Recent results of stability in functional inequalities

Jean Dolbeault

Ceremade, CNRS & Université Paris-Dauphine
http://www.ceremade.dauphine.fr/∼dolbeaul

January 15, 2023

First International Conference on Geometric and Functional Inequalities and their Applications in PDEs
Beijing Institute of Technology
(January 14-16, 2023)

Outline

1. Constructive stability estimates on $\mathbb{R}^d$
   - Entropy methods and flow estimates for some GNS inequalities
   - Stability results for the Sobolev inequality
2. LSI and GNS inequalities on the sphere: results
   - Stability results
   - Results based on the carré du champ method (heat flow)
   - Result based on the generalized carré du champ method (nonlinear diffusions)
3. LSI and GNS inequalities on the sphere: proofs
   - Nonlinear carré du champ: main steps of the method
   - Improved interpolation inequalities under orthogonality constraints
   - Proof of the main results

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

A joint work with M. Bonforte, B. Nazaret and N. Simonov
Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method
arXiv:2007.03674, to appear in *Memoirs of the AMS*

Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\geq \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \tag{GNS}$$

Range of exponents:

$$1<p\leq\frac{d}{d-2}\quad\Longleftrightarrow\quad\frac{d-1}{d}=:m_1\leq m<1$$

- Sobolev inequality: $p=\frac{d}{d-2}$, $m=m_1$
- Logarithmic Sobolev inequality: $p=1$, $m=1$

Entropy – entropy production inequality

Fast diffusion equation (written in self-similar variables)

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \big(v\,(\nabla v^{m-1} - 2\,x)\big) = 0 \qquad (r\,\mathrm{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m}\int_{\mathbb{R}^d}\big(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1}\,(v-\mathcal{B})\big)\,dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v\,\big|\nabla v^{m-1} + 2\,x\big|^2\,dx$$

satisfy an *entropy – entropy production inequality*

$$\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$$

[del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[v_0]\,e^{-4\,t}$$

The *entropy – entropy production inequality* $\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$ is equivalent to the *Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

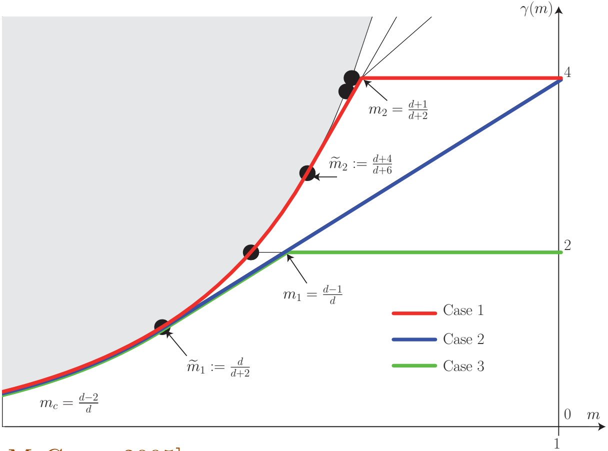
with equality if and only if

$$|f(x)|^{2p} = \mathcal{B}(x) = \left(1+|x|^2\right)^{\frac{1}{m-1}}$$

$$p = \frac{1}{2\,m-1} \quad \Longleftrightarrow \quad m = \frac{p+1}{2\,p} \in [m_1, 1) \quad \text{with} \quad m_1 = \frac{d-1}{d}$$

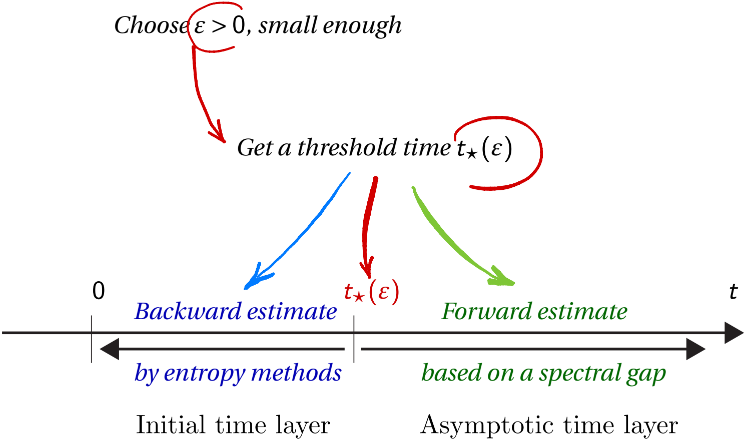
$u = f^{2\,p}$ so that $u^m = f^{p+1}$ and $u\left|\nabla u^{m-1}\right|^2 = (p-1)^2\,|\nabla f|^2$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Strategy of the method



