Sharp asymptotics for the subcritical Keller-Segel model

Jean Dolbeault

http://www.ceremade.dauphine.fr/~dolbeaul

Ceremade, CNRS and Université Paris-Dauphine

Nonlinear PDE day
Besançon (December 19, 2013)
1- Keller-Segel model: an introduction
1- An introduction
2- Functional framework and sharp asymptotics
3- Extensions, consequences

Warnings!

- Literature is huge
- Physics can be addressed in various ways: gravitation (Smoluchowski-Poisson) and statistics of gravitating systems, aggregation dynamics (sticky systems), biology (Patlak, Keller-Segel)
- Standard techniques have been reinvented many times: virial estimates, cumulated mass densities, matched asymptotics

⇒ some entry points in the literature

- do not specialize to radial solutions
- put emphasis on functional analysis
- insist on nonlinear evolution
- deal with the subcritical case: at least it gives some hint on how the bubble appears in the critical limit

J. Dolbeault
Sharp asymptotics for the subcritical Keller-Segel model
The parabolic-elliptic Keller – Segel system

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (u \nabla v) \quad x \in \mathbb{R}^2, \ t > 0 \\
-\Delta v &= u \quad x \in \mathbb{R}^2, \ t > 0 \\
u(\cdot, t = 0) &= n_0 \geq 0 \quad x \in \mathbb{R}^2
\end{aligned}
\]

We make the choice:

\[
\nu(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| \ u(t, y) \ dy
\]

and observe that

\[
\nabla \nu(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \ u(t, y) \ dy
\]

Mass conservation:

\[
\frac{d}{dt} \int_{\mathbb{R}^2} u(t, x) \ dx = 0
\]
Collapse [S. Childress, J.K. Percus 81] \( M = \int_{\mathbb{R}^2} n_0 \, dx > 8\pi \) and \( \int_{\mathbb{R}^2} |x|^2 \, n_0 \, dx < \infty \): blow-up in finite time

a solution \( u \) of

\[
\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla \nu)
\]

satisfies

\[
\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \, u(t, x) \, dx = - \int_{\mathbb{R}^2} 2x \cdot \nabla u \, dx + \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{2x \cdot (y-x)}{|x-y|^2} \, u(t, x) \, u(t, y) \, dx \, dy - 4M
\]

\[
= 4M - \frac{M^2}{2\pi} < 0 \quad \text{if} \quad M > 8\pi
\]
Blow-up and singular solutions: some results

- Formal asymptotic expansions in $\mathbb{R}^2$
  [Dejak, Lushnikov, Ovchinnikov, Sigal 2012], [Dejak, Egli, Lushnikov, Sigal 2013]

- Results in bounded domains: [Kavallaris, Souplet 2009]

- A first rigorous result in $\mathbb{R}^2$ (radial case)
  [Raphaël, Schweyer 2012-2013] stable chemotactic blow-up, universality of the bubble

- Other results in $\mathbb{R}^2$: [Montaru 2012-2013]

- Measure valued solutions: [Herrero, Velázquez 1997], [Luckhaus, Sugiyama, Velázquez 2012], [Seki, Sugiyama, Velázquez 2013]
  [Haškovec, Schmeiser 2009] the particle system, Wasserstein’s distance and free energy
  [Bedrossian, Masmoudi 2012] spectral gap and free energy
more results

1. [W. Jäger, S. Luckhaus], [A. Blanchet, JD, B. Perthame]
2. a review of related models: [D. Horstmann D (2003): ”From 1970 until present…”] Crowd modeling, social sciences
3. [L. Corrias et al.], [V. Calvez et al.] when other terms are taken into account. Limits: [P. Biler, L. Brandolese]
4. The $8\pi$ case: [A. Blanchet, J.A. Carrillo, N. Masmoudi], [E.A. Carlen, J. A. Carrillo, and M. Loss], [E.A. Carlen and A. Figalli],
5. Complex blow-up patterns [Y. Seki, Y. Sugiyama, J.J.L. Velázquez]
6. exploration of the blow-up by formal methods: [J.J.L. Velázquez, M.A. Herrero], [J.J.L. Velázquez et al.]... [S. Luckhaus, Y. Sugiyama, J.J.L. Velázquez 2012]
7. models with nonlinear diffusion terms: [Y. Sugiyama], [A. Blanchet and P. Laurençot],
8. models with prevention of overcrowding: [C. Schmeiser et al.]
9. models with more than one species: [E.E Espejo, K. Vilches, C. Carlos 13], [F. Dickstein 13]
10. and many more !... e.g. in bounded domains...
The super-critical range: life after blow-up
Regularization

Regularize the Poisson kernel

\[ (-\Delta)^{-1} \ast \rho(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x - y| + \varepsilon) \rho(y) \, dy \]


**Proposition (JD, C. Schmeiser 2009)**

For every \( \varepsilon > 0 \), the regularized problem has a global solution satisfying

\[ \| \rho^\varepsilon(\cdot, t) \|_{L^1(\mathbb{R}^2)} = \| \rho_I \|_{L^1(\mathbb{R}^2)} := M \]

\[ \| \rho^\varepsilon(t, \cdot) \|_{L^\infty(\mathbb{R}^2)} \leq c \left( 1 + \frac{1}{\varepsilon^2} \right) \]

with an \( \varepsilon \)-independent constant \( c \)
The nonlinear term

\[ m^\varepsilon(t, x) := \int_{\mathbb{R}^2} K^\varepsilon(x - y) \rho^\varepsilon(t, x) \rho^\varepsilon(t, y) dy \quad \text{with} \quad K^\varepsilon(x) = \frac{x \otimes x}{|x|(|x| + \varepsilon)} \]

Lemma (Poupaud)

The families \( \{\rho^\varepsilon(t)\}_{\varepsilon > 0} \) and \( \{m^\varepsilon(t)\}_{\varepsilon > 0} \) are tightly bounded locally uniformly in \( t \), and \( \{\rho^\varepsilon(t)\}_{\varepsilon > 0} \) is tightly equicontinuous in \( t \)

Tight boundedness and equicontinuity of \( \rho^\varepsilon(t) \Rightarrow \) compactness

