Sharp asymptotics for the subcritical Keller-Segel model

Jean Dolbeault

http://www.ceremade.dauphine.fr/∼dolbeaul

Ceremade, CNRS and Université Paris-Dauphine

Nonlinear PDE day
Besançon (December 19, 2013)

1- Keller-Segel model: an introduction

Warnings !

1. Literature is huge
2. Physics can be addressed in various ways: gravitation (Smoluchowski-Poisson) and statistics of gravitating systems, aggregation dynamics (sticky systems), biology (Patlak, Keller-Segel)
3. Standard techniques have been reinvented many times: virial estimates, cumulated mass densities, matched asymptotics

$\Longrightarrow$ some entry points in the literature

- do not specialize to radial solutions
- put emphasis on functional analysis
- insist on nonlinear evolution
- deal with the subcritical case: at least it gives some hint on how the *bubble* appears in the critical limit

The parabolic-elliptic Keller – Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \ t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \ t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| \, u(t,y) \, dy$$

and observe that

$$\nabla v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \, u(t,y) \, dy$$

Mass conservation: $\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^2} u(t,x) \, dx = 0$

Blow-up: the virial computation

Collapse [S. Childress, J.K. Percus 81] $M = \int_{\mathbb{R}^2} n_0\, dx > 8\pi$ and $\int_{\mathbb{R}^2} |x|^2\, n_0\, dx < \infty$: blow-up in finite time
a solution u of

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u\, \nabla v)$$

satisfies

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2\, u(t,x)\, dx$$

$$= \underbrace{- \int_{\mathbb{R}^2} 2\, x \cdot \nabla u\, dx}_{-4\, M} + \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \underbrace{\tfrac{2x\cdot(y-x)}{|x-y|^2}\, u(t,x)\, u(t,y)\, dx\, dy}_{\frac{(x-y)\cdot(y-x)}{|x-y|^2}\, u(t,x)\, u(t,y)\, dx\, dy}$$

$$= 4\, M - \frac{M^2}{2\pi} < 0 \quad \text{if} \quad M > 8\pi$$

Blow-up and singular solutions: some results

1. Formal asymptotic expansions in $\mathbb{R}^2$
   [Herrero, Velázquez 1997], [Chavanis, Sire 2002-2005], [Campos, PhD thesis, 2012]
   [Dejak, Lushnikov, Ovchinnikov, Sigal 2012], [Dejak, Egli, Lushnikov, Sigal 2013]
2. Results in bounded domains: [Kavallaris, Souplet 2009]
3. A first rigorous result in $\mathbb{R}^2$ (radial case)
   [Raphaël, Schweyer 2012-2013] stable chemotactic blow-up, universality of the bubble
4. Other results in $\mathbb{R}^2$: [Montaru 2012-2013]
5. Measure valued solutions: [Herrero, Velázquez 1997], [Luckhaus, Sugiyama, Velázquez 2012], [Seki, Sugiyama, Velázquez 2013]
   [Haškovec, Schmeiser 2009] the particle system, Wasserstein's distance and free energy
   [Bedrossian, Masmoudi 2012] spectral gap and free energy

more results

1. [W. Jäger, S. Luckhaus], [A. Blanchet, JD, B. Perthame]
2. a review of related models: [D. Horstmann D (2003): "From 1970 until present..."] Crowd modeling, social sciences
3. [L. Corrias et al.], [V. Calvez et al.] when other terms are taken into account. Limits: [P. Biler, L. Brandolese]
4. The 8π case: [A. Blanchet, J.A. Carrillo, N. Masmoudi], [E.A. Carlen, J. A. Carrillo, and M. Loss], [E.A. Carlen and A. Figalli],
5. Complex blow-up patterns [Y. Seki, Y. Sugiyama, J.J.L. Velázquez]
6. exploration of the blow-up by formal methods: [J.J.L. Velázquez, M.A. Herrero], [J.J.L. Velázquez et al.]... [S. Luckhaus, Y. Sugiyama, J.J.L. Velázquez 2012]
7. models with nonlinear diffusion terms: [Y. Sugiyama], [A. Blanchet and P. Laurençot],
8. models with prevention of overcrowding: [C. Schmeiser et al.]
9. models with more than one species: [E.E Espejo, K. Vilches, C. Carlos 13], [F. Dickstein 13]
10. and many more !... e.g. in bounded domains...

The super-critical range: life after blow-up

Regularization

Regularize the Poisson kernel

$$(-\Delta)_\varepsilon^{-1} * \rho\,(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|+\varepsilon)\,\rho(y)\,dy$$

[F. Poupaud, Diagonal defect measures, adhesion dynamics and Euler equations, Meth. Appl. Anal. **9** (2002), pp. 533–561]

Proposition (JD, C. Schmeiser 2009)

For every $\varepsilon > 0$, the regularized problem has a global solution satisfying

$$\|\rho^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^2)} = \|\rho_I\|_{L^1(\mathbb{R}^2)} := M$$

$$\|\rho^\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le c\left(1+\frac{1}{\varepsilon^2}\right)$$

with an ε-independent constant c

The nonlinear term

$$m^\varepsilon(t,x) := \int_{\mathbb{R}^2} \mathcal{K}^\varepsilon(x-y)\,\rho^\varepsilon(t,x)\,\rho^\varepsilon(t,y)dy \quad \text{with } \mathcal{K}^\varepsilon(x) = \frac{x^{\otimes 2}}{|x|(|x|+\varepsilon)}$$

Lemma (Poupaud)

The families $\{\rho^\varepsilon(t)\}_{\varepsilon>0}$ and $\{m^\varepsilon(t)\}_{\varepsilon>0}$ are tightly bounded locally uniformly in t, and $\{\rho^\varepsilon(t)\}_{\varepsilon>0}$ is tightly equicontinuous in t

Tight boundedness and equicontinuity of $\rho^\varepsilon(t) \Longrightarrow$ compactness

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \phi(x,y)\,\rho^\varepsilon(t,x)\,\rho^\varepsilon(t,y)\,dx\,dy \to \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \phi(x,y)\,\rho(t,x)\,\rho(t,y)\,dx\,dy$$

