frac{1}{2} |v|^2 + \phi_{f_M} - \frac{3E_M}{M} \right) g \, dx \, dv + O(t^2)$$

Since  $f_M$  minimizes  $E(\cdot) - E_M$  on  $\Gamma_M$ , we have that  $E(g(t)) - E_M \geq 0$  for any  $t \in [0, T]$  and consequently

$$\int_{\mathbb{R}^6} \left( \frac{1}{2} |v|^2 + \phi_{f_M} - \frac{3E_M}{M} \right) g \, dx \, dv \geq 0$$

for every  $g$  and  $\epsilon$ .

(i)  $g \geq 0$  on  $\mathbb{R}^6 \setminus \text{supp}(f_M)$ :

$$\frac{1}{2}|v|^2 + \phi_{f_M}(x) \geq \frac{3E_M}{M} \quad \forall (x, v) \in \mathbb{R}^6 \setminus \text{supp}(f_M)$$

or equivalently

$$\left\{ (x, v) \in \mathbb{R}^6 : \frac{1}{2}|v|^2 + \phi_{f_M}(x) \leq \frac{3E_M}{M} \right\} \subset \text{supp}(f_M)$$

(ii) On the other hand,  $g$  has no determined sign on  $S_\epsilon$

$$\frac{1}{2}|v|^2 + \phi_{f_M}(x) = \frac{3E_M}{M} \quad \forall (x, v) \in S_\epsilon \cap \text{supp}(f_M)$$

This set and  $S_\epsilon$  and have 0 Lebesgue measure:

$$f_M \equiv 1 \quad \text{on} \quad \text{supp}(f_M)$$

3)  $f_M$  minimizes  $\{E(f) : f \in \gamma_M\}$  where

$$\gamma_M = \left\{ f \in \Gamma_M : f \equiv 1 \text{ a.e. on } \text{supp}(f), \int_{\mathbb{R}^3} f(x, v) dv = \rho_M(x) \quad \forall x \in \mathbb{R}^3 \right\}$$

The problem is therefore reduced to the minimization of

$$\{E_{KIN}(f) : f \in \gamma_M\} \quad :$$

$$\int_{\mathbb{R}^3} |v|^2 f^{*v} dv \leq \int_{\mathbb{R}^3} |v|^2 f dv$$

with a strict inequality unless  $f \equiv f^{*v}$  a.e. Thus

$$f_M \equiv \chi_{|v| \leq \left(\frac{3}{4\pi} \rho_M(x)\right)^{1/3}} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^3 \text{ a.e.}$$

□

**Lemma 10** Let  $f_M$  be a minimizing function for  $E$  on  $\Gamma_M$  with radial mass density. Then  $\rho_M = \int_{\mathbb{R}^3} f_M(\cdot, v) dv$  and  $\phi_{f_M} = |\cdot|^{-1} * \rho_M$  are related by

$$\rho_M(x) = \begin{cases} \frac{4\pi}{3} \left[ 2 \left( \frac{7}{3} \frac{E_M}{M} - \phi_{f_M}(x) \right) \right]^{3/2} & \text{if } \phi_{f_M}(x) \leq \frac{7}{3} \frac{E_M}{M} \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Let  $\rho(x) \in L^1(\mathbb{R}^3) \cap L^{5/3}(\mathbb{R}^3)$  be a nonnegative function such that  $\|\rho\|_{L^1(\mathbb{R}^3)} = M$  and define  $f_\rho(x, v) := \chi_{\left\{ |v| \leq \left( \frac{3}{4\pi} \rho(x) \right)^{1/3} \right\}}$

$$E_{POT}(f_\rho) = \frac{1}{2} \int_{\mathbb{R}^6} \frac{\rho(y) \rho(x)}{|x - y|} dx dy$$

$$E_{KIN}(f_\rho) = \frac{3^{5/3}}{10 (4\pi)^{2/3}} \int_{\mathbb{R}^3} \left[ \rho(x) \right]^{5/3} dx$$

and write the Euler-Lagrange equations.