A constructive stability result (critical case only)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \geq 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\;\; |x|\, f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d)\right\}$$

Theorem

Let $d \geq 3$ and $A > 0$. For any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathsf{g}\, dx \text{ and } \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \leq A$$

we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\mathcal{C}_\star(A)}{4+\mathcal{C}_\star(A)} \int_{\mathbb{R}^d} \left|\nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathsf{g}^{-\frac{2}{d-2}}\right|^2 dx$$

$\mathcal{C}_\star(A) = \mathcal{C}_\star(0)\,\big(1+A^{1/(2\,d)}\big)^{-1}$ *and* $\mathcal{C}_\star(0) > 0$ *depends only on* d

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

A joint work with JD, M.J. Esteban, A. Figalli, R. Frank, M. Loss

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

arXiv: 2209.08651

Stability results for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g(x) = c\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,,\quad a \in (0,\infty)\,,\quad b \in \mathbb{R}^d\,,\quad c \in \mathbb{R}$$

Theorem

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

[JD, Esteban,Figalli, Frank, Loss] Cf. R. Frank's lecture yesterday
▷ The "far away" regime and the "neighborhood" of $\mathcal{M}$
▷ Competing symmetries and a notion of a continuous flow (based on Steiner's symmetrization)

Logarithmic Sobolev and Gagliardo-Nirenberg-Sobolev on the sphere

A joint work with G. Brigati and N. Simonov
Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results
arXiv:2211.13180

(Improved) logarithmic Sobolev inequality

On the sphere $\mathbb{S}^d$ with $d \geq 1$

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \geq \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log \left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \tag{LSI}$$

$d\mu$: uniform probability measure; equality case: constant functions
Optimal constant: test functions $F_\varepsilon(x) = 1 + \varepsilon\, x \cdot \nu$, $\nu \in \mathbb{S}^d$, $\varepsilon \to 0$
▷ *improved inequality* under an appropriate *orthogonality condition*

Theorem

Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ such that $\int_{\mathbb{S}^d} x\, F \, d\mu = 0$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log \left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu \geq \frac{2}{d+2} \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu$$

Improved ineq. $\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu \geq \left(\frac{d}{2} + 1 \right) \int_{\mathbb{S}^d} F^2 \log \left(F^2 / \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) d\mu$

Logarithmic Sobolev inequality: stability (1)

What if $\int_{\mathbb{S}^d} x\,F\,d\mu \neq 0$? Take $F_\varepsilon(x) = 1 + \varepsilon\,x\cdot\nu$and let $\varepsilon \to 0$

$$\|\nabla F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{d}{2}\int_{\mathbb{S}^d} F_\varepsilon^2\,\log\left(\frac{F_\varepsilon^2}{\|F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)d\mu = O(\varepsilon^4) = O\left(\|\nabla F_\varepsilon\|^4_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Such a behaviour is in fact optimal: *carré du champ* method

Proposition

Let $d \geq 1$, $\gamma = 1/3$ if $d = 1$ and $\gamma = (4\,d-1)\,(d-1)^2/(d+2)^2$ if $d \geq 2$. Then, for any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ with $\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$ we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu - \frac{d}{2}\int_{\mathbb{S}^d} F^2\,\log F^2\,d\mu \geq \frac{1}{2}\,\frac{\gamma\,\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\gamma\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + d}$$

In other words, if $\|\nabla F\|_{\mathrm{L}^2(\mathbb{S}^d)}$ is small

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu - \tfrac{d}{2}\int_{\mathbb{S}^d} F^2\,\log F^2\,d\mu \geq \tfrac{\gamma}{2\,d}\,\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)} + o\left(\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Logarithmic Sobolev inequality: stability (2)

Let $\Pi_1 F$ denote the orthogonal projection of a function $F \in \mathrm{L}^2(\mathbb{S}^d)$ on the spherical harmonics corresponding to the first eigenvalue on $\mathbb{S}^d$

$$\Pi_1 F(x) = \frac{x}{d+1} \cdot \int_{\mathbb{S}^d} y\, F(y)\, d\mu(y) \quad \forall\, x \in \mathbb{S}^d$$