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \phi(t,x)\,m^\varepsilon(t,x)\,dx\,dt \to \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \phi(t,x)\,m(t,x)\,dx\,dt$$

for all $\phi \in C_b([t_1,t_2]\times\mathbb{R}^2)$

Defect measure

$$\nu(t,x) = m(t,x) - \int_{\mathbb{R}^2} \mathcal{K}(x-y)\,\rho(t,x)\,\rho(t,y)\,dy\,, \quad \mathcal{K}(x) = \frac{x^{\otimes 2}}{|x|^2}$$

Atomic support

The limit is characterized by the pair (ρ, ν), the atomic support of ρ is an at most countable set

Lemma (Poupaud 2002)

ν is symmetric, nonnegative, and satisfies

$$\mathrm{tr}(\nu(t,x)) \leq \sum_{a \in S_{at}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x-a)$$

$\mathcal{M}$: spaces of Radon measures
$\mathcal{M}_1^+$: subset of nonnegative bounded measures

$$\mathcal{DM}^+(I;\mathbb{R}^2) \;=\; \Big\{(\rho,\nu):\; \rho(t) \in \mathcal{M}_1^+(\mathbb{R}^2)\ \forall t \in I,\ \nu \in \mathcal{M}(I \times \mathbb{R}^2)^{2\times 2}$$

ρ is tightly continuous with respect to t

ν is a nonnegative, symmetric, matrix valued measure

$$\mathrm{tr}(\nu(t,x)) \leq \sum_{a \in S_{at}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x-a)\Big\}$$

Limiting problem

$$\int_0^T \int_{\mathbb{R}^2} \phi(t,x)\, j[\rho,\nu](t,x)\; dx\, dt$$
$$= -\frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^4} (\phi(t,x) - \phi(t,y))\, K(x-y)\, \rho(t,x)\, \rho(t,y)\; dx\, dy\, dt$$
$$- \frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2} \nu(t,x) \nabla\phi(t,x)\; dx\, dt$$

for $\phi \in C^1_b((0,T) \times \mathbb{R}^2)$

Theorem (JD, C. Schmeiser 2009)

For every $T > 0$, ρ^ε converges tightly and uniformly in time to $\rho(t)$ and there exists $\nu(t)$ such that $(\rho,\nu) \in \mathcal{DM}^+((0,T);\mathbb{R}^2)$ is a generalized solution of

$$\partial_t \rho + \nabla \cdot (j[\rho,\nu] - \nabla\rho) = 0$$

$\rho(t=0) = \rho_I$ holds in the sense of tight continuity

Strong formulation (formal) : an *ansatz*

- $\rho = \overline{\rho} + \hat{\rho},\ \hat{\rho}(t,x) = \sum_{n \in N} M_n(t)\,\delta_n(t,x),\ \delta_n(t,x) = \delta(x - x_n(t))$
- $(\rho, \nu) \in \mathcal{DM}^+((0,T);\mathbb{R}^2)$

$$\Longrightarrow\ \nu(t,x) = \sum_{n \in N} \nu_n(t)\,\delta_n(t,x),\ \ \mathrm{tr}(\nu_n) \le M_n^2$$

$$j[\rho,\nu] = \overline{\rho}\,\nabla S_0[\overline{\rho}+\hat{\rho}] + \sum_n M_n\,\delta_n\,\nabla S_0\left[\overline{\rho} + \sum_{m \neq n} M_m\,\delta_m\right] + \frac{1}{4\pi}\sum_n M_n\,\nu_n\,\nabla\delta_n$$

$$\begin{aligned}
\partial_t\overline{\rho} + \nabla\cdot(\overline{\rho}\,\nabla S_0[\overline{\rho}] - \nabla\overline{\rho}) + \nabla\overline{\rho}\cdot\nabla S_0[\hat{\rho}] & \\
+ \sum_n \delta_n(\dot{M}_n - \overline{\rho}\,M_n) & \\
- \sum_n M_n\,\nabla\delta_n\left(\dot{x}_n - \nabla S_0\left[\overline{\rho} + \textstyle\sum_{m\neq n} M_m\,\delta_m\right]\right) & \\
+ \sum_n \left(\tfrac{1}{4\pi}\nu_n : \nabla^2\delta_n - M_n\,\Delta\delta_n\right) &= 0
\end{aligned}$$

$$\nu_n = 4\pi M_n \, \mathrm{id}$$

As a consequence of $\mathrm{tr}(\nu_n) = 8\pi M_n \leq M_n^2$, point masses have to be at least 8π (there is only a finite number of them)

$$\partial_t \overline{\rho} + \nabla \cdot (\overline{\rho} \, \nabla S_0[\overline{\rho}] - \nabla \overline{\rho}) - \frac{1}{2\pi} \nabla \overline{\rho} \cdot \sum_n M_n \, \frac{x - x_n}{|x - x_n|^2} = 0$$

$$\dot{M}_n = \overline{\rho}(x = x_n) \, M_n$$

$$\dot{x}_n = \nabla S_0[\overline{\rho}](x = x_n) - \frac{1}{2\pi} \sum_{m \neq n} M_m \frac{x_n - x_m}{|x_n - x_m|^2}$$

Note that $\frac{d}{dt} \left(\int_{\mathbb{R}^2} \overline{\rho} \, dx + \sum_n M_n \right) = 0$

... Comparison with Velázquez' results

Long time behaviour

Assume again

$$\nu(t,x) = 4\pi \,\mathrm{id} \sum_{a \in S_{at}(\rho(t))} \rho(t)(\{a\})\, \delta(x-a)$$

and

$$\int_{\mathbb{R}^2} |x|^2 \rho_I \, dx < \infty$$

With $\hat{M} = \sum_{a \in S_{at}(\rho(t))} \rho(t)(\{a\})$ and $\bar{M} = M - \hat{M}$

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \rho \, dx = 4M - \frac{1}{2\pi} \int_{\mathbb{R}^4} (1-\chi_D) \rho \otimes \rho \, dy \, dx - \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{tr}(\nu) \, dx$$

$$= \bar{M} \left(4 - \frac{M}{2\pi} - \frac{\hat{M}}{2\pi} \right) - \frac{1}{2\pi} \sum_{a \neq b,\ a,b \in S_{at}(\rho(t))} \rho(t)(\{a\})\, \rho(t)(\{b\})$$

... compatible with Wasserstein's framework [Haškovec, Schmeiser 2009]