**Lemma 11**  $\phi_{f_M}$  is unique and continuously differentiable. If  $f$  is another minimum of  $E$  on  $\Gamma_M$ , then there exists  $y \in \mathbb{R}^3$  such that

$$\int_{\mathbb{R}^3} f(x, v) dv = \rho_M(x - y) \quad \text{a.e. } x \in \mathbb{R}^3.$$

*Proof.* Rewrite the Poisson equation for  $\phi_{f_M}$ :

$$\Delta \phi_{f_M} = 4\pi \rho_M(x) = \begin{cases} \frac{1}{3} (4\pi)^2 \left[ 2 \left( \frac{7E_M}{3M} - \phi_{f_M} \right) \right]^{3/2} & \text{if } \phi_{f_M}(x) \leq \frac{7E_M}{3M} \\ 0 & \text{otherwise} \end{cases}$$

with  $w(r) := \frac{7E_M}{3M} - \phi_{f_M}(r/\sqrt{c})$ ,  $c = \frac{1}{3} 32\sqrt{2} \pi^2$ :

$$(r^2 w'(r))' + r^2 w_+^{3/2}(r) = 0$$

At  $R = \frac{3M^2}{7|E_M|} \sqrt{c}$ ,  $w(R) = 0$  and  $w'(R) = -\frac{1}{\sqrt{c}} \phi'_{f_M} \left( \frac{R}{\sqrt{c}} \right) = -\frac{M\sqrt{c}}{R^2}$

### Nonlinear stability for the evolution problem

We follow the strategy of Guo in [Guo99]. Consider for any  $g, h \in \Gamma_M$  the distance  $d$  defined by

$$d(g, h) = E(g) - E(h) + \frac{1}{4\pi} \|\nabla\phi_g - \nabla\phi_h\|_{L^2(\mathbb{R}^3)}^2$$

**Theorem 5** *For every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that, if  $f$  is a solution of the VP system with an initial condition  $f_0 \in \Gamma_M$ , then*

$$d(f_0, f_M) \leq \delta \implies d(f^*(t), f_M) \leq \epsilon \quad \forall t \geq 0$$

*Proof.* The result is easily achieved by contradiction since  $E(f^*(t)) - E(f_M) \leq E(f_0) - E(f_M) \searrow 0$  implies  $\|\nabla\phi_{f^*(t)} - \nabla\phi_{f_M}\|_{L^2(\mathbb{R}^3)} \searrow 0$   $\square$

### 3. LARGE TIME BEHAVIOUR

The *Galilean invariance* of a classical solution  $f$  to the VP system with initial data  $f_0(x, v)$  means that for any  $\mathbf{u} \in \mathbb{R}^3$ , the solution with initial data  $f_0^{\mathbf{u}}(x, v) = f_0(x, v - \mathbf{u})$  is given by

$$f^{\mathbf{u}}(t, x, v) = f(t, x - t\mathbf{u}, v - \mathbf{u}) \quad \forall (t, x, v) \in (0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$$

$$\langle v \rangle(f^{\mathbf{u}}) := \int_{\mathbb{R}^6} v f^{\mathbf{u}}(t, x, v) dx dv = \langle v \rangle(f) + \mathbf{u} \|f(t)\|_{L^1(\mathbb{R}^6)}$$

Total momentum:

$$\langle v \rangle(f^{\mathbf{u}}) := \int_{\mathbb{R}^6} v f^{\mathbf{u}}(t, x, v) dx dv = \langle v \rangle(f) + \mathbf{u} \|f(t)\|_{L^1(\mathbb{R}^6)}$$

Total energy:

$$E(f^{\mathbf{u}}) = E(f) + \mathbf{u} \cdot \langle v \rangle(f) + \frac{1}{2} |\mathbf{u}|^2 \|f\|_{L^1(\mathbb{R}^6)}.$$



Among the family  $(f^u)_{u \in \mathbb{R}^3}$ ,  $\min_{u \in \mathbb{R}^3} E_{KIN}(f^u)$  is reached by

$$E_{KIN}(f^{\bar{u}}) = E_{KIN}(f) - \frac{\langle v \rangle^2(f)}{2 \|f\|_{L^1(\mathbb{R}^6)}} \quad \text{for } \bar{u} = -\frac{\langle v \rangle(f)}{\|f\|_{L^1(\mathbb{R}^6)}}$$

Galilean invariance and asymptotic behaviour

**Lemma 12**  $E(f) < \frac{1}{2} \frac{\langle v \rangle^2(f)}{\|f\|_{L^1(\mathbb{R}^6)}} \implies E(f^{\bar{u}}) < 0$

**Proposition 13** *Under the above condition, there exists three constants  $C_1, C_2, C_3 > 0$  such that*

$$C_1 \leq E_{POT}(f(t, \cdot, \cdot)) \leq C_2$$

$$\|\rho_f(t, \cdot)\|_{L^{5/3}(\mathbb{R}^3)} \geq C_3$$

Variance and dispersion estimates The solutions to the VP system in the gravitational case have a qualitative behaviour which strongly differs from the behaviour in the plasma physics case since, for instance, stationary solutions exist. Assume that

$$E(f) > \frac{1}{2} \frac{\langle v \rangle^2(f)}{\|f\|_{L^1(\mathbb{R}^6)}}$$

$$\begin{aligned} \langle (\Delta x)^2 \rangle &:= \int_{\mathbb{R}^6} |x|^2 f(t, x, v) \, dx \, dv - \left( \int_{\mathbb{R}^6} x f(t, x, v) \, dx \, dv \right)^2 \\ \langle (\Delta v)^2 \rangle &:= \int_{\mathbb{R}^6} |v|^2 f(t, x, v) \, dx \, dv - \left( \int_{\mathbb{R}^6} v f(t, x, v) \, dx \, dv \right)^2 \end{aligned}$$