\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x, y) \rho^\varepsilon(t, x) \rho^\varepsilon(t, y) \, dx \, dy \to \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x, y) \rho(t, x) \rho(t, y) \, dx \, dy
\]

\[
\int_{t_1}^{t_2} \int_{\mathbb{R}^2} \phi(t, x) m^\varepsilon(t, x) \, dx \, dt \to \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \phi(t, x) m(t, x) \, dx \, dt
\]

for all \( \phi \in C_b([t_1, t_2] \times \mathbb{R}^2) \)

Defect measure

\[ \nu(t, x) = m(t, x) - \int_{\mathbb{R}^2} K(x - y) \rho(t, x) \rho(t, y) \, dy, \quad K(x) = \frac{x \otimes x}{|x|^2} \]
Atomic support

The limit is characterized by the pair \((\rho, \nu)\), the atomic support of \(\rho\) is an at most countable set

Lemma (Poupaud 2002)

\[
\nu \text{ is symmetric, nonnegative, and satisfies } \\
\text{tr}(\nu(t,x)) \leq \sum_{a \in S_{\text{at}}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x - a)
\]

\(\mathcal{M}\): spaces of Radon measures
\(\mathcal{M}_1^+:\) subset of nonnegative bounded measures

\[
\mathcal{D}\mathcal{M}^+(I; \mathbb{R}^2) = \left\{ (\rho, \nu) : \rho(t) \in \mathcal{M}_1^+(\mathbb{R}^2) \ \forall t \in I, \ \nu \in \mathcal{M}(I \times \mathbb{R}^2)^{2 \times 2} \right\}
\]

- \(\rho\) is tightly continuous with respect to \(t\)
- \(\nu\) is a nonnegative, symmetric, matrix valued measure

\[
\text{tr}(\nu(t,x)) \leq \sum_{a \in S_{\text{at}}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x - a)
\]
Limiting problem

\[ \int_0^T \int_{\mathbb{R}^2} \phi(t,x) j[\rho, \nu](t,x) \, dx \, dt \]

\[ = -\frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^4} (\phi(t,x) - \phi(t,y)) K(x - y) \rho(t,x) \rho(t,y) \, dx \, dy \, dt \]

\[ - \frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2} \nu(t,x) \nabla \phi(t,x) \, dx \, dt \]

for \( \phi \in C^1((0, T) \times \mathbb{R}^2) \)

**Theorem (JD, C. Schmeiser 2009)**

*For every \( T > 0 \), \( \rho^\varepsilon \) converges tightly and uniformly in time to \( \rho(t) \) and there exists \( \nu(t) \) such that \((\rho, \nu) \in DM^+((0, T); \mathbb{R}^2)\) is a generalized solution of*

\[ \partial_t \rho + \nabla \cdot (j[\rho, \nu] - \nabla \rho) = 0 \]

\[ \rho(t = 0) = \rho_I \text{ holds in the sense of tight continuity} \]
Strong formulation (formal) : an \textit{ansatz}

\begin{equation}
\begin{aligned}
\rho &= \bar{\rho} + \hat{\rho}, \quad \hat{\rho}(t, x) = \sum_{n \in \mathbb{N}} M_n(t) \delta_n(t, x), \quad \delta_n(t, x) = \delta(x - x_n(t)) \\
(\rho, \nu) &\in \mathcal{D}\mathcal{M}^+((0, T); \mathbb{R}^2) \\
\implies \nu(t, x) &= \sum_{n \in \mathbb{N}} \nu_n(t) \delta_n(t, x), \quad \operatorname{tr}(\nu_n) \leq M_n^2
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\rho \nabla S_0[\bar{\rho} + \hat{\rho}] + \sum_n M_n \delta_n \nabla S_0 \left[ \bar{\rho} + \sum_{m \neq n} M_m \delta_m \right] + \frac{1}{4\pi} \sum_n M_n \nu_n \nabla \delta_n
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\partial_t \bar{\rho} + \nabla \cdot (\bar{\rho} \nabla S_0[\bar{\rho}] - \nabla \bar{\rho}) + \nabla \bar{\rho} \cdot \nabla S_0[\hat{\rho}]
+ \sum_n \delta_n (\dot{M}_n - \bar{\rho} M_n)
- \sum_n M_n \nabla \delta_n \left( \dot{x}_n - \nabla S_0 \left[ \bar{\rho} + \sum_{m \neq n} M_m \delta_m \right] \right)
+ \sum_n \left( \frac{1}{4\pi} \nu_n : \nabla^2 \delta_n - M_n \Delta \delta_n \right) = 0
\end{aligned}
\end{equation}
\[ \nu_n = 4\pi M_n \text{id} \]

As a consequence of \( \text{tr}(\nu_n) = 8\pi M_n \leq M_n^2 \), point masses have to be at least \( 8\pi \) (there is only a finite number of them)

\[
\partial_t \bar{\rho} + \nabla \cdot (\bar{\rho} \nabla S_0[\bar{\rho}] - \nabla \bar{\rho}) - \frac{1}{2\pi} \nabla \bar{\rho} \cdot \sum_n M_n \frac{x - x_n}{|x - x_n|^2} = 0
\]

\[
\dot{M}_n = \bar{\rho}(x = x_n) M_n
\]

\[
\dot{x}_n = \nabla S_0[\bar{\rho}](x = x_n) - \frac{1}{2\pi} \sum_{m \neq n} M_m \frac{x_n - x_m}{|x_n - x_m|^2}
\]

Note that
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^2} \bar{\rho} \, dx + \sum_n M_n \right) = 0
\]

... Comparison with Velázquez’ results.
Long time behaviour

Assume again

\[ \nu(t, x) = 4\pi \text{id} \sum_{a \in S_{at}(\rho(t))} \rho(t)\{a\} \delta(x - a) \]

and

\[ \int_{\mathbb{R}^2} |x|^2 \rho_I \, dx < \infty \]

With \( \hat{M} = \sum_{a \in S_{at}(\rho(t))} \rho(t)\{a\} \) and \( \bar{M} = M - \hat{M} \)

\[ \frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \rho \, dx = 4M - \frac{1}{2\pi} \int_{\mathbb{R}^4} (1 - \chi_D) \rho \otimes \rho \, dy \, dx - \frac{1}{2\pi} \int_{\mathbb{R}^2} \text{tr}(\nu) \, dx \]

\[ = \bar{M} \left( 4 - \frac{M}{2\pi} - \frac{\hat{M}}{2\pi} \right) - \frac{1}{2\pi} \sum_{a \neq b, a, b \in S_{at}(\rho(t))} \rho(t)\{a\} \rho(t)\{b\} \]

