Nonlinear diffusions as limits of BGK-type kinetic equations

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Outline of the talk

1. Kinetic BGK Model: Formulation
2. Motivations and references
3. Main results and assumptions
4. Existence and uniqueness
5. Drift diffusion limit
6. Convergence to equilibrium
7. Examples
    - Porous medium flow
    - Fast diffusion
    - Fermi-Dirac statistics
    - Bose-Einstein statistics

Outline of the talk, II

Application to a flat rotating system of gravitating particles (coll. J. Fernández

- Kinetic description of a system in rotation at constant angular speed
- Polytropes, critical points, reduced variational problem
- Results and open questions

BGK models

- BGK model of gas dynamics

$$\partial_t f + v \cdot \nabla_x f = \frac{\rho(x,t)}{(2\pi T)^{n/2}} \exp\left(\frac{-|v-u(x,t)|^2}{2T(x,t)}\right) - f \,,$$

  where $\rho(x,t)$ (position density), $u(x,t)$ (local mean velocity) and $T(x,t)$ (temperature) are chosen such that they equal the corresponding quantities associated to f.

  [Perthame,Pulvirenti]: Weighted L^∞ bounds and uniqueness for the Boltzmann BGK model, 1993

- Linear BGK model in semiconductor physics

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \frac{\rho(x,t)}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}|v|^2\right) - f \,,$$

  where $\rho(x,t)$ equals the position density of f.

  [Poupaud]: Mathematical theory of kinetic equations for transport modelling in semiconductors, 1994

BGK-type kinetic equation

$$\begin{aligned} \varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon - \varepsilon \nabla_x V(x) \cdot \nabla_v f^\varepsilon &= G_{f^\varepsilon} - f^\varepsilon \,, \\ f^\varepsilon(x,v,t=0) &= f_I(x,v)\,, \quad x,v \in \mathbb{R}^3\,, \end{aligned}$$

with the Gibbs equilibrium $G_f := \gamma\left(\frac{|v|^2}{2} + V(x) - \mu_{\rho_f}(x,t)\right)$.

The Fermi energy $\mu_{\rho_f}(x,t)$ is implicitly defined by

$$\int_{\mathbb{R}^3} \gamma\left(\frac{|v|^2}{2} + V(x) - \mu_{\rho_f}(x,t)\right) dv = \int_{\mathbb{R}^3} f(x,v,t)\,dv =: \rho_f(x,t).$$

$f^\varepsilon(x,v,t)$ ... phase space particle density

$V(x)$ ... potential

ε ... mean free path.

Motivations, I

- Local Gibbs states in stellar dynamics (polytropic distribution functions) and semiconductor theory (Fermi-Dirac distributions).
  Collisions : short time scale
- Monotone energy profiles are natural for the study of stability: monotonicity $\Leftrightarrow$ convex Lyapunov functional, Global Gibbs states
- Goal: derive the nonlinear diffusion limit consistently with the Gibbs state: a relaxation-time kernel

Motivations, II

- Gibbs states $\Longleftrightarrow$ generalized entropies
- nonlinear diffusion equations are difficult to justify directly
- global Gibbs states have the same macroscopic density at the kinetic / diffusion levels
- they have the 'same' Lyapunov functionals

References

- Formal expansions (generalized Smoluchowski equation):
  [Ben Abdallah, J.D.], [Chavanis-Laurençot, Lemou], [Chavanis et al.], [Degond, Ringhofer]
- Astrophysics:
  [Binney,Tremaine], [Guo,Rein], [Chavanis et al.]
- Fermi-Dirac statistics in semiconductors models:
  [Goudon-Poupaud]

Main result

Theorem 1. *For any* $\varepsilon > 0$*, the equation has a unique weak solution* $f^\varepsilon \in C(0,\infty; L^1 \cap L^p(\mathbb{R}^6))$ *for all* $p < \infty$*. As* $\varepsilon \to 0$*,* f^ε *weakly converges to a local Gibbs state* f^0 *given by*

$$f^0(x,v,t) = \gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}(\rho(x,t))\right)$$