Local density profiles

For fixed t and $a \in S_{at}(\rho(t))$, let $\varepsilon\xi = x - a$ and $\varepsilon^2 \rho^\varepsilon = R^\varepsilon$

$$\varepsilon^2 \partial_t R^\varepsilon + \nabla_\xi \cdot (R^\varepsilon \nabla_\xi S_1[R^\varepsilon] - \nabla_\xi R^\varepsilon) = 0$$

R^ε is uniformly bounded, implying compactness of $\nabla_\xi S_1[R^\varepsilon]$. The L^∞-weak* limit R of R^ε (take subsequences, formal) satisfies

$$\nabla_\xi \cdot (R \nabla_\xi S_1[R] - \nabla_\xi R) = 0$$

Observe that

$$\int_{\mathbb{R}^2} R(\xi) d\xi = \frac{1}{8\pi} \int_{\mathbb{R}^4} \frac{|\xi - \eta|}{|\xi - \eta| + 1} R(\xi) R(\eta) d\eta d\xi \le \frac{1}{8\pi} \left(\int_{\mathbb{R}^2} R(\xi) d\xi \right)^2$$

This shows that either R vanishes or its mass is not smaller than 8π

Free energy (1/2)

$$F_\varepsilon[\rho] := \int_{\mathbb{R}^2} \left(\rho \log \rho - \frac{1}{2} \rho S_\varepsilon[\rho] \right) dx$$
$$= \int_{\mathbb{R}^2} \rho \log \rho \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^4} \log(|x-y|+\varepsilon) \rho(x) \rho(y) dy \, dx$$

and

$$\frac{d}{dt} F_\varepsilon[\rho^\varepsilon] = - \int_{\mathbb{R}^2} \rho^\varepsilon |\nabla(\log \rho^\varepsilon - S_\varepsilon[\rho^\varepsilon])|^2 dx$$

With an arbitrary $a \in \mathbb{R}^2$ and $R(\xi) = \varepsilon^2 \rho(a + \varepsilon \xi)$ we have

$$F_\varepsilon[\rho] = \left(2M - \frac{M^2}{4\pi} \right) \log \frac{1}{\varepsilon} + F_1[R]$$

Free energy (2/2)

Lemma

Let $R \in L^1_+(\mathbb{R}^2)$ be radial, $\int_{\mathbb{R}^2} \log(1+|x|)\, R(x)\, dx < \infty$, $M = \int_{\mathbb{R}^2} R\, dx$

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \log(1+|x-y|)\, R(y)\, dy \geq \frac{M}{4\pi} \log|x| \quad \forall\, x \in \mathbb{R}^2$$

$L^1_{+,M} := \{R \in L^1_+(\mathbb{R}^2) : \int_{\mathbb{R}^2} R\, d\xi = M\}$, $J_M := \inf_{R \in L^1_{+,M}} F_1[R] \geq -\infty$

Theorem

$J_M = -\infty$ for $M < 8\pi$, and $J_M > -\infty$ for $M \geq 8\pi$. If $M > 8\pi$, there exists a radial nonincreasing minimizer

Keller-Segel model: the subcritical range

Existence and free energy

$M = \int_{\mathbb{R}^2} n_0\,dx \leq 8\pi$: global existence [W. Jäger, S. Luckhaus 1992], [JD, B. Perthame 2004], [A. Blanchet, JD, B. Perthame 2006]

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u\,(\nabla\,(\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u\;dx - \frac{1}{2} \int_{\mathbb{R}^2} u\,v\;dx$$

satisfies

$$\frac{d}{dt} F[u(t,\cdot)] = -\int_{\mathbb{R}^2} u\,\left|\nabla\,(\log u) - \nabla v\right|^2\;dx$$

(log HLS) inequality [E. Carlen, M. Loss 1992]:
F is bounded from below if $M \leq 8\pi$

... $M = 8\pi$ the critical case [A. Blanchet, J.A. Carrillo, N. Masmoudi 2008], [A. Blanchet et al.]

The existence setting for the subcritical regime

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \; t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \; t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\, dx) \,, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx) \,, \quad M := \int_{\mathbb{R}^2} n_0(x)\, dx < 8\,\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\, dx \;\forall\, t \geq 0$

$v = -\frac{1}{2\pi} \log|\cdot| * u$

Time-dependent rescaling

$$u(x,t) = \frac{1}{R^2(t)}\, n\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad v(x,t) = c\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1+2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

[A. Blanchet, JD, B. Perthame] Convergence in self-similar variables

$$\lim_{t\to\infty} \|n(\cdot,\cdot+t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,\cdot+t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

means *intermediate asymptotics* in original variables:

$$\left\| u(x,t) - \tfrac{1}{R^2(t)}\, n_\infty\left(\tfrac{x}{R(t)}, \tau(t)\right) \right\|_{L^1(\mathbb{R}^2)} \searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M \, \frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty \;, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

- Radial symmetry [Y. Naito]
- Uniqueness [P. Biler, G. Karch, P. Laurençot, T. Nadzieja]
- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon \in (0,1)$ [A. Blanchet, JD, B. Perthame]
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M

$$\lim_{M\to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[D.D. Joseph, T.S. Lundgren] [JD, R. Stańczy]
(The bifurcation diagramwill be shown later)

The stationary solution when mass varies

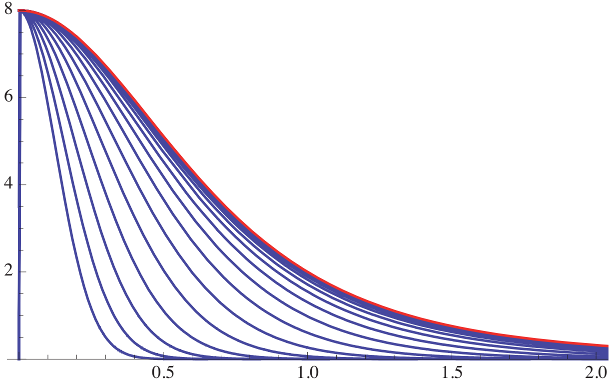


Figure: Representation of the solution appropriately scaled so that the 8π case appears as a limit (in red)