... compatible with Wasserstein’s framework

[Haškovec, Schmeiser 2009]
Local density profiles

For fixed $t$ and $a \in S_{at}(\rho(t))$, let $\varepsilon \xi = x - a$ and $\varepsilon^2 \rho^\varepsilon = R^\varepsilon$

$$\varepsilon^2 \partial_t R^\varepsilon + \nabla_\xi \cdot (R^\varepsilon \nabla_\xi S_1[R^\varepsilon] - \nabla_\xi R^\varepsilon) = 0$$

$R^\varepsilon$ is uniformly bounded, implying compactness of $\nabla_\xi S_1[R^\varepsilon]$. The $L^\infty$-weak* limit $R$ of $R^\varepsilon$ (take subsequences, formal) satisfies

$$\nabla_\xi \cdot (R \nabla_\xi S_1[R] - \nabla_\xi R) = 0$$

Observe that

$$\int_{\mathbb{R}^2} R(\xi) d\xi = \frac{1}{8\pi} \int_{\mathbb{R}^4} \frac{|\xi - \eta|}{|\xi - \eta| + 1} R(\xi) R(\eta) d\eta d\xi \leq \frac{1}{8\pi} \left( \int_{\mathbb{R}^2} R(\xi) d\xi \right)^2$$

This shows that either $R$ vanishes or its mass is not smaller than $8\pi$. 
Free energy (1/2)

\[
F_\varepsilon[\rho] := \int_{\mathbb{R}^2} \left( \rho \log \rho - \frac{1}{2} \rho S_\varepsilon[\rho] \right) \, dx
\]

\[
= \int_{\mathbb{R}^2} \rho \log \rho \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^4} \log(|x - y| + \varepsilon) \rho(x)\rho(y) \, dy \, dx
\]

and

\[
\frac{d}{dt} F_\varepsilon[\rho^\varepsilon] = -\int_{\mathbb{R}^2} \rho^\varepsilon |\nabla (\log \rho^\varepsilon - S_\varepsilon[\rho^\varepsilon])|^2 \, dx
\]

With an arbitrary \( a \in \mathbb{R}^2 \) and \( R(\xi) = \varepsilon^2 \rho(a + \varepsilon \xi) \) we have

\[
F_\varepsilon[\rho] = \left( 2M - \frac{M^2}{4\pi} \right) \log \frac{1}{\varepsilon} + F_1[R]
\]
Lemma

Let \( R \in L^1_+(\mathbb{R}^2) \) be radial, \( \int_{\mathbb{R}^2} \log(1 + |x|) R(x) \, dx < \infty \), \( M = \int_{\mathbb{R}^2} R \, dx \)

\[
\frac{1}{4\pi} \int_{\mathbb{R}^2} \log(1 + |x - y|) R(y) \, dy \geq \frac{M}{4\pi} \log |x| \quad \forall \, x \in \mathbb{R}^2
\]

\( L^1_{+,M} := \{ R \in L^1_+(\mathbb{R}^2) : \int_{\mathbb{R}^2} R \, d\xi = M \} \), \( J_M := \inf_{R \in L^1_{+,M}} F_1[R] \geq -\infty \)

Theorem

\( J_M = -\infty \) for \( M < 8\pi \), and \( J_M > -\infty \) for \( M \geq 8\pi \). If \( M > 8\pi \), there exists a radial nonincreasing minimizer
Keller-Segel model: the subcritical range
Existence and free energy

$M = \int_{\mathbb{R}^2} n_0 \, dx \leq 8\pi$: global existence \cite{JagerLuckhaus92}, \cite{JerrardPerthame04}, \cite{BlanchetJerrardPerthame06}

If $u$ solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u \,(\nabla (\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u \, dx - \frac{1}{2} \int_{\mathbb{R}^2} u \, v \, dx$$

satisfies

$$\frac{d}{dt} F[u(t, \cdot)] = - \int_{\mathbb{R}^2} u \, |\nabla (\log u) - \nabla v|^2 \, dx$$

(log HLS) inequality \cite{CarlenLoss92}: $F$ is bounded from below if $M \leq 8\pi$

... $M = 8\pi$ the critical case \cite{BlanchetCarrilloMasmoudi08}, \cite{BlanchetDuongJaffardPerthameTadmori10}
The existence setting for the subcritical regime

\[
\begin{cases}
\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2, \ t > 0 \\
-\Delta v = u & x \in \mathbb{R}^2, \ t > 0 \\
u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2
\end{cases}
\]

Initial conditions

\[n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2) \, dx), \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx), \quad M := \int_{\mathbb{R}^2} n_0(x) \, dx < 8 \pi\]

Global existence and mass conservation: \[M = \int_{\mathbb{R}^2} u(x, t) \, dx \ \forall \ t \geq 0\]

\[\nu = -\frac{1}{2\pi} \log |\cdot| * u\]
Time-dependent rescaling

\[ u(x, t) = \frac{1}{R^2(t)} n \left( \frac{x}{R(t)}, \tau(t) \right) \quad \text{and} \quad v(x, t) = c \left( \frac{x}{R(t)}, \tau(t) \right) \]

with \( R(t) = \sqrt{1 + 2t} \) and \( \tau(t) = \log R(t) \)

\[
\begin{align*}
\frac{\partial n}{\partial t} &= \Delta n - \nabla \cdot (n (\nabla c - x)) \\
c &= -\frac{1}{2\pi} \log |\cdot| * n \\
n(\cdot, t = 0) &= n_0 \geq 0
\end{align*}
\]

[A. Blanchet, JD, B. Perthame] Convergence in self-similar variables

\[
\lim_{t \to \infty} \| n(\cdot, \cdot + t) - n_\infty \|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t \to \infty} \| \nabla c(\cdot, \cdot + t) - \nabla c_\infty \|_{L^2(\mathbb{R}^2)} = 0
\]

means intermediate asymptotics in original variables:

\[
\| u(x, t) - \frac{1}{R^2(t)} n_\infty \left( \frac{x}{R(t)}, \tau(t) \right) \|_{L^1(\mathbb{R}^2)} \searrow 0
\]
The stationary solution in self-similar variables

\[ n_\infty = M \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2} \, dx} = -\Delta c_\infty, \quad c_\infty = -\frac{1}{2\pi} \log |\cdot| * n_\infty \]