where ρ *is a solution of the nonlinear diffusion equation*

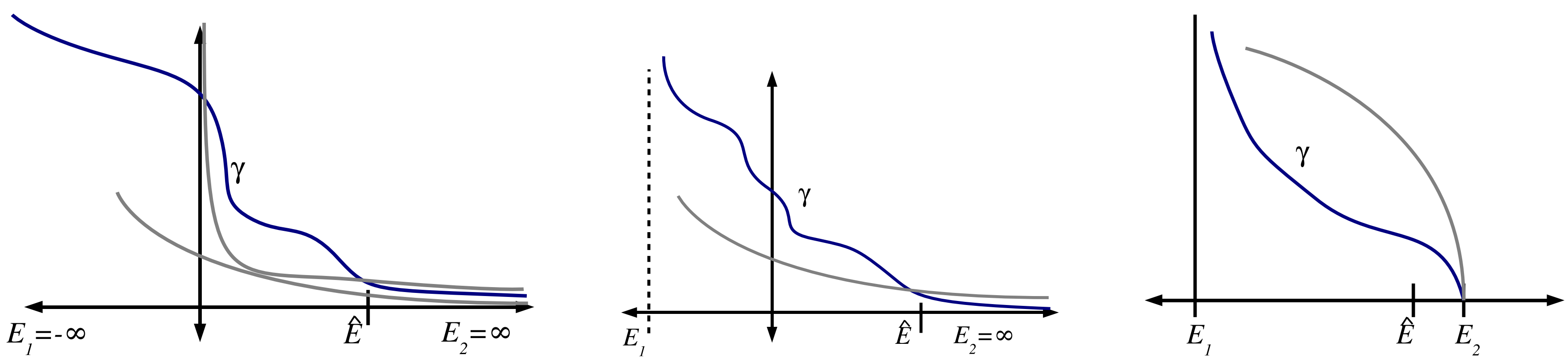
$$\partial_t \rho = \nabla_x \cdot (\nabla_x \nu(\rho) + \rho \nabla_x V(x))$$

with initial data $\rho(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v)\,dv$

$$\nu(\rho) = \int_0^\rho s\,\bar{\mu}'(s)\,ds$$

Assumptions on the energy profile

- $\gamma(E) \in \mathcal{C}^1((E_1, E_2), \mathbb{R}^+)$ where $-\infty \le E_1 < E_2 \le \infty$.
- γ monotonically decreasing and $\lim_{E \to E_2} \gamma(E) = 0$.



(a) Asymptotically exponential lower bound.

(b) Asymptotically exponential upper bound..

(c) $E_2 < \infty$.

Initial condition

- $f(x, v, t = 0) = f_I(x, v)$
- The total mass $M := \iint_{\mathbb{R}^6} f_I(x, v)\, dv\, dx$ is preserved by the evolution.
- $\exists \mu^*$ s.t. $0 \leq f_I(x, v) \leq f^*(x, v) := \gamma\left(\frac{|v|^2}{2} + V(x) - \mu^*\right)$
- Maximal macroscopic density

$$\bar{\rho} := \lim_{\theta \to -E_1^+} \int_{\mathbb{R}^3} \gamma\left(\frac{1}{2}|v|^2 - \theta\right) dv\ .$$

  Observe $\bar{\rho} = \infty$ if $E_1 = -\infty$.
- If $\bar{\rho} < \infty$ we require $\rho^*(x) := \int_{\mathbb{R}^3} f^*\, dv \leq \bar{\rho}\ \forall\, x \in \mathbb{R}^3$.

Fermi energy

The Fermi-energy $\mu_{\rho_f}(x,t)$ ensures local mass conservation,

$$\int_{\mathbb{R}^3} G_f\, dv = \int_{\mathbb{R}^3} \gamma\Big(\frac{|v|^2}{2} + \underbrace{V(x) - \mu_{\rho_f}(x,t)}_{=:-\bar{\mu}(\rho_f(x,t))\ \text{('quasi Fermi level')}}\Big) dv = \rho_f(x,t)$$

- Compute $\bar{\mu}$ in terms of γ

  $$(\bar{\mu}^{-1})(\theta) = 4\pi\sqrt{2}\int_0^\infty \gamma(p-\theta)\sqrt{p}\, dp$$