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla \Big[n \left(\log n - x + \nabla c \right) \Big]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \, dx + \int_{\mathbb{R}^2} \frac{1}{2} \, |x|^2 \, n \, dx - \frac{1}{2} \int_{\mathbb{R}^2} n \, c \, dx$$

satisfies

$$\frac{d}{dt} F[n(t,\cdot)] = - \int_{\mathbb{R}^2} n \left| \nabla \left(\log n \right) + x - \nabla c \right|^2 \, dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2 \, n(\lambda \, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2) \, dx + \int_{\mathbb{R}^2} \tfrac{\lambda^{-2}-1}{2} \, |x|^2 \, n \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) \, n(y) \, \log \frac{1}{\lambda} \, dx \, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi} \right)}_{>0 \text{ if } M < 8\pi} \log \lambda + \frac{\lambda^{-2} - 1}{2} \int_{\mathbb{R}^2} |x|^2 \, n \, dx$$

Keller-Segel with subcritical mass in self-similar variables

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

$$\lim_{t\to\infty} \|n(\cdot,\cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,\cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

$$n_\infty = M\,\frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

First result: small mass case

Theorem (A. Blanchet, JD, M. Escobedo, J. Fernández)

There exists a positive constant M^* *such that, for any initial data* $n_0 \in L^2(n_\infty^{-1}\,dx)$ *of mass* $M < M^*$ *satisfying the above assumptions, there is a unique solution* $n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau,\infty)\times\mathbb{R}^2)$ *for any* $\tau > 0$

Moreover, there are two positive constants, C *and* δ*, such that*

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \,\frac{dx}{n_\infty} \le C\,e^{-\,\delta\,t} \quad \forall\; t > 0$$

As a function of M, δ *is such that* $\lim_{M\to 0_+} \delta(M) = 1$

Four steps proof

The condition $M \leq 8\,\pi$ is necessary and sufficient for the global existence of the solutions, but there are two extra smallness conditions in our proof:

- Uniform estimate: the *method of the trap*
- L^p and H^1 estimates in the self-similar variables
- *Spectral gap* of a linearized operator $\mathcal{L}$
- Duhamel formula and nonlinear estimates

Linearization

We can introduce two functions f and g such that

$$n = n_\infty\,(1+f) \quad \text{and} \quad c = c_\infty(1+g)$$

and rewrite the Keller-Segel model as

$$\frac{\partial f}{\partial t} = \mathcal{L}\,f + \frac{1}{n_\infty}\,\nabla(f\,n_\infty\,\nabla(c_\infty\,g))$$

where the linearized operator is

$$\mathcal{L}\,f = \frac{1}{n_\infty}\nabla\cdot\big(n_\infty\nabla(f - c_\infty\,g)\big)$$

and

$$-\Delta(c_\infty\,g) = n_\infty\,f$$

2 – Keller-Segel model: functional framework and sharp asymptotics

- bifurcation diagrams
- spectrum of the linearized operator
- symmetrization
- nonlinear estimates
- rates of convergence for subcritical masses

... some preliminaries are needed

A parametrization of the solutions and the linearized operator

[J. Campos, JD]

$$-\Delta c = M \, \frac{e^{-\frac{1}{2}\,|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}\,|x|^2+c}\,dx}$$

Solve

$$-\phi'' - \frac{1}{r}\,\phi' = e^{-\frac{1}{2}\,r^2+\phi}\,, \quad r>0$$

with initial conditions $\phi(0)=a$, $\phi'(0)=0$ and get with $r=|x|$

$$M(a) := 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2}\,r^2+\phi_a}\,dx$$

$$n_a(x) = M(a)\,\frac{e^{-\frac{1}{2}\,r^2+\phi_a(r)}}{2\pi\int_{\mathbb{R}^2} r\,e^{-\frac{1}{2}\,r^2+\phi_a}\,dx} = e^{-\frac{1}{2}\,r^2+\phi_a(r)}$$

Mass

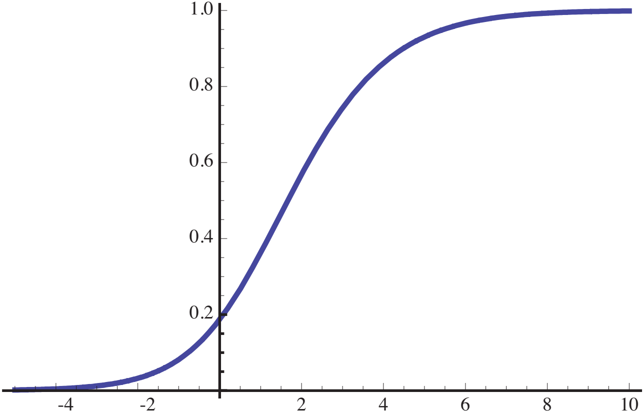


Figure: *The mass can be computed as* $M(a) = 2\pi \int_0^\infty n_a(r)\, r\, dr$*. Plot of* $a \mapsto M(a)/8\pi$

Bifurcation diagram

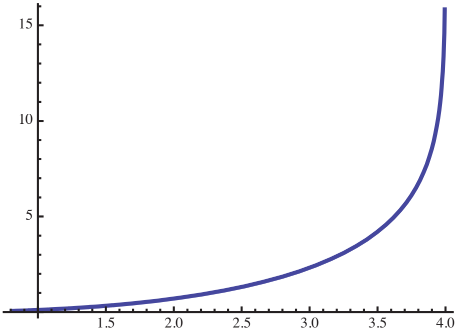


Figure: *The bifurcation diagram can be parametrized by*
$a \mapsto \left(\frac{1}{2\pi} M(a), \|c_a\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\right)$ *with* $\|c_a\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = c_a(0) = a - b(a)$
(cf. Keller-Segel system in a ball with no flux boundary conditions)

Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

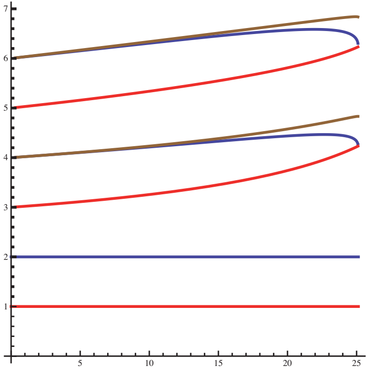


Figure: The lowest eigenvalues of $-\mathcal{L} = (-\Delta)^{-1}(n_a\, f)$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

[A. Blanchet, JD, M. Escobedo, J. Fernández], [J. Campos, JD], [V. Calvez, J.A. Carrillo], [J. Bedrossian, N. Masmoudi]

Spectral analysis in the functional framework determined by the relative entropy method