- Radial symmetry [Y. Naito]
- Uniqueness [P. Biler, G. Karch, P. Laurençot, T. Nadzieja]
- As \(|x| \to +\infty\), \(n_\infty\) is dominated by \(e^{-(1-\epsilon)|x|^2/2}\) for any \(\epsilon \in (0, 1)\) [A. Blanchet, JD, B. Perthame]
- Bifurcation diagram of \(\|n_\infty\|_{L^\infty(\mathbb{R}^2)}\) as a function of \(M\)

\[ \lim_{M \to 0^+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0 \]

[D.D. Joseph, T.S. Lundgren] [JD, R. Stańczy]
(The bifurcation diagram will be shown later)
The stationary solution when mass varies

Figure: Representation of the solution appropriately scaled so that the $8\pi$ case appears as a limit (in red)
The free energy in self-similar variables

\[
\frac{\partial n}{\partial t} = \nabla \left[ n \left( \log n - x + \nabla c \right) \right]
\]

\[
F[n] := \int_{\mathbb{R}^2} n \log n \, dx + \int_{\mathbb{R}^2} \frac{1}{2} |x|^2 n \, dx - \frac{1}{2} \int_{\mathbb{R}^2} n \, c \, dx
\]

satisfies

\[
\frac{d}{dt} F[n(t, \cdot)] = - \int_{\mathbb{R}^2} n |\nabla (\log n) + x - \nabla c|^2 \, dx
\]

A last remark on $8\pi$ and scalings: $n^\lambda(x) = \lambda^2 n(\lambda x)$

\[
F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2) \, dx + \int_{\mathbb{R}^2} \frac{\lambda^{-2} - 1}{2} |x|^2 n \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) n(y) \log \frac{1}{\lambda} \, dx \, dy
\]

\[
F[n^\lambda] - F[n] = \left( 2M - \frac{M^2}{4\pi} \right) \log \lambda + \frac{\lambda^{-2} - 1}{2} \int_{\mathbb{R}^2} |x|^2 n \, dx
\]

$> 0$ if $M < 8\pi$
Keller-Segel with subcritical mass in self-similar variables

\[
\begin{cases}
\frac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n(\nabla c - x)) & x \in \mathbb{R}^2, t > 0 \\
c = -\frac{1}{2\pi} \log |\cdot| * n & x \in \mathbb{R}^2, t > 0 \\
n(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \\
\lim_{t \to \infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t \to \infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0 \\
n_\infty = M \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2} \, dx} = -\Delta c_\infty \quad \text{and} \quad c_\infty = -\frac{1}{2\pi} \log |\cdot| * n_\infty
\end{cases}
\]
First result: small mass case

**Theorem (A. Blanchet, JD, M. Escobedo, J. Fernández)**

*There exists a positive constant $M^*$ such that, for any initial data $n_0 \in L^2(n^{-1} \, dx)$ of mass $M < M^*$ satisfying the above assumptions, there is a unique solution $n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau, \infty) \times \mathbb{R}^2)$ for any $\tau > 0$.*

Moreover, there are two positive constants, $C$ and $\delta$, such that

$$
\int_{\mathbb{R}^2} |n(t, x) - n_\infty(x)|^2 \frac{dx}{n_\infty} \leq C \, e^{-\delta t} \quad \forall \ t > 0
$$

As a function of $M$, $\delta$ is such that $\lim_{M \to 0^+} \delta(M) = 1$. 

J. Dolbeault
Sharp asymptotics for the subcritical Keller-Segel model
Four steps proof

The condition $M \leq 8\pi$ is necessary and sufficient for the global existence of the solutions, but there are two extra smallness conditions in our proof:

- Uniform estimate: the *method of the trap*
- $L^p$ and $H^1$ estimates in the self-similar variables
- *Spectral gap* of a linearized operator $\mathcal{L}$
- Duhamel formula and nonlinear estimates
We can introduce two functions $f$ and $g$ such that

$$n = n_\infty (1 + f) \quad \text{and} \quad c = c_\infty (1 + g)$$

and rewrite the Keller-Segel model as

$$\frac{\partial f}{\partial t} = \mathcal{L} f + \frac{1}{n_\infty} \nabla (f n_\infty \nabla (c_\infty g))$$

where the linearized operator is

$$\mathcal{L} f = \frac{1}{n_\infty} \nabla \cdot (n_\infty \nabla (f - c_\infty g))$$

and

$$-\Delta (c_\infty g) = n_\infty f$$
2 – Keller-Segel model: functional framework and sharp asymptotics

- bifurcation diagrams
- spectrum of the linearized operator
- symmetrization
- nonlinear estimates
- rates of convergence for subcritical masses

... some preliminaries are needed
A parametrization of the solutions and the linearized operator

[J. Campos, JD]

$$-\Delta c = M \frac{e^{-\frac{1}{2}|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}|x|^2+c} \, dx}$$

Solve

$$-\phi'' - \frac{1}{r} \phi' = e^{-\frac{1}{2} r^2+\phi}, \quad r > 0$$

with initial conditions $\phi(0) = a, \phi'(0) = 0$ and get with $r = |x|$

$$M(a) := 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2} r^2+\phi_a} \, dx$$

$$n_a(x) = M(a) \frac{e^{-\frac{1}{2} r^2+\phi_a(r)}}{2\pi \int_{\mathbb{R}^2} r \, e^{-\frac{1}{2} r^2+\phi_a} \, dx} = e^{-\frac{1}{2} r^2+\phi_a(r)}$$
The mass can be computed as $M(a) = 2\pi \int_0^\infty n_a(r) r \, dr$. Plot of $a \mapsto M(a)/8\pi$.
The bifurcation diagram can be parametrized by $a \mapsto \left( \frac{1}{2\pi} M(a), \|c_a\|_{L^\infty(\mathbb{R}^d)} \right)$ with $\|c_a\|_{L^\infty(\mathbb{R}^d)} = c_a(0) = a - b(a)$ (cf. Keller-Segel system in a ball with no flux boundary conditions).
Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