  $\Rightarrow \bar{\mu}(\rho) : (0,\bar{\rho}) \to (-E_2, -E_1)$, increasing.
- Differentiation leads to an Abelian equation $\Rightarrow \gamma$ in terms of $\bar{\mu}$:

  $$\gamma(E) = \frac{1}{\sqrt{2}\, 2\pi^2}\frac{d^2}{dE^2}\int_{-\infty}^{-E} \frac{(\bar{\mu}^{-1})(\theta)}{\sqrt{-E-\theta}}\, d\theta$$

Assumptions on the potential

- Boundedness from below

$$V(x) \geq V_{\text{min}} = 0,$$

- Regularity

$$V \in C^{1,1}(\mathbb{R}^3)\ .$$

- Potential is confining in the sense that

$$\iint_{\mathbb{R}^6} \left(1 + \frac{|v|^2}{2} + V(x)\right) \underbrace{\gamma\left(\frac{|v|^2}{2} + V(x) - \mu^*\right)}_{=f^*} dv\, dx < \infty.$$

Existence and uniqueness

Proposition 1. *Let $1 \leq p < \infty$, then the problem has a unique solution in $\mathcal{V} := \{f \in \mathcal{C}(0,\infty;(L^1 \cap L^p)(\mathbb{R}^6)) : 0 \leq f \leq f^*, \forall t > 0$ a.e.$\}$.*

The proof uses a fixpoint argument on the map $f \mapsto g$, where g satisfies

$$\begin{aligned} \varepsilon^2 \partial_t g + \varepsilon\, v \cdot \nabla_x g - \varepsilon\, \nabla_x V \cdot \nabla_v g &= \gamma\left(\frac{|v|^2}{2} - \bar{\mu}(\rho_f)\right) - g\,, \\ g(t=0,x,v) &= f_I(x,v)\,, \\ \text{where} \quad \rho_f(x,t) &:= \int_{\mathbb{R}^3} f(x,v,t)\, dv\,. \end{aligned}$$

$f \leq f^* \Rightarrow f \in L^\infty_{x,v,t}$, $\rho \in L^\infty_{x,t}$ and if f^* has compact support in $\mathbb{R}^3_v$, this will also be true for f (porous medium case).

Formal asymptotics

$$\varepsilon^2 \partial_t f + \varepsilon v \cdot \nabla_x f - \varepsilon \nabla_x V(x) \cdot \nabla_v f = Q[f]$$

Expand $f = \sum_{i=0}^{\infty} f^i \varepsilon^i$, $\rho^i = \int_{\mathbb{R}^3} f^i \, dv$, $G_f = \sum_{i=1}^{\infty} G^i \varepsilon^i$. Then $G^0 = \gamma(|v|^2/2 - \bar{\mu}(\rho^0)) = \gamma(|v|^2/2 + V - \mu^0)$.

$$\mathcal{O}(1) : G^0 = f^0.$$

$$\mathcal{O}(\varepsilon) : v \cdot \nabla_x f^0 - \nabla_x V \cdot \nabla_v f^0 = G^1 - f^1$$

$$\Rightarrow \quad f^1 = v \cdot \nabla_x \mu^0 \, \gamma' \left(\tfrac{1}{2} v^2 + V(x) - \mu^0(x,t)\right) + G^1$$

$$\Rightarrow \quad \int_{\mathbb{R}^3} v f^1 \, dv = -\rho^0 \nabla_x \mu^0$$

$$\mathcal{O}(\varepsilon^2) : \partial_t f^0 + v \cdot \nabla_x f^1 - \nabla_x V \cdot \nabla_v f^1 = G^2 - f^2$$

$$\Rightarrow \quad \partial_t \rho^0 = \nabla \cdot (\rho^0 \nabla \mu^0) = \Delta \nu(\rho^0) + \nabla \cdot (\rho^0 \nabla_x V)$$

where $\rho^0(x,t) = \int_{\mathbb{R}^3} f^0(x,v,t) dv, \quad \rho^0(x,0) = \int_{\mathbb{R}^3} f_I(x,v) dv$.
The nonlinearity ν is given by $\nu(\rho) := \displaystyle\int_0^{\rho} \tilde{\rho} \, \bar{\mu}'(\tilde{\rho}) \, d\tilde{\rho}$

Free energy

- Define the free energy (convex functional)

$$\mathcal{F}(f) := \iint_{\mathbb{R}^6} \left[\left(\frac{|v|^2}{2} + V(x) \right) f - \int_0^f \gamma^{-1}(\tilde{f})\, d\tilde{f} \right] dv\, dx.$$