Simple eigenfunctions

Kernel Let $f_0 = \frac{\partial}{\partial M} c_\infty$ be the solution of

$$-\Delta\, f_0 = n_\infty\, f_0$$

and observe that $g_0 = f_0 / c_\infty$ is such that

$$\frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f_0 - c_\infty\, g_0) \big) =: \mathcal{L}\, f_0 = 0$$

Lowest non-zero eigenvalues $f_1 := \frac{1}{n_\infty}\, \frac{\partial n_\infty}{\partial x_1}$ associated with $g_1 = \frac{1}{c_\infty}\, \frac{\partial c_\infty}{\partial x_1}$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L}\, f_1 = f_1$

With $D := x \cdot \nabla$, let $f_2 = 1 + \frac{1}{2}\, D \log n_\infty = 1 + \frac{1}{2\, n_\infty}\, D\, n_\infty$. Then

$$-\Delta \left(D\, c_\infty \right) + 2\, \Delta c_\infty = D\, n_\infty = 2\, (f_2 - 1)\, n_\infty$$

and so $g_2 := \frac{1}{c_\infty}\, (-\Delta)^{-1} (n_\infty\, f_2)$ is such that $-\mathcal{L}\, f_2 = 2\, f_2$

Functional setting...

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality [A. Blanchet, JD, B. Perthame]

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)\,(c - c_\infty)\, dx \geq 0$$

achieves its minimum for $n = n_\infty$

$$\mathsf{Q}_1[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} F[n_\infty(1 + \varepsilon\, f)] \geq 0$$

if $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$. Notice that f_0 generates the kernel of Q_1

Lemma (J. Campos, JD)

Poincaré type inequality For any $f \in H^1(\mathbb{R}^2, n_\infty\, dx)$ such that $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$, we have

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f\, n_\infty)|^2\, n_\infty\, dx = \int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, n_\infty\, dx \leq \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

... and eigenvalues

With g such that $-\Delta(g\,c_\infty) = f\,n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1\,f_2\,n_\infty\,dx - \int_{\mathbb{R}^2} f_1\,n_\infty\,(g_2\,c_\infty)\,dx$$

on the orthogonal space to f_0 in $L^2(n_\infty\,dx)$

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\,c_\infty)|^2\,n_\infty\,dx \quad \text{with} \quad g = -\frac{1}{c_\infty}\,\frac{1}{2\pi}\,\log|\cdot| * (f\,n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L}\,f \rangle = \mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma (J. Campos, JD)

$\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is 1

Linearized Keller-Segel theory

$$\mathcal{L}\, f = \frac{1}{n_\infty} \nabla \cdot \left(n_\infty \nabla (f - c_\infty\, g) \right)$$

Corollary (J. Campos, JD)

$$\langle f, f \rangle \leq \langle \mathcal{L}\, f, f \rangle$$

The linearized problem takes the form

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f$$

where $\mathcal{L}$ is a self-adjoint operator on the orthogonal of f_0 equipped with $\langle \cdot, \cdot \rangle$. A solution of

$$\frac{d}{dt} \langle f, f \rangle = -\,2 \langle \mathcal{L}\, f, f \rangle$$

has therefore exponential decay

More on functional inequalities

A subcritical logarithmic HLS inequality

Recall that

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality [A. Blanchet, JD, B. Perthame]

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)\,(c - c_\infty)\,dx \geq 0$$

achieves its minimum for $n = n_\infty$

Lemma (J. Campos, JD)

Poincaré type inequality For any $f \in H^1(\mathbb{R}^2, n_\infty\,dx)$ such that $\int_{\mathbb{R}^2} f\,n_\infty\,dx = 0$, we have

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f\,n_\infty)|^2\,n_\infty\,dx = \int_{\mathbb{R}^2} |\nabla(g\,c_\infty)|^2\,n_\infty\,dx \leq \int_{\mathbb{R}^2} |f|^2\,n_\infty\,dx$$

... Legendre duality

An Onofri type inequality

Theorem (J. Campos, JD)

For any $M \in (0, 8\pi)$, if $n_\infty = M \frac{e^{c_\infty - \frac{1}{2}|x|^2}}{\int_{\mathbb{R}^2} e^{c_\infty - \frac{1}{2}|x|^2}\,dx}$ with $c_\infty = (-\Delta)^{-1} n_\infty$, $d\mu_M = \frac{1}{M} n_\infty\,dx$, we have the inequality

$$\log\left(\int_{\mathbb{R}^2} e^{\phi}\,d\mu_M\right) - \int_{\mathbb{R}^2} \phi\,d\mu_M \le \frac{1}{2\,M}\int_{\mathbb{R}^2} |\nabla\phi|^2\,dx \quad \forall\,\phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$$

Corollary (J. Campos, JD)

The following Poincaré inequality holds

$$\int_{\mathbb{R}^2} \left|\psi - \overline{\psi}\right|^2 n_\infty\,dx \le \int_{\mathbb{R}^2} |\nabla\psi|^2\,dx \quad \text{where} \quad \overline{\psi} = \int_{\mathbb{R}^2} \psi\,d\mu_M$$

An improved interpolation inequality (coercivity estimate)

Lemma (J. Campos, JD)

For any $f \in L^2(\mathbb{R}^2, n_\infty\,dx)$ *such that* $\int_{\mathbb{R}^2} f\, f_0\, n_\infty\, dx = 0$ *holds, we have*

$$- \frac{1}{2\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\, n_\infty(x)\, \log|x-y|\, f(y)\, n_\infty(y)\, dx\, dy \le (1-\varepsilon) \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

for some $\varepsilon > 0$, *where* $g\, c_\infty = G_2 * (f\, n_\infty)$ *and, if* $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$ *holds,*

$$\int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, dx \le (1-\varepsilon) \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

Equivalence of the norms

As a consequence

$$\langle f , f \rangle := \int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx - \int_{\mathbb{R}^2} f \, n_\infty \, (g \, c_\infty) \, dx$$

is equivalent to

$$\int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx$$

under the condition that $\int_{\mathbb{R}^2} f \, f_0 \, n_\infty \, dx = 0$

A similar result is true in the critical case:
[J. Bedrossian, N. Masmoudi], [P. Raphaël, R. Schweyer]

A spectral gap estimate

Theorem (J. Campos, JD)

For any function $f \in \mathcal{D}(\mathsf{L}_2)$, we have

$$\langle f, f \rangle = \mathsf{Q}_1[f] \le \mathsf{Q}_2[f] = \langle f, \mathrm{L}\, f \rangle \,.$$