Figure: The lowest eigenvalues of $-\mathcal{L} = (-\Delta)^{-1}(n_a f)$ (shown as a function of the mass) are $0, 1$ and $2$, thus establishing that the spectral gap of $-\mathcal{L}$ is $1$

[A. Blanchet, JD, M. Escobedo, J. Fernández], [J. Campos, JD], [V. Calvez, J.A. Carrillo], [J. Bedrossian, N. Masmoudi]
Spectral analysis in the functional framework determined by the relative entropy method
Simple eigenfunctions

**Kernel** Let $f_0 = \frac{\partial}{\partial M} c_\infty$ be the solution of

$$-\Delta f_0 = n_\infty f_0$$

and observe that $g_0 = f_0 / c_\infty$ is such that

$$\frac{1}{n_\infty} \nabla \cdot (n_\infty \nabla (f_0 - c_\infty g_0)) =: \mathcal{L} f_0 = 0$$

**Lowest non-zero eigenvalues** $f_1 := \frac{1}{n_\infty} \frac{\partial n_\infty}{\partial x_1}$ associated with $g_1 = \frac{1}{c_\infty} \frac{\partial c_\infty}{\partial x_1}$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L} f_1 = f_1$

With $D := x \cdot \nabla$, let $f_2 = 1 + \frac{1}{2} D \log n_\infty = 1 + \frac{1}{2n_\infty} D n_\infty$. Then

$$-\Delta (D c_\infty) + 2 \Delta c_\infty = D n_\infty = 2 (f_2 - 1) n_\infty$$

and so $g_2 := \frac{1}{c_\infty} (-\Delta)^{-1}(n_\infty f_2)$ is such that $-\mathcal{L} f_2 = 2 f_2$
Lemma (A. Blanchet, JD, B. Perthame)

**Sub-critical HLS inequality** [A. Blanchet, JD, B. Perthame]

\[
F[n] := \int_{\mathbb{R}^2} n \log \left( \frac{n}{n_\infty} \right) \, dx - \frac{1}{2} \int_{\mathbb{R}^2} (n - n_\infty)(c - c_\infty) \, dx \geq 0
\]

achieves its minimum for \( n = n_\infty \)

\[
Q_1[f] = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} F[n_\infty(1 + \epsilon f)] \geq 0
\]

if \( \int_{\mathbb{R}^2} f n_\infty \, dx = 0 \). Notice that \( f_0 \) generates the kernel of \( Q_1 \)

Lemma (J. Campos, JD)

**Poincaré type inequality** For any \( f \in H^1(\mathbb{R}^2, n_\infty \, dx) \) such that

\[
\int_{\mathbb{R}^2} f n_\infty \, dx = 0,
\]

we have

\[
\int_{\mathbb{R}^2} |\nabla (-\Delta)^{-1}(f n_\infty)|^2 n_\infty \, dx = \int_{\mathbb{R}^2} |\nabla (g c_\infty)|^2 n_\infty \, dx \leq \int_{\mathbb{R}^2} |f|^2 n_\infty \, dx
\]
... and eigenvalues

With $g$ such that $-\Delta(g c_\infty) = f n_\infty$, $Q_1$ determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1 f_2 n_\infty \, dx - \int_{\mathbb{R}^2} f_1 n_\infty (g_2 c_\infty) \, dx$$

on the orthogonal space to $f_0$ in $L^2(n_\infty \, dx)$

$$Q_2[f] := \int_{\mathbb{R}^2} |\nabla (f - g c_\infty)|^2 n_\infty \, dx \quad \text{with} \quad g = -\frac{1}{c_\infty} \frac{1}{2\pi} \log |\cdot| \ast (f n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L} f \rangle = Q_2[f] \quad \forall \ f \in \mathcal{D}(L_2)$$

**Lemma (J. Campos, JD)**

$L$ has pure discrete spectrum and its lowest eigenvalue is 1
Linearized Keller-Segel theory

\[ \mathcal{L} f = \frac{1}{n_\infty} \nabla \cdot (n_\infty \nabla (f - c_\infty g)) \]

**Corollary (J. Campos, JD)**

\[ \langle f, f \rangle \leq \langle \mathcal{L} f, f \rangle \]

The linearized problem takes the form

\[ \frac{\partial f}{\partial t} = \mathcal{L} f \]

where \( \mathcal{L} \) is a self-adjoint operator on the orthogonal of \( f_0 \) equipped with \( \langle \cdot , \cdot \rangle \). A solution of

\[ \frac{d}{dt} \langle f, f \rangle = -2 \langle \mathcal{L} f, f \rangle \]

has therefore exponential decay.
More on functional inequalities
A subcritical logarithmic HLS inequality

Recall that

**Lemma (A. Blanchet, JD, B. Perthame)**

*Sub-critical HLS inequality [A. Blanchet, JD, B. Perthame]*

\[ F[n] := \int_{\mathbb{R}^2} n \log \left( \frac{n}{n_\infty} \right) \, dx - \frac{1}{2} \int_{\mathbb{R}^2} (n - n_\infty)(c - c_\infty) \, dx \geq 0 \]

achieves its minimum for \( n = n_\infty \)

**Lemma (J. Campos, JD)**

*Poincaré type inequality* For any \( f \in H^1(\mathbb{R}^2, n_\infty \, dx) \) such that \( \int_{\mathbb{R}^2} f n_\infty \, dx = 0 \), we have

\[ \int_{\mathbb{R}^2} |\nabla (-\Delta)^{-1}(f \, n_\infty)|^2 n_\infty \, dx = \int_{\mathbb{R}^2} |\nabla (g \, c_\infty)|^2 n_\infty \, dx \leq \int_{\mathbb{R}^2} |f|^2 n_\infty \, dx \]

... Legendre duality
An Onofri type inequality

**Theorem (J. Campos, JD)**

For any $M \in (0, 8\pi)$, if $n_\infty = M \frac{e^{c_\infty} - \frac{1}{2} |x|^2}{\int_{\mathbb{R}^2} e^{c_\infty} - \frac{1}{2} |x|^2 \, dx}$ with $c_\infty = (-\Delta)^{-1} n_\infty$, $d\mu_M = \frac{1}{M} n_\infty \, dx$, we have the inequality

$$
\log \left( \int_{\mathbb{R}^2} e^{\phi} \, d\mu_M \right) - \int_{\mathbb{R}^2} \phi \, d\mu_M \leq \frac{1}{2M} \int_{\mathbb{R}^2} |\nabla \phi|^2 \, dx \quad \forall \phi \in D^{1,2}_0(\mathbb{R}^2)
$$