- Production of free energy

$$\varepsilon^2 \frac{d}{dt}\mathcal{F}(f^\varepsilon) = \iint_{\mathbb{R}^6} \left(\gamma(E_{f^\varepsilon}) - f^\varepsilon\right) \left(E_{f^\varepsilon} - (\gamma^{-1})(f^\varepsilon)\right) dv\, dx \le 0,$$

with $\quad E_{f^\varepsilon} := \dfrac{|v|^2}{2} + V(x) - \mu_{\rho_{f^\varepsilon}}(x,t)\ , \qquad G_{f^\varepsilon} = \gamma(E_{f^\varepsilon})$

- Free energy is finite, $\forall\, t \in \mathbb{R}_+$:

$$-\infty < \mathcal{F}(f^\infty) \le \mathcal{F}(G_{f^\varepsilon}(.,.,t)) \le \mathcal{F}(f^\varepsilon(.,.,t)) \le \mathcal{F}(f_I) < \infty$$

as $\mathcal{F}(f^\infty) = \iint_{\mathbb{R}^6} \gamma \left(\frac{|v|^2}{2} + V - \mu^\infty \right) \left(\mu^\infty - \frac{|v|^2}{3} \right) < \infty$ by assumptions on the potential.

Perturbations of moments

- Perturbations of 1st and 2nd moments

$$j^{\varepsilon} := \int_{\mathbb{R}^3} v \, \frac{f^{\varepsilon} - G_{f^{\varepsilon}}}{\varepsilon} \, dv \quad \text{and} \quad \kappa^{\varepsilon} := \int_{\mathbb{R}^3} v \otimes v \, \frac{f^{\varepsilon} - G_{f^{\varepsilon}}}{\varepsilon} \, dv.$$

- $\Rightarrow \forall\, U$ open and bounded $\exists$ uniform bounds,

$$\|j^{\varepsilon}\|_{L^2_{x,t}(U)} \leq M^1_U \quad \text{and} \quad \|\kappa^{\varepsilon}\|_{L^2_{x,t}(U)} \leq M^2_U$$

- Proof uses production of free energy

$$\mathcal{O}(\varepsilon^2) = \iiint_{\{G_{f^\varepsilon} > 0\}} (G_{f^\varepsilon} - f^{\varepsilon})^2 (-\gamma^{-1})'(f^*) dx\, dv\, dt +$$

$$+ \iiint_{\{G_{f^\varepsilon} = 0\}} \big(\underbrace{E_{f^\varepsilon} - E_2}_{\geq 0} + \underbrace{E_2 - \gamma^{-1}(f^{\varepsilon})}_{\geq 0}\big) f^{\varepsilon} \, dx\, dv\, dt.$$

2^{nd} moments of local Gibbs states

- Let

$$\nu(\rho) := \int_{\mathbb{R}^3} v_i^2\,\gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}(\rho)\right)\,dv\ .$$
$$\Rightarrow \nu'(\rho) = \rho\,\bar{\mu}'(\rho)\ .$$

- On $[0, \rho^{\max} := \bar{\mu}^{-1}(\mu^*)]$ for some $C > 0$:

either $\quad \nu'(\rho) > C \quad$ or $\quad 1/\nu'(\rho) > C\ .$

- If $E_2 < \infty$ ("porous medium case"): $\lim_{\rho\to 0}\nu'(\rho) = 0$.

Strong convergence of ρ, I

Proposition 2. $\rho^\varepsilon \longrightarrow \rho^0$ *in* L^p_{loc} *strongly for all* $p \in (1,\infty)$*.*

The proof uses compensated compactness theory applying the Div-Curl-Lemma to

$$U^\varepsilon := (\rho^\varepsilon, j^\varepsilon), \qquad V^\varepsilon := (\nu(\rho^\varepsilon), 0, 0, 0).$$

Rewrite the equations for the mass and momentum densities (using $(\text{curl}\, w)_{ij} := w^i_{x_j} - w^j_{x_i}$)

$$\begin{cases} \text{div}_{t,x} U^\varepsilon = \partial_t \rho^\varepsilon + \nabla_x \cdot j^\varepsilon = 0, \\ (\text{curl}_{t,x} V^\varepsilon)_{1,2\ldots4} = \nabla_x \nu(\rho^\varepsilon) = \underbrace{-j^\varepsilon - \rho^\varepsilon \nabla_x V - \varepsilon \nabla_x \cdot \kappa^\varepsilon - \varepsilon^2 \partial_t j^\varepsilon}_{\text{precompact in } H^{-1,\text{loc}}_{x,t}}, \end{cases}$$

as j^ε, κ^ε and $\rho^\varepsilon \in L^{2,\text{loc}}_{x,t}$.