The nonlinear Keller-Segel model, a functional analysis approach

Exponential convergence for any mass $M \leq 8\pi$

If $n_{0,*}(\sigma)$ stands for the symmetrized function associated to n_0, assume that for any $s \geq 0$

$$(H) \quad \exists\, \varepsilon \in (0, 8\,\pi - M) \quad \text{such that} \quad \int_0^s n_{0,*}(\sigma)\, d\sigma \leq \int_{B\left(0,\sqrt{s/\pi}\right)} n_{\infty, M+\varepsilon}(x)\, dx$$

Theorem (J. Campos, JD)

Under the above assumption, if $n_0 \in L^2_+(n_\infty^{-1}\, dx)$ and $M := \int_{\mathbb{R}^2} n_0\, dx < 8\,\pi$, then any solution with initial datum n_0 is such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \frac{dx}{n_\infty(x)} \leq C\, e^{-\,2\,t} \quad \forall\, t \geq 0$$

for some positive constant C, where n_∞ is the unique stationary solution with mass M

Sketch of the proof

- [J. Campos, JD] Uniform convergence of $n(t,\cdot)$ to n_∞ can be established for any $M \in (0, 8\pi)$ by an adaptation of the symmetrization techniques of [J.I. Díaz, T. Nagai, J.M. Rakotoson]
- Uniform estimates (with no rates) easily result
- Estimates based on Duhammel's formula inspired by [M. Escobedo, E. Zuazua] allow to prove uniform convergence
- Spectral estimates can be incorporated to the relative entropy approach
- Exponential convergence of the relative entropy follows

Step 1: symmetrization (1/2)

To any measurable function $u : \mathbb{R}^2 \mapsto [0, +\infty)$,
we associate the distribution function defined by $\mu(t, \tau) := |\{u > \tau\}|$
and its decreasing rearrangement given by

$$u_* : [0, +\infty) \rightarrow [0, +\infty] , \quad s \mapsto u_*(s) = \inf\{\tau \geq 0 : \mu(t, \tau) \leq s\} .$$

1. For every measurable function $F : \mathbb{R}^+ \mapsto \mathbb{R}^+$, we have

$$\int_{\mathbb{R}^2} F(u)\, dx = \int_{\mathbb{R}^+} F(u_*)\, ds$$

2. If $u \in W^{1,q}(0, T; L^p(\mathbb{R}^N))$ is a nonnegative function, with $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then $u_* \in W^{1,q}(0, T; L^p(0, \infty))$ and the formula

$$\int_0^{\mu(t,\tau)} \frac{\partial u_*}{\partial t}(t, \sigma)\, d\sigma = \int_{\{u(t,\cdot)>\tau\}} \frac{\partial u}{\partial t}(t, x)\, dx$$

holds for almost every $t \in (0, T)$ [J.I. Díaz, T. Nagai, J.M. Rakotoson]

Step 1: symmetrization (2/2)

Lemma

If n is a solution, then the function

$$k(t,s) := \int_0^s n_*(t,\sigma)\,d\sigma$$

satisfies $k \in L^\infty\left([0,+\infty)\times(0,+\infty)\right) \cap H^1\left([0,+\infty); W^{1,p}_{\mathrm{loc}}(0,+\infty)\right) \cap L^2\left([0,+\infty); W^{2,p}_{\mathrm{loc}}(0,+\infty)\right)$ *and*

$$\begin{cases} \frac{\partial k}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 k}{\partial s^2} - (k+2\,s)\,\frac{\partial k}{\partial s} \le 0 & a.e.\ \ in\ (0,+\infty)\times(0,+\infty) \\ k(t,0)=0\ , \quad k(t,+\infty)=\int_{\mathbb{R}^2} n_0\,dx & for\ t\in(0,+\infty) \\ k(0,s)=\int_0^s (n_0)_*\,d\sigma & for\ s\ge 0 \end{cases}$$

Step 2: Uniform estimates

Proposition (J.I. Díaz, T. Nagai, J.M. Rakotoson)

Let f, g be two continuous functions on $Q=\mathbb{R}^+\times(0,+\infty)$ such that …

$$\left\{\begin{array}{c}\frac{\partial f}{\partial t}-4\,\pi\,s\,\frac{\partial^2 f}{\partial s^2}-(f+2\,s)\,\frac{\partial f}{\partial s}\le\frac{\partial g}{\partial t}-4\,\pi\,s\,\frac{\partial^2 g}{\partial s^2}-(g+2\,s)\,\frac{\partial g}{\partial s}\ \ a.e.\ \ in\ Q\\ f(t,0)=0=g(t,0)\quad and\quad f(t,+\infty)\le g(t,+\infty)\ for\ any\ t\in(0,+\infty)\\ f(0,s)\le g(0,s)\ for\ s\ge 0\,,\ and\ g(t,s)\ge 0\ \ in\ Q\end{array}\right.$$

then $f\le g$ on Q

Corollary

Assume that $n_0\in L^2_+(n_\infty^{-1}\,dx)$ satisfies (H) and $M:=\int_{\mathbb{R}^2}n_0\,dx<8\,\pi$. Then there exist positive constants $C_1=C_1(M,p)$ and $C_2=C_2(M,p)$ such that

$$\|n\|_{L^p(\mathbb{R}^2)}\le C_1\quad and\quad \|\nabla c\|_{L^\infty(\mathbb{R}^2)}\le C_2$$

Step 3: Estimates based on Duhammel's formula

Consider the kernel associated to the Fokker-Planck equation

$$K(t,x,y) := \frac{1}{2\pi\left(1-e^{-2t}\right)}\, e^{-\frac{1}{2}\frac{|x-e^{-t}y|^2}{1-e^{-2t}}} \quad x\in\mathbb{R}^2\,, \quad y\in\mathbb{R}^2\,, \quad t>0$$

If n is a solution, then

$$n(t,x) = \int_{\mathbb{R}^2} K(t,x,y)\, n_0(y)\, dy + \int_0^t \int_{\mathbb{R}^2} \nabla_x K(t-s,x,y)\cdot n(s,y)\, \nabla c(s,y)\, dy\, ds$$