▷ a global (and detailed) stability result

Theorem

Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$
$$\geq \mathscr{S}_d \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{2}\,\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla(\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for some explicit stability constant $\mathscr{S}_d > 0$

Gagliardo-Nirenberg(-Sobolev) inequalities

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu \ge \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\,F\in\mathrm{H}^1(\mathbb{S}^d,d\mu) \tag{GNS}$$

for any $p\in[1,2)\cup(2,2^*)$, with $d\mu$: uniform probability measure
$2^*:=2\,d/(d-2)$ if $d\ge3$ and $2^*=+\infty$ otherwise
Optimal constant: test functions $F_\varepsilon(x)=1+\varepsilon\,x\cdot\nu$, $\nu\in\mathbb{S}^d$, $\varepsilon\to0$
logarithmic Sobolev inequality: obtained by taking the limit as $p\to2$

Theorem

Let $d\ge1$. For any $F\in\mathrm{H}^1(\mathbb{S}^d,d\mu)$ such that $\int_{\mathbb{S}^d} x\,F\,d\mu=0$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu - \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \ge \mathscr{C}_{d,p}\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu$$

with $\mathscr{C}_{d,p}=\frac{2\,d-p\,(d-2)}{2\,(d+p)}$

Gagliardo-Nirenberg inequalities: stability (1)

With $F_\varepsilon(x) = 1 + \varepsilon\, x \cdot \nu$, the deficit is of order ε^4 as $\varepsilon \to 0$

Proposition

Let $d \geq 1$ and $p \in (1,2) \cup (2, 2^*)$. There is a convex function ψ on $\mathbb{R}^+$ with $\psi(0) = \psi'(0) = 0$ such that, for any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu - \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \geq \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\, \psi\left(\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)$$

This is also a consequence of the *carré du champ* method, with an explicit construction of ψ

There is no orthogonality constraint

Gagliardo-Nirenberg inequalities: stability (2)

As in the case of the logarithmic Sobolev inequality, the improved inequality under orthogonality constraint and the stability inequality arising from the *carré du champ* method can be combined

Theorem

Let $d \geq 1$ and $p \in (1,2) \cup (2, 2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{p-2} \left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) \geq \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1) F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

Generalized entropy functionals

$$\mathcal{E}_p[F] := \frac{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2} \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[F] := \frac{1}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$

▷ The key idea is to evolve these quantities by a diffusion flow and prove the inequalities as a consequence of a monotonicity along the flow

Heat flow estimates: fixing parmaeters

Let us consider the constant γ given by

$$\gamma := \left(\frac{d-1}{d+2}\right)^2 (p-1)\,(2^\# - p) \quad \text{if} \quad d \geq 2\,, \quad \gamma := \frac{p-1}{3} \quad \text{if} \quad d = 1$$

and the *Bakry-Emery exponent*

$$2^\# := \frac{2\,d^2+1}{(d-1)^2}$$

Let us define

$$s_\star := \frac{1}{p-2} \quad \text{if} \quad p > 2 \quad \text{and} \quad s_\star := +\infty \quad \text{if} \quad p \leq 2$$

For any $s \in [0, s_\star)$, let

$$\begin{aligned}
\varphi(s) &= \tfrac{1-(p-2)\,s-(1-(p-2)\,s)^{-\frac{\gamma}{p-2}}}{2-p-\gamma} && \text{if} \quad \gamma \neq 2-p \quad \text{and} \quad p \neq 2\\
\varphi(s) &= \tfrac{1}{2-p}\,(1+(2-p)\,s)\log\left(1+(2-p)\,s\right) && \text{if} \quad \gamma = 2-p \neq 0\\
\varphi(s) &= \tfrac{1}{\gamma}\,(e^{\gamma\,s}-1) && \text{if} \quad p = 2
\end{aligned}$$

Heat flow: stability estimates

[JD, Esteban, Kowalczyk, Loss], [JD, Esteban 2020]

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\varphi\left(\frac{\mathcal{E}_p[F]}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\,F\in\mathrm{H}^1(\mathbb{S}^d)$$