**Corollary (J. Campos, JD)**

The following Poincaré inequality holds

$$
\int_{\mathbb{R}^2} \left| \psi - \overline{\psi} \right|^2 n_\infty \, dx \leq \int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx \quad \text{where} \quad \overline{\psi} = \int_{\mathbb{R}^2} \psi \, d\mu_M
$$
Lemma (J. Campos, JD)

For any $f \in L^2(\mathbb{R}^2, n_{\infty} \, dx)$ such that $\int_{\mathbb{R}^2} f \, f_{\infty} \, n_{\infty} \, dx = 0$ holds, we have

$$- \frac{1}{2\pi} \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) \, n_{\infty}(x) \log |x - y| \, f(y) \, n_{\infty}(y) \, dx \, dy$$

$$\leq (1 - \varepsilon) \int_{\mathbb{R}^2} |f|^2 \, n_{\infty} \, dx$$

for some $\varepsilon > 0$, where $g_{\infty} = G_2 \ast (f \, n_{\infty})$ and, if $\int_{\mathbb{R}^2} f \, n_{\infty} \, dx = 0$ holds,

$$\int_{\mathbb{R}^2} |\nabla (g_{\infty})|^2 \, dx \leq (1 - \varepsilon) \int_{\mathbb{R}^2} |f|^2 \, n_{\infty} \, dx$$
Equivalence of the norms

As a consequence

$$\langle f, f \rangle := \int_{\mathbb{R}^2} |f|^2 n_\infty \, dx - \int_{\mathbb{R}^2} f n_\infty (g c_\infty) \, dx$$

is equivalent to

$$\int_{\mathbb{R}^2} |f|^2 n_\infty \, dx$$

under the condition that $\int_{\mathbb{R}^2} f f_0 n_\infty \, dx = 0$

A similar result is true in the critical case: [J. Bedrossian, N. Masmoudi], [P. Raphaël, R. Schweyer]
A spectral gap estimate

Theorem (J. Campos, JD)

For any function $f \in D(L_2)$, we have

$$\langle f, f \rangle = Q_1[f] \leq Q_2[f] = \langle f, Lf \rangle.$$
The nonlinear Keller-Segel model, a functional analysis approach
Exponential convergence for any mass $M \leq 8\pi$

If $n_{0,*}(\sigma)$ stands for the symmetrized function associated to $n_0$, assume that for any $s \geq 0$

$$(H) \exists \varepsilon \in (0, 8\pi - M) \text{ such that } \int_0^s n_{0,*}(\sigma) \, d\sigma \leq \int_{B(0, \sqrt{s/\pi})} n_{\infty,M+\varepsilon}(x) \, dx$$

**Theorem (J. Campos, JD)**

*Under the above assumption, if $n_0 \in L_+^2(n_{\infty}^{-1} \, dx)$ and $M := \int_{\mathbb{R}^2} n_0 \, dx < 8\pi$, then any solution with initial datum $n_0$ is such that*

$$\int_{\mathbb{R}^2} |n(t, x) - n_{\infty}(x)|^2 \frac{dx}{n_{\infty}(x)} \leq C \, e^{-2\,t} \quad \forall \; t \geq 0$$

*for some positive constant $C$, where $n_{\infty}$ is the unique stationary solution with mass $M$*
Sketch of the proof

- [J. Campos, JD] Uniform convergence of $n(t, \cdot)$ to $n_\infty$ can be established for any $M \in (0, 8\pi)$ by an adaptation of the symmetrization techniques of [J.I. Díaz, T. Nagai, J.M. Rakotoson]

- Uniform estimates (with no rates) easily result

- Estimates based on Duhammel’s formula inspired by [M. Escobedo, E. Zuazua] allow to prove uniform convergence

- Spectral estimates can be incorporated to the relative entropy approach

- Exponential convergence of the relative entropy follows
Step 1: symmetrization (1/2)

To any measurable function $u : \mathbb{R}^2 \mapsto [0, +\infty)$, we associate the distribution function defined by $\mu(t, \tau) := |\{u > \tau\}|$ and its decreasing rearrangement given by

$$u_*(s) = \inf\{\tau \geq 0 : \mu(t, \tau) \leq s\}.$$

For every measurable function $F : \mathbb{R}^+ \mapsto \mathbb{R}^+$, we have

$$\int_{\mathbb{R}^2} F(u) \, dx = \int_{\mathbb{R}^+} F(u_*) \, ds.$$

If $u \in W^{1,q}(0, T; L^p(\mathbb{R}^N))$ is a nonnegative function, with $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then $u_* \in W^{1,q}(0, T; L^p(0, \infty))$ and the formula

$$\int_0^{\mu(t, \tau)} \frac{\partial u_*}{\partial t}(t, \sigma) \, d\sigma = \int_{\{u(t, \cdot) > \tau\}} \frac{\partial u}{\partial t}(t, x) \, dx$$

holds for almost every $t \in (0, T)$ [J.I. Díaz, T. Nagai, J.M. Rakotoson].
Lemma

If \( n \) is a solution, then the function

\[
k(t, s) := \int_0^s n_*(t, \sigma) \, d\sigma
\]

satisfies \( k \in L^\infty ([0, +\infty) \times (0, +\infty)) \cap H^1 \left([0, +\infty); W^{1,p}_{loc}(0, +\infty)\right) \cap L^2 \left([0, +\infty); W^{2,p}_{loc}(0, +\infty)\right) \) and

\[
\begin{cases}
\frac{\partial k}{\partial t} - 4\pi s \frac{\partial^2 k}{\partial s^2} - (k + 2s) \frac{\partial k}{\partial s} \leq 0 & \text{a.e. in } (0, +\infty) \times (0, +\infty) \\
k(t, 0) = 0, \quad k(t, +\infty) = \int_{\mathbb{R}^2} n_0 \, d\chi & \text{for } t \in (0, +\infty) \\
k(0, s) = \int_0^s (n_0)_* \, d\sigma & \text{for } s \geq 0
\end{cases}
\]
Step 2: Uniform estimates