Corollary

Assume that n is a solution. Then

$$\lim_{t\to\infty} \|n(t,\cdot)-n_\infty\|_{\mathrm{L}^p(\mathbb{R}^d)} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|\nabla c(t,\cdot)-\nabla c_\infty\|_{\mathrm{L}^q(\mathbb{R}^d)} = 0$$

for any $p\in[1,\infty]$ and any $q\in[2,\infty]$

Step 4: Spectral estimates can be incorporated

With $\mathsf{Q}_1[f] = \langle f, f \rangle$ and $\mathsf{Q}_2[f] = \langle f, \mathcal{L} f \rangle$

1. For any function f in the orthogonal of the kernel of $\mathcal{L}$, we have

$$\mathsf{Q}_1[f] \leq \mathsf{Q}_2[f]$$

2. For any radial function $f \in \mathcal{D}(\mathsf{L}_2)$, we have

$$2\,\mathsf{Q}_1[f] \leq \mathsf{Q}_2[f]$$

Cf. [V. Calvez, J.A. Carrillo]

Step 5: Exponential convergence of the relative entropy

$$\frac{\partial f}{\partial t} = \mathcal{L} f - \frac{1}{n_\infty} \nabla \left[n_\infty \, f \, \nabla (g \, c_\infty) \right]$$

$$\frac{d}{dt} \mathsf{Q}_1[f(t,\cdot)] = -\,2\,\mathsf{Q}_2[f(t,\cdot)] + \int_{\mathbb{R}^2} \nabla (f - g\,c_\infty)\, f\, n_\infty \cdot \nabla (g\,c_\infty)\, dx$$

$$\frac{d}{dt} \mathsf{Q}_1[f(t,\cdot)] \le -\,2\,\mathsf{Q}_2[f(t,\cdot)] + \delta(t,\varepsilon)\,\sqrt{\mathsf{Q}_1[f(t,\cdot)]\,\mathsf{Q}_2[f(t,\cdot)]}$$

$$\mathsf{Q}_1[f(t,\cdot)] \le \mathcal{Q} \quad \forall\, t \ge 0$$

$$\frac{d}{dt} \mathsf{Q}_1[f(t,\cdot)] \le -\,\mathsf{Q}_1[f(t,\cdot)] \left[2 - \,\delta(t,\varepsilon) \left(\left(\mathsf{Q}_1[f(t,\cdot)]\right)^{\frac{1-\varepsilon}{2-\varepsilon}} + \mathsf{Q}_1[f(t,\cdot)]\right)^{\frac{1}{2+\varepsilon}} \right) \right]$$

As a consequence, we finally get that

$$\limsup_{t\to\infty} e^{2\,t}\, \mathsf{Q}_1[f(t,\cdot)] < \infty$$

Some key ideas

1. Lyapunov / Entropy functionals and functional inequalities
2. Linearization and best constants
3. Functional framework for linearized operators can be deduced from the entropy functional

- [G. Egaña, S. Mischler, 2013]

- weak notion of solution (based on free energy estimates)
- uniqueness, smoothing
- linearized and nonlinear stability in rescaled variables and exponential convergence under weaker assumptions - sharp rates in $\mathrm{L}^{4/3}(\mathbb{R}^2)$

3- Extensions, consequences

- parabolic-parabolic models
  [JD, G. Jankowiak, P. Markowich]
  [G. Jankowiak]
- improved functional inequalities
  [JD, G. Jankowiak]

Parabolic-parabolic models

Parabolic-parabolic models for crowd motion

[JD, G. Jankowiak, P. Markowich] A model for crowd motion

$$\partial_t \rho = \Delta \rho - \nabla \cdot \big(\rho \, (1-\rho) \, \nabla D \big)$$

$$\partial_t D = \kappa \, \Delta D - \delta \, D + g(\rho)$$

on a bounded domain Ω with no-flux boundary conditions

$$\big(\nabla \rho - \rho \, (1-\rho) \, \nabla D \big) \cdot \nu = 0 \quad \text{on} \quad \partial \Omega$$

Model (I): $g(\rho) = \rho \, (1-\rho)$ or Model (II): $g(\rho) = \rho$
Any stationary solution solves

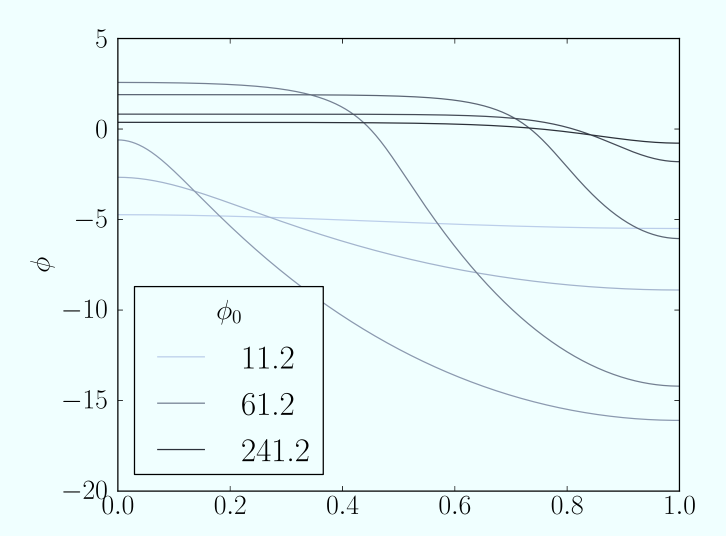
$$\nabla \rho - \rho \, (1-\rho) \, \nabla D = 0 \quad \text{on} \quad \Omega \quad \Longleftrightarrow \quad \rho = \frac{1}{1+e^{-\phi}}$$

where $\phi = D - \phi_0$ and $\int_\Omega \frac{1}{1+e^{\phi_0 - D}} \, dx = M$

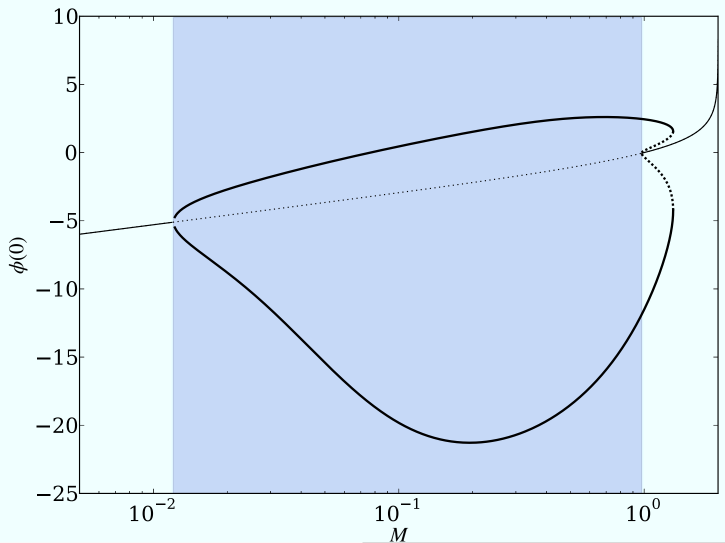
$$-\kappa \, \Delta \phi + \delta \, (\phi + \phi_0) - f(\phi) = 0 \quad \text{on} \quad \Omega$$