Since $\varphi(0)=0$, $\varphi'(0)=1$, and φ is convex increasing, with an asymptote at $s=s_\star$ if $p\in(2,2^\#)$, we know that $\varphi:[0,s_\star)\to\mathbb{R}^+$ is invertible and $\psi:\mathbb{R}^+\to[0,s_\star)$, $s\mapsto\psi(s):=s-\varphi^{-1}(s)$, is convex increasing with $\psi(0)=\psi'(0)=0$, $\lim_{t\to+\infty}\big(t-\psi(t)\big)=s_\star$, and

$$\psi''(0)=\varphi''(0)=\tfrac{(d-1)^2}{(d+2)^2}\left(2^\#-p\right)(p-1)>0\quad\forall\,p\in(1,2^\#)$$

First stability estimates for Gagliardo-Nirenberg inequalities

Proposition

With the above notations, $d\ge1$ and $p\in(1,2^\#)$, we have

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-d\,\mathcal{E}_p[F]\ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)\quad\forall\,F\in\mathrm{H}^1(\mathbb{S}^d)$$

A simpler reformulation

Let $d \geq 1$, $\gamma \neq 2-p$ as above

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \frac{d}{2-p-\gamma}\left(\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|F\|^{2-\frac{2\gamma}{2-p}}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|F\|^{\frac{2\gamma}{2-p}}_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d)$$

[JD, Esteban 2020]
which is a refinement of the standard Gagliardo-Nirenberg inequality

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu \geq \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

... with the restriction $p < 2^\# := \frac{2\,d^2+1}{(d-1)^2} < 2^* := \frac{2\,d}{d-2}$ if $d \geq 3$

So far, we considered only the case $1 \le p < 2^\#$. Our goal is to cover also the subcritical range $p \in [2^\#, 2^*)$

$$\varphi_{m,p}(s) := \int_0^s \exp\left[-\zeta\left(\left(1-(p-2)\,z\right)^{1-\delta} - \left(1-(p-2)\,s\right)^{1-\delta}\right)\right] dz$$

provided m is admissible, that is,

$$m \in \mathscr{A}_p := \mathscr{A}_p := \left\{ m \in [m_-(d,p), m_+(d,p)] \,:\, \tfrac{2}{p} \le m < 1 \text{ if } p < 4 \right\}$$

$$m_\pm(d,p) := \frac{1}{(d+2)\,p}\left(d\,p + 2 \pm \sqrt{d\,(p-1)\,\big(2\,d-(d-2)\,p\big)}\right)$$

The parameters δ and ζ are defined by

$$\delta := 1 + \tfrac{(m-1)\,p^2}{4\,(p-2)}$$

$$\zeta := \frac{(d+2)^2\,p^2\,m^2 - 2\,p\,(d+2)\,(d\,p+2)\,m + d^2\big(5\,p^2-12\,p+8\big) + 4\,d\,(3-2\,p)\,p + 4}{(1-m)\,(d+2)^2\,p^2}$$

Nonlinear diffusion flow: stability estimates

We consider the inverse function $\varphi_{m,p}^{-1}:\mathbb{R}^+\to[0,s_\star)$ and $\psi_{m,p}(s):=s-\varphi_{m,p}^{-1}(s)$. Exactly as in the case $m=1$, we have the *improved entropy – entropy production inequality*

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\ge d\;\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\;\varphi_{m,p}\left(\frac{\mathcal{E}_p[F]}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)\quad\forall\,F\in\mathrm{H}^1(\mathbb{S}^d)$$

Proposition

With above notations, $d\ge1$, $p\in(2,2^*)$ and $m\in\mathscr{A}_p$, we have

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-d\,\mathcal{E}_p[F]\ge d\;\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\;\psi_{m,p}\left(\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{d\;\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)\quad\forall\,F\in\mathrm{H}^1(\mathbb{S}^d)$$

The function $\varphi_{m,p}$ can be expressed in terms of the *incomplete* Γ *function* while $\psi_{m,p}$ is known only implicitly

(Improved) logarithmic Sobolev inequality

Where is the flow ?

▷ The case of the logarithmic Sobolev inequality is a limit case corresponding to $p = 2$ of the Gagliardo-Nirenberg-Sobolev inequalities for $p \neq 2$
▷ We use the fast diffusion flow ($m < 1$), porous medium flow ($m > 1$) and as a limit case the heat flow ($m = 1$) given by

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

where Δ is the Laplace-Beltrami operator on $\mathbb{R}^d$

... how do we relate ρ and F ?