Strong convergence of ρ, II

The Div-Curl-Lemma yields

$$\overline{\rho\,\nu} = \overline{\rho}\;\overline{\nu}.$$

where

$$\begin{cases} \nu(\rho^{\varepsilon_i}) \overset{*}{\rightharpoonup} \overline{\nu} = \displaystyle\int_0^{\rho^{\max}} \nu(\rho)\, d\eta_{x,t}(\rho), \\ \rho^{\varepsilon_i} \overset{*}{\rightharpoonup} \overline{\rho} = \displaystyle\int_0^{\rho^{\max}} \rho\, d\eta_{x,t}(\rho), \\ \rho^{\varepsilon_i}\,\nu(\rho^{\varepsilon_i}) \overset{*}{\rightharpoonup} \overline{\rho\nu} = \displaystyle\int_0^{\rho^{\max}} \rho\nu(\rho)\, d\eta_{x,t}(\rho). \end{cases}$$

$\eta_{x,t}$ ... Young measure associated with $\rho^{\varepsilon_i} \overset{*}{\rightharpoonup} \overline{\rho}$

Strong convergence of ρ, III

The mean value theorem yields

$$\nu(\rho) = \nu(\overline{\rho}) + \nu'(\tilde{\rho})(\rho - \overline{\rho})$$

for some $\tilde{\rho} \in (0, \rho^{\text{max}})$. Conclude

$$0 = \overline{\rho\,\nu} - \overline{\rho}\;\overline{\nu} =$$

$$= \int_0^{\rho^{\text{max}}} \nu(\rho)(\rho - \overline{\rho}) d\eta_{x,t}(\rho) = \underbrace{\int_0^{\rho^{\text{max}}} \nu(\overline{\rho})(\rho - \overline{\rho}) d\eta_{x,t}(\rho)}_{=0} +$$

$$+ \int_0^{\rho^{\text{max}}} \nu'(\tilde{\rho})(\rho - \overline{\rho})^2 d\eta_{x,t}(\rho) \geq C \int_0^{\rho^{\text{max}}} (\rho - \overline{\rho})^2 d\eta_{x,t}(\rho),$$

assuming $\nu'(\tilde{\rho}) \geq C \Rightarrow \eta_{x,t} = \delta_{\overline{\rho}(x,t)} \Rightarrow \nu(\overline{\rho}) = \overline{\nu}$.

Weak formulation of the pde I

Lemma 1. *Let* $f^{\varepsilon_i} \rightharpoonup f^0$*, then* $f^0 = G_{f^0}$ *a.e. .*

Lemma 2. *Let* $j^{\varepsilon_i} \to j^0$ *in* $\mathcal{D}'_{x,t}$*, then* $j^0 = -\nabla_x \nu(\rho^0) - \rho^0 \nabla_x V$*.*

Proof. Multiply the kinetic equation by $\frac{1}{\varepsilon}$,

$$
\begin{aligned}
\varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon - \nabla_x V \cdot \nabla_v f^\varepsilon \quad &= \quad -\frac{f^\varepsilon - G_{f^\varepsilon}}{\varepsilon} \\
\downarrow \text{in} \quad \mathcal{D}'(\mathbb{R}^7) \\
v \cdot \nabla_x f^0 - \nabla_x V \cdot \nabla_v f^0 \quad &= \\
= v \cdot \nabla_x G_{f^0} - \nabla_x V \cdot \nabla_v G_{f^0} \quad &=: \quad -r^0
\end{aligned}
$$

Using uniform boundedness of κ^ε we prove

$$
j^{\varepsilon_i} \xrightarrow{\mathcal{D}'_{x,t}} \int_{\mathbb{R}^3} v\, r^0 \, dv = -\Big(\rho^0 \nabla_x V + \nabla_x \nu(\rho^0)\Big) \; .
$$