Proposition (J.I. Díaz, T. Nagai, J.M. Rakotoson)

Let \( f, g \) be two continuous functions on \( Q = \mathbb{R}^+ \times (0, +\infty) \) such that ...

\[
\begin{align*}
\frac{\partial f}{\partial t} - 4 \pi s \frac{\partial^2 f}{\partial s^2} - (f + 2s) \frac{\partial f}{\partial s} & \leq \frac{\partial g}{\partial t} - 4 \pi s \frac{\partial^2 g}{\partial s^2} - (g + 2s) \frac{\partial g}{\partial s} \quad \text{a.e. in } Q \\
f(t, 0) = 0 = g(t, 0) \quad &\text{and} \quad f(t, +\infty) \leq g(t, +\infty) \text{ for any } t \in (0, +\infty) \\
f(0, s) \leq g(0, s) \text{ for } s \geq 0, \text{ and } g(t, s) \geq 0 \text{ in } Q
\end{align*}
\]

then \( f \leq g \) on \( Q \)

Corollary

Assume that \( n_0 \in L^2_+ (n_\infty^{-1} \, dx) \) satisfies (H) and \( M := \int_{\mathbb{R}^2} n_0 \, dx < 8 \pi \).

Then there exist positive constants \( C_1 = C_1(M, p) \) and \( C_2 = C_2(M, p) \) such that

\[
\|n\|_{L^p(\mathbb{R}^2)} \leq C_1 \quad \text{and} \quad \|\nabla c\|_{L^\infty(\mathbb{R}^2)} \leq C_2
\]
Consider the kernel associated to the Fokker-Planck equation

\[ K(t, x, y) := \frac{1}{2\pi (1 - e^{-2t})} e^{-\frac{1}{2} \frac{|x - e^{-t}y|^2}{1 - e^{-2t}}} \quad x \in \mathbb{R}^2, \; y \in \mathbb{R}^2, \; t > 0 \]

If \( n \) is a solution, then

\[ n(t, x) = \int_{\mathbb{R}^2} K(t, x, y) n_0(y) \, dy + \int_0^t \int_{\mathbb{R}^2} \nabla_x K(t-s, x, y) \cdot n(s, y) \nabla c(s, y) \, dy \, ds \]

**Corollary**

Assume that \( n \) is a solution. Then

\[ \lim_{t \to \infty} \| n(t, \cdot) - n_\infty \|_{L^p(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t \to \infty} \| \nabla c(t, \cdot) - \nabla c_\infty \|_{L^q(\mathbb{R}^d)} = 0 \]

for any \( p \in [1, \infty] \) and any \( q \in [2, \infty] \)
Step 4: Spectral estimates can be incorporated

With $Q_1[f] = \langle f, f \rangle$ and $Q_2[f] = \langle f, \mathcal{L} f \rangle$

- For any function $f$ in the orthogonal of the kernel of $\mathcal{L}$, we have
  $$Q_1[f] \leq Q_2[f]$$

- For any radial function $f \in \mathcal{D}(L_2)$, we have
  $$2Q_1[f] \leq Q_2[f]$$

Cf. [V. Calvez, J.A. Carrillo]
Step 5: Exponential convergence of the relative entropy

\[ \frac{\partial f}{\partial t} = \mathcal{L} f - \frac{1}{n_\infty} \nabla [n_\infty f \nabla (g c_\infty)] \]

\[ \frac{d}{dt} Q_1[f(t, \cdot)] = -2 Q_2[f(t, \cdot)] + \int_{\mathbb{R}^2} \nabla (f - g c_\infty) f n_\infty \cdot \nabla (g c_\infty) \, dx \]

\[ \frac{d}{dt} Q_1[f(t, \cdot)] \leq -2 Q_2[f(t, \cdot)] + \delta(t, \varepsilon) \sqrt{Q_1[f(t, \cdot)]} Q_2[f(t, \cdot)] \]

\[ Q_1[f(t, \cdot)] \leq Q \quad \forall \ t \geq 0 \]

\[ \frac{d}{dt} Q_1[f(t, \cdot)] \leq -Q_1[f(t, \cdot)] \left[ 2 - \delta(t, \varepsilon) \left( Q_1[f(t, \cdot)] \right)^{\frac{1-\varepsilon}{2-\varepsilon}} + Q_1[f(t, \cdot)]^{\frac{1}{2+\varepsilon}} \right] \]

As a consequence, we finally get that

\[ \limsup_{t \to \infty} e^{2t} Q_1[f(t, \cdot)] < \infty \]
Some key ideas

- Lyapunov / Entropy functionals and functional inequalities
- Linearization and best constants
- Functional framework for linearized operators can be deduced from the entropy functional

[ G. Egaña, S. Mischler, 2013]
- weak notion of solution (based on free energy estimates)
- uniqueness, smoothing
- linearized and nonlinear stability in rescaled variables and exponential convergence under weaker assumptions - sharp rates in $L^{4/3}(\mathbb{R}^2)$
3- Extensions, consequences

- parabolic-parabolic models
  [JD, G. Jankowiak, P. Markowich]
  [G. Jankowiak]

- improved functional inequalities
  [JD, G. Jankowiak]
Parabolic-parabolic models
Parabolic-parabolic models for crowd motion

[JD, G. Jankowiak, P. Markowich] A model for crowd motion

\[ \partial_t \rho = \Delta \rho - \nabla \cdot (\rho (1 - \rho) \nabla D) \]

\[ \partial_t D = \kappa \Delta D - \delta D + g(\rho) \]

on a bounded domain \( \Omega \) with no-flux boundary conditions

\[ (\nabla \rho - \rho (1 - \rho) \nabla D) \cdot \nu = 0 \quad \text{on} \quad \partial \Omega \]

Model (I): \( g(\rho) = \rho (1 - \rho) \) or Model (II): \( g(\rho) = \rho \)