Algebraic preliminaries

$$\mathrm{L}v := \mathrm{H}v - \frac{1}{d}\,(\Delta v)\,g_d \quad\text{and}\quad \mathrm{M}v := \frac{\nabla v\otimes\nabla v}{v} - \frac{1}{d}\,\frac{|\nabla v|^2}{v}\,g_d$$

With $a : b = a^{ij}\,b_{ij}$ and $\|a\|^2 := a : a$, we have

$$\|\mathrm{L}v\|^2 = \|\mathrm{H}v\|^2 - \tfrac{1}{d}\,(\Delta v)^2\,,\quad \|\mathrm{M}v\|^2 = \left\|\tfrac{\nabla v\otimes\nabla v}{v}\right\|^2 - \tfrac{1}{d}\,\tfrac{|\nabla v|^4}{v^2} = \tfrac{d-1}{d}\,\tfrac{|\nabla v|^4}{v^2}$$

A first identity

$$\int_{\mathbb{S}^d}\Delta v\,\frac{|\nabla v|^2}{v}\,d\mu = \frac{d}{d+2}\left(\frac{d}{d-1}\int_{\mathbb{S}^d}\|\mathrm{M}v\|^2\,d\mu - 2\int_{\mathbb{S}^d}\mathrm{L}v : \frac{\nabla v\otimes\nabla v}{v}\,d\mu\right)$$

Second identity (Bochner-Lichnerowicz-Weitzenböck formula

$$\int_{\mathbb{S}^d}(\Delta v)^2\,d\mu = \frac{d}{d-1}\int_{\mathbb{S}^d}\|\mathrm{L}v\|^2\,d\mu + d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu$$

An estimate

With $b=(\kappa+\beta-1)\,\frac{d-1}{d+2}$ and $c=\frac{d}{d+2}\,(\kappa+\beta-1)+\kappa\,(\beta-1)$

$$\mathscr{K}[v]:=\int_{\mathbb{S}^d}\left(\Delta v+\kappa\,\frac{|\nabla v|^2}{v}\right)\left(\Delta v+(\beta-1)\,\frac{|\nabla v|^2}{v}\right)d\mu$$
$$=\frac{d}{d-1}\,\|\mathrm{L}v-\,b\,\mathrm{M}v\|^2+\left(c-b^2\right)\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu+d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu$$

Let $\kappa=\beta\,(p-2)+1$. The condition $\gamma:=c-b^2\geq 0$ amounts to

$$\gamma=\tfrac{d}{d+2}\,\beta\,(p-1)+\big(1+\beta\,(p-2)\big)\,(\beta-1)-\left(\tfrac{d-1}{d+2}\,\beta\,(p-1)\right)^2$$

Lemma

$$\mathscr{K}[v]\geq\gamma\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu+d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu$$

Hence $\mathscr{K}[v]\geq d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu$ if $\gamma\geq 0$, which is a condition on β

... and finally, here is the flow

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u} \right)$$

Check: if $m = 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right)$, then $\rho = u^{\beta p}$ solves $\frac{\partial \rho}{\partial t} = \Delta \rho^m$

$$\frac{d}{dt}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} = 0\,, \quad \frac{d}{dt}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 2\,(p-2) \int_{\mathbb{S}^d} u^{-\,p\,(1-m)}\,|\nabla u|^2\,d\mu\,,$$

$$\frac{d}{dt}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = -\,2 \int_{\mathbb{S}^d} \left(\beta\, v^{\beta-1}\,\frac{\partial v}{\partial t} \right) \left(\Delta v^\beta \right) d\mu = -\,2\,\beta^2\,\mathscr{K}[v]$$

Lemma

Assume that $p \in (1, 2^*)$ and $m \in [m_-(d,p), m_+(d,p)]$. Then

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt} \left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[u] \right) \le -\,\gamma \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2}\,d\mu \le 0$$