Weak formulation of the pde II

Proposition 3. $\rho^0 := \int_{\mathbb{R}^3} f^0\, dv$ *satisfies a weak formulation of the formal macroscopic limit.*

Integrate the kinetic equation w.r. to v,

$$\partial_t \rho^\varepsilon + \nabla_x \cdot \int_{\mathbb{R}^3} v \frac{f^\varepsilon - G_{f^\varepsilon}}{\varepsilon} dv = \partial_t \rho^\varepsilon + \nabla_x \cdot j^\varepsilon = 0\,.$$

In the limit as $\varepsilon \to 0$ we obtain

$$\begin{aligned} \partial_t \rho^0 &= \Delta \nu\left(\rho^0\right) + \nabla_x \cdot \left(\rho^0\, \nabla_x V\right)\,, \\ \rho^0(x, t=0) &= \int_{\mathbb{R}^3} f_I(x,v)\, dv\,. \end{aligned}$$

with $\nu(\rho) = \int_0^\rho \tilde{\rho}\, \bar{\mu}'(\tilde{\rho})\, d\tilde{\rho}\,.$

Convergence to equilibrium, I

If $E_2 < \infty$ we additionally require that V is uniformly convex.

We consider the evolution in time of solutions of the problem with $\varepsilon = 1$

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = G_f - f \ .$$

Proposition 4. *For every sequence $t_n \longrightarrow \infty$, there exists a subsequence (again denoted by t_n) such that*

$$f^n(t,x,v) := f(t_n + t, x, v) \rightharpoonup f^\infty = G^\infty := \gamma\left(\frac{|v|^2}{2} + V(x) - \mu^\infty\right)$$

where μ^∞ is the unique constant Fermi energy which satisfies $\int_{\mathbb{R}^3} \bar{\mu}^{-1}(\mu^\infty - V(x))\, dx = M = \int_{\mathbb{R}^6} f_I \, dv\, dx$.

Velocity averaging

- Let $\phi \in \mathcal{D}_{x,v,t}$, then $(\phi f) \in L^2_{x,v,t}$ and

$$\partial_t(\phi f^n) + v \cdot \nabla_x(\phi f^n) =$$
$$= \phi G^n + f^n \left(\partial_t \phi + v \cdot \nabla_x \phi - \phi - \nabla_x V \cdot \nabla_v \phi\right) +$$
$$+ \nabla_v \cdot (\phi f^n \nabla_x V) =: g^n \ \in \ L^2_{x,t}(H_v^{-1}) \ .$$

- Golse, Perthame, Sentis '85:

$$\rho_R^n := \int_{|v| \le R} f^n \, dv \xrightarrow{L^2_{x,t}(U)} \rho_R^\infty \ .$$

- As $(f^n)_n$ is weakly precompact in $L^1(U \times \mathbb{R}^3)$:

$$\exists \rho^\infty = \lim_{R \to \infty} \rho_R^\infty = \lim_{n \to \infty} \rho^n \quad \text{in} \quad L^2_{x,t}(U) \ .$$

Convergence to equilibrium, II

By boundedness of the free energy from below and integrating the production of free energy we obtain

$$0 \le -\int_0^\infty \iint_{\mathbb{R}^6} (\gamma(E_f) - f)(E_f - \gamma^{-1}(f))\, dv\, dx\, dt < \infty\ .$$

Hence

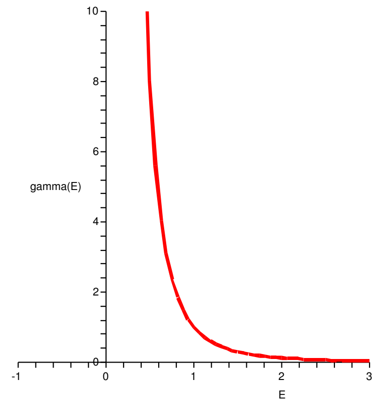
$$0 = \lim_{n\to\infty} \int_0^\infty \iint_{\mathbb{R}^6} (\gamma(E_{f^n}) - f^n)(E_{f^n} - \gamma^{-1}(f^n))\, dv\, dx\, dt\ .$$