### Lemma 14

$$\frac{1}{2} \frac{d^2}{dt^2} \langle (\Delta x)^2 \rangle = E(f) + \frac{1}{2} \langle (\Delta v)^2 \rangle - \frac{1}{2} \langle v \rangle^2(f)$$

*Proof.* A straightforward calculation using the VP system gives

$$\begin{aligned}
\frac{1}{2} \frac{d^2}{dt^2} \int_{\mathbb{R}^6} |x|^2 f(t, x, v) \, dx \, dv &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^6} |x|^2 (-v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f) \, dx \, dv \\
&= \int_{\mathbb{R}^6} (v \cdot x) (-v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f) \, dx \, dv \\
&= \int_{\mathbb{R}^6} |v|^2 f \, dx \, dv - \frac{1}{4\gamma\pi} \int_{\mathbb{R}^3} (x \cdot \nabla_x \phi) \Delta \phi \, dx \\
&= \int_{\mathbb{R}^6} |v|^2 f \, dx \, dv - \frac{1}{8\gamma\pi} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx \\
&= E(f) + \frac{1}{2} \int_{\mathbb{R}^6} |v|^2 f \, dx \, dv
\end{aligned}$$

This equivalent to the *pseudo-conformal law*

$$\frac{d}{dt} \left( \int_{\mathbb{R}^6} |x - tv|^2 f(t, x, v) \, dx \, dv - \frac{t^2}{4\pi\gamma} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx \right) = -\frac{t}{4\pi\gamma} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx$$

established by R. Illner, G. Rein [IllRe96], and B. Perthame [Pert96].

**Proposition 15** *If  $E(f) > \frac{1}{2} \frac{\langle v \rangle^2(f)}{\|f\|_{L^1(\mathbb{R}^6)}}$ , then there exists constants  $C, C_1, C_2 > 0$  such that for some  $t_0 > 0$ ,*

$$C_1 t^2 \leq \int_{\mathbb{R}^6} |x|^2 f(t, x, v) dx dv \leq C_2 t^2 \quad \forall t \geq t_0 > 0$$

*and, for any  $p \in [1, \infty)$ ,*

$$\|\rho(t, x)\|_{L^p(\mathbb{R}^3)} \geq \frac{C}{t^{3(p-1)/p}} \quad \forall t > t_0 > 0$$

*Proof.*  $\frac{1}{2} \frac{d^2}{dt^2} \left( \int_{\mathbb{R}^6} |x|^2 f(t, x, v) dx dv \right) = 2E(f) + E_{POT}(f)$

$$\begin{aligned} \int_{\mathbb{R}^6} f(t, x, v) dx dv &\leq \int_{\mathbb{R}^3} \int_{|x| \leq R} f(t, x, v) dx dv + \int_{\mathbb{R}^3} \int_{|x| > R} f(t, x, v) dx dv \\ &\leq C \|\rho(t, x)\|_{L^p(\mathbb{R}^3)}^{\frac{2p}{5p-3}} \left( \int_{\mathbb{R}^6} |x|^2 f(t, x, v) dx dv \right)^{\frac{3p-3}{5p-3}} \quad \square \end{aligned}$$

## CONCLUSION

- Rotating systems [Guo, Rein], [Schaeffer]: mathematically, rotations do not induce a loss of compactness
- Other norms (or convex functionals) than  $L^\infty$ : stability w.r.t. other stationary solutions
- Stability in  $L^1$  [Sánchez, Soler] can be achieved by concentration-compactness methods (no rotation)

- **Concentration -compactness** offers 3 scenarii (after concentration has been discarded) [J.D., Poupaud, Sánchez, Soler, in progress]

- 1) compactness

- 2) vanishing

- 3) dichotomy

Notion of “eternal” solution

- Several notions of **dispersion** :

- 1) local convergence to 0 in  $L^1$

- 2) local convergence to 0 in  $L^q$ ,  $q > 1$ , or in  $L_t^p(L_x^q(L_v^1))$

- 3) statistical dispersion:  $\langle x^2 \rangle - \langle x \rangle^2 \sim t^2$  as  $t \rightarrow +\infty$ . This occurs whenever  $E(f\bar{u}) > 0$