with homogeneous Neumann boundary conditions

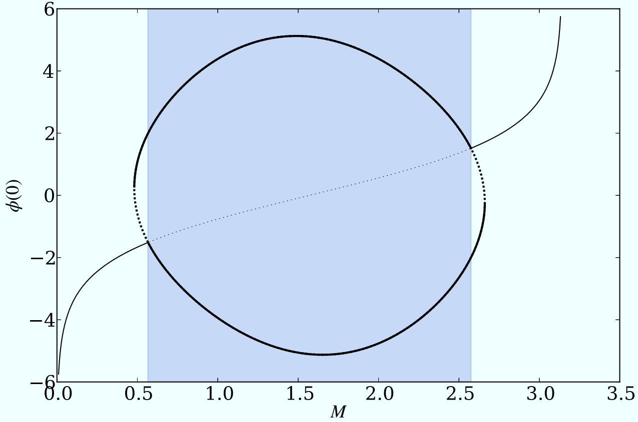
Model (I), $d = 1$, $\delta = 10^{-3}$



Model (I), $\kappa = 5 \times 10^{-4}$, $\delta = 10^{-3}$



Model (II), $\kappa = 10^{-2}$, $\delta = 10^{-3}$



Parabolic-parabolic Keller-Segel model

[G. Jankowiak] Analysis of the stability of self-similar solutions, including for masses larger than 8π

Logarithmic Hardy-Littlewood-Sobolev and Onofri inequalities: duality, flows

Critical case: the logarithmic HLS inequality

The classical logarithmic Hardy-Littlewood-Sobolev (logHLS) in $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} n \log\left(\frac{n}{M}\right) dx + \frac{2}{M}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\, n(y)\, \log|x-y|\, dx\, dy + M\,(1+\log\pi) \geq 0$$

Equality is achieved by

$$\mu(x) := \frac{1}{\pi\,(1+|x|^2)^2} \quad \forall\, x\in\mathbb{R}^2$$

Notice that $-\Delta \log\mu = 8\,\pi\,\mu$ can be inverted as

$$(-\Delta)^{-1}\mu = \frac{1}{8\,\pi}\,\log(\pi\,\mu)$$

With $M = 8\,\pi$ and $n_\infty = 8\,\pi\,\mu$ (logHLS) can be rewritten as

$$\int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx \geq \frac{1}{2}\int_{\mathbb{R}^2} (n-n_\infty)\,(-\Delta)^{-1}(n-n_\infty)\, dx$$

Subritical case: the logarithmic HLS inequality

The minimum of

$$\int_{\mathbb{R}^2} n \log\left(\frac{n}{M}\right) dx + \frac{2}{M} \int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\, n(y)\, \log|x-y|\; dx\, dy + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2\, n\, dx$$

is achieved by the stationary solution n_∞ of the Keller-Segel model and can again be written as

$$\int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx \geq \frac{1}{2} \int_{\mathbb{R}^2} (n - n_\infty)\,(-\Delta)^{-1}(n - n_\infty)\; dx$$

Critical case: Legendre duality

Onofri's inequality

$$F_1[u] := \log\left(\int_{\mathbb{R}^2} e^u\, d\mu\right) \le \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\, dx + \int_{\mathbb{R}^2} u\,\mu\, dx =: F_2[u]$$

By duality: $F_i^*[v] = \sup\left(\int_{\mathbb{R}^2} v\, u\, d\mu - F_i[u]\right)$ we can relate Onofri's inequality with (logHLS)

For any $v \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} v\, dx = 1$, such that $v \log v$ and $(1+\log|x|^2)\, v \in \mathrm{L}^1(\mathbb{R}^2)$, we have

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v\, \log\left(\frac{v}{\mu}\right) dx - 4\,\pi \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\, dx \ge 0$$

[E. Carlen, M. Loss 1992 & V. Calvez, L. Corrias 2008]
The same property holds in the subcritical case

The two-dimensional case: (logHLS) and flows

[E. Carlen, J. Carrillo, M. Loss 2010]

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\;dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx$$

is related to Gagliardo-Nirenberg inequalities if $v_t = \Delta\sqrt{v}$

- Alternatively, assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta\log\left(\frac{v}{\mu}\right)\quad t>0\,,\quad x\in\mathbb{R}^2$$

Proposition (JD 2011)

If v is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\;dx = 1$, $v_0\,\log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu \in \mathrm{L}^1(\mathbb{R}^2)$, then

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\;dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)\,u\;d\mu \ge F_2[u]-F_1[u]$$

with $\log(v/\mu) = u/2$

Hierarchies of inequalities, improved inequalities

Theorem (JD, Jankowiak 2013)

If $d \geq 3$, with $q = \frac{d+2}{d-2}$

$$\mathsf{S}_d \, \|u^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} u^q \, (-\Delta)^{-1} u^q \; dx \\ \leq \mathsf{S}_d \, \|u\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \\ \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)$$

and, when $d = 2$, for any function $f \in \mathcal{D}(\mathbb{R}^2)$

$$\left(\int_{\mathbb{R}^2} e^{\,f} \, d\mu\right)^2 - 4\pi \int_{\mathbb{R}^d} e^{f} \, \mu \, (-\Delta)^{-1} \, e^{f} \, \mu \; dx \\ \leq \left(\int_{\mathbb{R}^2} e^{\,f} \, d\mu\right)^2 \left[\frac{1}{16\,\pi} \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \int_{\mathbb{R}^2} f \; d\mu - \log\left(\int_{\mathbb{R}^2} e^{\,f} \, d\mu\right)\right]$$

Thank you for your attention !