Admissible parameters

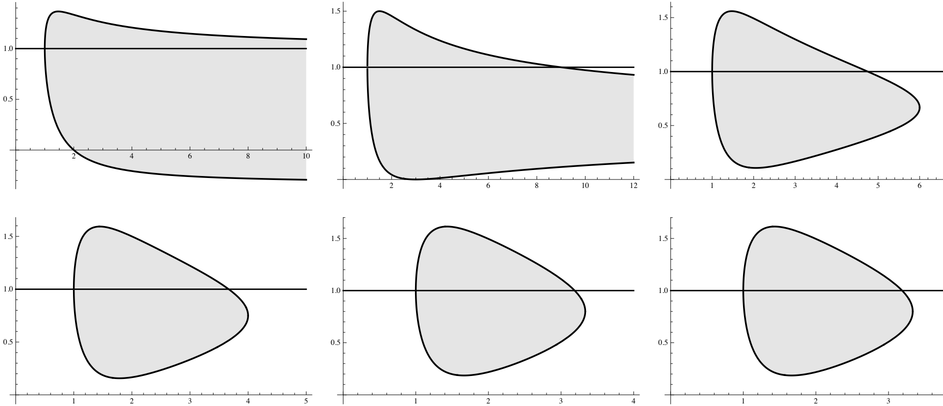


Figure: $d = 1, 2, 3$ (first line) and $d = 4, 5$ and 10 (second line): the curves $p \mapsto m_\pm(p)$ determine the admissible parameters (p, m) [JD, Esteban, 2019]

Inequalities and improved inequalities

From $\frac{1}{2\,\beta^2}\,\frac{d}{dt}\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[u]\right) \le -\gamma\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu \le 0$ and $\lim_{t\to+\infty}\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[u]\right) = 0$, we deduce the inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[u]$$

[Bakry-Emery, 1984], [Bidaut-Véron, Véron, 1991], [Beckner,1993]

... but we can do better

[Demange, 2008], [JD, Esteban, Kowalczyk, Loss]

Improved inequalities: flow estimates

With $\|u\|_{\mathrm{L}^p(\mathbb{S}^d)}=1$, consider the *entropy* and the *Fisher information*

$$\mathsf{e}:=\frac{1}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)\quad\text{and}\quad \mathsf{i}:=\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

Lemma

With $\delta:=\frac{2-(4-p)\,\beta}{2\,\beta\,(p-2)}$ *if* $p>2$, $\delta:=1$ *if* $p\in[1,2]$

$$(\mathsf{i}-d\,\mathsf{e})'\le\frac{\gamma\,\mathsf{i}\,\mathsf{e}'}{\big(1-(p-2)\,\mathsf{e}\big)^\delta}$$

$$\Longrightarrow\ \|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-d\,\mathcal{E}_p[F]\ge d\,\psi\left(\tfrac{1}{d}\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)\quad\forall\,F\in\mathrm{H}^1(\mathbb{S}^d)\ \text{s.t.}\ \|F\|_{\mathrm{L}^p(\mathbb{S}^d)}$$

With $\overline{F}:=\int_{\mathbb{S}^d}F\,d\mu$, this improves upon [Frank, 2022]

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-d\,\mathcal{E}_p[F]\ge\mathsf{c}_\star(d,p)\,\frac{\left(\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\|F-\overline{F}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^2}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\frac{d}{p-2}\,\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

Improved interpolation inequalities under orthogonality

Decomposition of $\mathrm{L}^2(\mathbb{S}^d, d\mu)$ into spherical harmonics

$$\mathrm{L}^2(\mathbb{S}^d, d\mu) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell$$

Let Π_k be the orthogonal projection onto $\bigoplus_{\ell=1}^{k} \mathcal{H}_\ell$

Theorem

Assume that $d \geq 1$, $p \in (1, 2^*)$ and $k \in \mathbb{N} \setminus \{0\}$ be an integer. For some $\mathscr{C}_{d,p,k} \in (0,1)$ with $\mathscr{C}_{d,p,k} \leq \mathscr{C}_{d,p,1} = \frac{2\,d - p\,(d-2)}{2\,(d+p)}$

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\, \mathcal{E}_p[F] \geq \mathscr{C}_{d,p,k} \int_{\mathbb{S}^d} \left|\nabla(\mathrm{Id} - \Pi_k)\, F\right|^2 d\mu$$

Proof

Using the Funk-Hecke formula as in [Lieb, 1983] and following [Beckner,1993], we learn that

$$\mathcal{E}_p[F] \le \sum_{j=1}^{\infty} \zeta_j(p) \int_{\mathbb{S}^d} |F_j|^2 \, d\mu \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

hold for any $p \in (1,2) \cup (2,2^*)$ with

$$\zeta_j(p) := \frac{\gamma_j\big(\frac{d}{p}\big) - 1}{p-2} \quad \text{and} \quad \gamma_j(x) := \frac{\Gamma(x)\,\Gamma(j+d-x)}{\Gamma(d-x)\,\Gamma(x+j)}$$