Any stationary solution solves

\[ \nabla \rho - \rho (1 - \rho) \nabla D = 0 \quad \text{on} \quad \Omega \quad \iff \quad \rho = \frac{1}{1 + e^{-\phi}} \]

where \( \phi = D - \phi_0 \) and \( \int_\Omega \frac{1}{1+e^{\phi_0-D}} \, dx = M \)

\[ -\kappa \Delta \phi + \delta (\phi + \phi_0) - f(\phi) = 0 \quad \text{on} \quad \Omega \]

with homogeneous Neumann boundary conditions
Model (I), $d = 1, \delta = 10^{-3}$
Model (I), $\kappa = 5 \times 10^{-4}$, $\delta = 10^{-3}$
Model (II), $\kappa = 10^{-2}, \delta = 10^{-3}$
Parabolic-parabolic Keller-Segel model

[G. Jankowiak] Analysis of the stability of self-similar solutions, including for masses larger than $8\pi$
Logarithmic Hardy-Littlewood-Sobolev and Onofri inequalities: duality, flows
Critical case: the logarithmic HLS inequality

The classical logarithmic Hardy-Littlewood-Sobolev (logHLS) in $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} n \log \left( \frac{n}{M} \right) \, dx + \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) n(y) \log |x-y| \, dx \, dy + M \left( 1 + \log \pi \right) \geq 0$$

Equality is achieved by

$$\mu(x) := \frac{1}{\pi (1 + |x|^2)^2} \quad \forall \ x \in \mathbb{R}^2$$

Notice that $-\Delta \log \mu = 8 \pi \mu$ can be inverted as

$$(-\Delta)^{-1} \mu = \frac{1}{8 \pi} \log (\pi \mu)$$

With $M = 8 \pi$ and $n_{\infty} = 8 \pi \mu$ (logHLS) can be rewritten as

$$\int_{\mathbb{R}^2} n \log \left( \frac{n}{n_{\infty}} \right) \, dx \geq \frac{1}{2} \int_{\mathbb{R}^2} (n - n_{\infty}) (-\Delta)^{-1} (n - n_{\infty}) \, dx$$
Subcritical case: the logarithmic HLS inequality

The minimum of

$$\int_{\mathbb{R}^2} n \log \left( \frac{n}{M} \right) \, dx + \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) \, n(y) \log |x-y| \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \, n \, dx$$

is achieved by the stationary solution $n_\infty$ of the Keller-Segel model and can again be written as

$$\int_{\mathbb{R}^2} n \log \left( \frac{n}{n_\infty} \right) \, dx \geq \frac{1}{2} \int_{\mathbb{R}^2} (n - n_\infty) \, (-\Delta)^{-1} (n - n_\infty) \, dx$$
Critical case: Legendre duality

Onofri’s inequality

\[ F_1[u] := \log \left( \int_{\mathbb{R}^2} e^u \, d\mu \right) \leq \frac{1}{16 \pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} u \, \mu \, dx =: F_2[u] \]

By duality: \( F_i^*[v] = \sup \left( \int_{\mathbb{R}^2} v \, u \, d\mu - F_i[u] \right) \) we can relate Onofri’s inequality with \( \log \text{HLS} \)

For any \( v \in L^1_+(\mathbb{R}^2) \) with \( \int_{\mathbb{R}^2} v \, dx = 1 \), such that \( v \log v \) and \( (1 + \log |x|^2) \) \( v \in L^1(\mathbb{R}^2) \), we have

\[ F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v \log \left( \frac{v}{\mu} \right) \, dx - 4 \pi \int_{\mathbb{R}^2} (v - \mu) \, (-\Delta)^{-1} (v - \mu) \, dx \geq 0 \]

The same property holds in the subcritical case
The two-dimensional case: (logHLS) and flows

[E. Carlen, J. Carrillo, M. Loss 2010]

\[
H_2[\nu] := \int_{\mathbb{R}^2} (\nu - \mu) (-\Delta)^{-1}(\nu - \mu) \, dx - \frac{1}{4\pi} \int_{\mathbb{R}^2} \nu \log \left( \frac{\nu}{\mu} \right) \, dx
\]

is related to Gagliardo-Nirenberg inequalities if \( \nu_t = \Delta \sqrt{\nu} \)

Alternatively, assume that \( \nu \) is a positive solution of

\[
\frac{\partial \nu}{\partial t} = \Delta \log \left( \frac{\nu}{\mu} \right) \quad t > 0, \quad x \in \mathbb{R}^2
\]

**Proposition (JD 2011)**

If \( \nu \) is a solution with nonnegative initial datum \( \nu_0 \) in \( L^1(\mathbb{R}^2) \) such that \( \int_{\mathbb{R}^2} \nu_0 \, dx = 1 \), \( \nu_0 \log \nu_0 \in L^1(\mathbb{R}^2) \) and \( \nu_0 \log \mu \in L^1(\mathbb{R}^2) \), then

\[
\frac{d}{dt} H_2[\nu(t, \cdot)] = \frac{1}{16 \pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx - \int_{\mathbb{R}^2} (e^{\frac{u}{2}} - 1) u \, d\mu \geq F_2[u] - F_1[u]
\]

with \( \log(\nu/\mu) = u/2 \)
Hierarchies of inequalities, improved inequalities

Theorem (JD, Jankowiak 2013)

If \( d \geq 3 \), with \( q = \frac{d+2}{d-2} \)

\[
S_d \| u^q \|_{L^\frac{2d}{d+2}(\mathbb{R}^d)}^2 - \int_{\mathbb{R}^d} u^q (-\Delta)^{-1} u^q \, dx
\]

\[
\leq S_d \| u^q \|_{L^\frac{2}{d-2}(\mathbb{R}^d)}^\frac{4}{d-2} \left[ S_d \| \nabla u \|_{L^2(\mathbb{R}^d)}^2 - \| u \|_{L^2(\mathbb{R}^d)}^2 \right]
\]

\( \forall u \in H^1(\mathbb{R}^d) \)

and, when \( d = 2 \), for any function \( f \in \mathcal{D}(\mathbb{R}^2) \)

\[
\left( \int_{\mathbb{R}^2} e^f \, d\mu \right)^2 - 4\pi \int_{\mathbb{R}^2} e^f \mu (-\Delta)^{-1} e^f \mu \, dx
\]

\[
\leq \left( \int_{\mathbb{R}^2} e^f \, d\mu \right)^2 \left[ \frac{1}{16\pi} \| \nabla f \|_{L^2(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^2} f \, d\mu - \log \left( \int_{\mathbb{R}^2} e^f \, d\mu \right) \right]
\]
Thank you for your attention!