Finally implying $f^\infty = G^\infty$. Boundedness in L^1 and L^∞ on $\mathbb{R}^6 \times [0,T)$ and choosing particular test-functions in the weak formulation of the problem yields

$$f^n \rightharpoonup G^\infty := \gamma\left(\frac{|v|^2}{2} - \bar{\mu}(\rho^\infty(x,t))\right) = \gamma\left(\frac{|v|^2}{2} + V(x) - \mu^\infty\right)\ .$$

Ex. 1, fast diffusion case

- Maxwellian is a negative power of the energy, $\gamma(E) := \frac{D}{E^k}$, $D > 0$ and $k > 5/2$.



- $$\Rightarrow \partial_t \rho = \nabla \cdot \left(\Theta(k) \nabla (\rho^{\frac{k-5/2}{k-3/2}}) + \rho \nabla V \right).$$

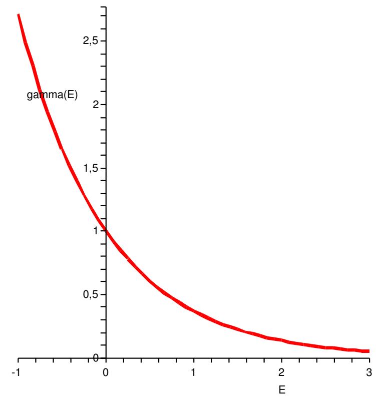
  Observe $0 < \frac{k-5/2}{k-3/2} < 1$ and $\nu'(\rho) = \Theta \frac{2k-5}{2k-3} \rho^{\frac{-1}{k-\frac{3}{2}}} \xrightarrow{\rho \to 0} \infty$.

- Sufficient confinement of the potential

$$V(x) \geq C|x|^q, \quad \text{a.e. for} \quad |x| > R \quad \text{with} \quad q > \frac{3}{k - \frac{5}{2}}.$$

Ex. 2, borderline case

- Maxwell distribution $\gamma(E) = \exp(-E)$ leads to the linear kinetic BGK model (simplified version).



- $\Rightarrow$ Linear drift-diffusion equation

$$\partial_t \rho = \nabla \cdot \left(\nabla \rho + \rho \nabla V\right).$$

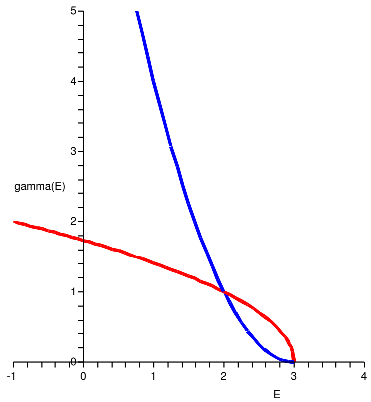
- $\nu(\rho) = \rho$ and the diffusivity $\nu'(\rho) \equiv 1$.
- Growth of the potential

$$V(x) \geq q \log(|x|), \quad \text{a.e. for} \quad |x| > R \quad \text{with} \quad q > 3.$$

Ex. 3, porous medium case

- Cut-off power as Gibbs state:

$$\gamma(E) = (E_2 - E)_+^k \,, \quad k > 0$$



- $\Rightarrow$ Porous medium equation

$$\partial_t \rho = \nabla \cdot \left(\Theta(k) \nabla (\rho^{\frac{k+5/2}{k+3/2}}) + \rho \nabla V \right)$$

$1 < \frac{k+5/2}{k+3/2} < \frac{5}{3}$ and $\nu'(\rho) = \Theta \frac{2k+5}{2k+3} \rho^{\frac{1}{k+\frac{3}{2}}} \xrightarrow{\rho \to 0} 0$.