▷ Use convexity estimates and monotonicity properties of the coefficients

Proof of the main results

It remains to combine the *improved entropy – entropy production inequality* (carré du champ method) and the *improved interpolation inequalities under orthogonality constraints*

Theorem

Let $d \geq 1$ and $p \in (1, 2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d \, \mathcal{E}_p[F] \geq \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1) \, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

N.B. This relies on the computations of [Frank, 2022] (Bianchi-Egnell) made quantitative

The "far away" regime and the "neighborhood" of $\mathcal{M}$

▷ If $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \,/\, \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \ge \vartheta_0 > 0$, by the convexity of $\psi_{m,p}$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\;\psi_{m,p}\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right) \ge \frac{d}{\vartheta_0}\,\psi_{m,p}\left(\frac{\vartheta_0}{d}\right)\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

▷ From now on, we assume that $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta_0\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$, take $\|F\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$, learn that

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta := \frac{d\,\vartheta_0}{d-(p-2)\,\vartheta_0} > 0$$

from the standard interpolation inequality and deduce from the Poincaré inequality that

$$\frac{d-\vartheta}{d} < \left(\int_{\mathbb{S}^d} F\,d\mu\right)^2 \le 1$$

Partial decomposition on spherical harmonics

With $\mathscr{M} = \Pi_0 F$ and $\Pi_1 F = \varepsilon\,\mathscr{Y}$ where $\mathscr{Y}(x) = \sqrt{\frac{d+1}{d}}\,x\cdot\nu$ for some given $\nu\in\mathbb{S}^d$

$$F = \mathscr{M}\,(1+\varepsilon\,\mathscr{Y}+\eta\,G)$$

For some explicit constants $a_{p,d}$, $b_{p,d}$ and $c_{p,d}^{(\pm)}$

$$c_{p,d}^{(-)}\,\varepsilon^6 \le \|1+\varepsilon\,\mathscr{Y}\|_{\mathrm{L}^p(\mathbb{S}^d)}^p - \left(1+a_{p,d}\,\varepsilon^2+b_{p,d}\,\varepsilon^4\right) \le c_{p,d}^{(+)}\,\varepsilon^6$$

We apply to $u = 1+\varepsilon\,\mathscr{Y}$ and $r = \eta\,G$ the estimate

$$\begin{aligned}\|u+r\|_{\mathrm{L}^p(\mathbb{S}^d)}^2 &\le \|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^2\\ &\quad + \frac{2}{p}\,\|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^{2-p}\left(p\int_{\mathbb{S}^d} u^{p-1}\,r\,d\mu + \frac{p}{2}\,(p-1)\int_{\mathbb{S}^d} u^{p-2}\,r^2\,d\mu\right.\\ &\qquad\left. + \sum_{2<k<p} C_k^p\int_{\mathbb{S}^d} u^{p-k}\,|r|^k\,d\mu + K_p\int_{\mathbb{S}^d}|r|^p\,d\mu\right)\end{aligned}$$

Estimate various terms like $\int_{\mathbb{S}^d}(1+\varepsilon\,\mathscr{Y})^{p-1}\,G\,d\mu$, $\int_{\mathbb{S}^d}(1+\varepsilon\,\mathscr{Y})^{p-2}\,|G|^2\,d\mu$, $\int_{\mathbb{S}^d}(1+\varepsilon\,\mathscr{Y})^{p-k}\,|G|^k\,d\mu$, etc.

... conclusion

With explicit expressions for all constants we obtain

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d \, \mathcal{E}_p[F] \geq \mathscr{M}^2 \left(A \, \varepsilon^4 - B \, \varepsilon^2 \, \eta + C \, \eta^2 - \mathcal{R}_{p,d} \left(\vartheta^p + \vartheta^{5/2} \right) \right)$$

under the condition that $\varepsilon^2 + \eta^2 < \vartheta$...

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/
▷ Lectures

More related papers can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/list/
▷ Preprints and papers

For final versions, use `Dolbeault` as login and `Jean` as password

E-mail: dolbeault@ceremade.dauphine.fr

Thank you for your attention !