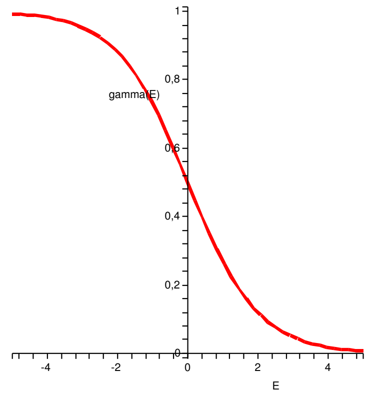
- Potential (μ^* is the upper bound for the Fermi enery)

$\left(E_2 + \mu^* - V(x)\right)_+ = \mathcal{O}\left(\frac{1}{|x|^q}\right)$ a.e., $q > \dfrac{3}{k + 3/2}$ as $|x| \to \infty$

Ex. 4, Fermi-Dirac statistics

For the Fermi-Dirac distribution

$$\gamma(E) = \frac{1}{\exp(E) + \alpha}$$

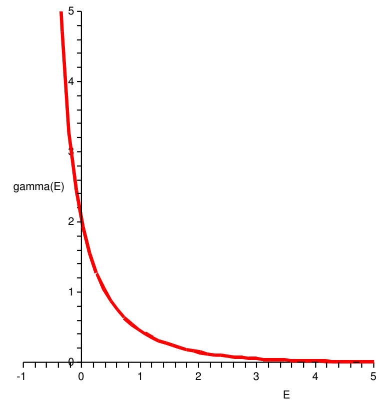


we obtain $\quad \partial_t \rho = \nabla \cdot \big(D(\rho) \nabla \rho + \rho \nabla V \big).$

$$\begin{aligned} D(\rho) &= \nu'(\rho) = \frac{-\alpha}{(2\pi)^{3/2}} \frac{\rho}{\mathrm{Li}_{1/2}\big((\mathrm{Li}_{3/2}^{-1})(\frac{-\alpha\rho}{(2\pi)^{3/2}})\big)} \\ &= 1 + \frac{\sqrt{2}}{4} \frac{\alpha\rho}{(2\pi)^{3/2}} + \mathcal{O}(\rho^2)\ , \quad \text{as} \quad \rho \to 0\ . \end{aligned}$$

with the polylogarithmic function $\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}$.

Ex. 5, Bose-Einstein statistics



For the Bose-Einstein distribution

$$\gamma(E) = \frac{1}{\exp(E) - \alpha}$$

the diffusivity is given by

$$\begin{aligned} D(\rho) &= \nu'(\rho) = \frac{+\alpha}{(2\pi)^{3/2}} \frac{\rho}{\mathrm{Li}_{1/2}\big((\mathrm{Li}_{3/2}^{-1})(\frac{+\alpha\rho}{(2\pi)^{3/2}})\big)} \\ &= 1 - \frac{\sqrt{2}}{4} \frac{\alpha\rho}{(2\pi)^{3/2}} + \mathcal{O}(\rho^2)\,, \quad \text{as} \quad \rho \to 0\,. \end{aligned}$$

The maximal density $\bar{\rho}$ is given by $\bar{\rho} = \frac{(2\pi)^{3/2}\zeta(\frac{3}{2})}{\alpha}$. (Riemann Zeta function $\zeta(s) := \mathrm{Li}_s(1) = \sum_{k=1}^{\infty} \frac{1}{k^s}$).

Observe: $\lim_{\rho\to\bar{\rho}} \nu'(\rho) = 0$ and $\lim_{\rho\to 0} \nu'(\rho) = 1$

Extension

- An extended model with local energy conservation:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f =$$
$$= \gamma \left(\alpha_f(x,t) \left(\frac{1}{2}|v|^2 + V(x) \right) + \mu_f(x,t) \right) - f \ ,$$

  where the parameter functions $\mu_f(x,t)$ and $\alpha_f(x,t)$ are adjusted to the position density and to the energy density of f.

- The diffusion limit of this equation is an energy transport model, see [Degond, Génieys, Jüngel, 1997].

Application to a flat rotating system of gravitating particles

Preliminaries: a kinetic description

Consider the gravitational Vlasov-Poisson-Boltzmann system

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \psi \cdot \nabla_v F = \mathcal{Q}_\omega(F)$$

where the potential ψ is given as a solution of the Poisson equation

$$\Delta \psi = \int_{\mathbb{R}^2 \times \mathbb{R}} F \, dv \, dw$$

the distribution function is concentrated on $\{((x,z),(v,w)) \in (\mathbb{R}^2 \times \mathbb{R}) \times (\mathbb{R}^2 \times \mathbb{R}) \ : \ z = 0\,, \ w = 0\}$ and $\mathcal{Q}_\omega(F)$ is a collision kernel which depends on the angular velocity ω, to be specified later

$$\psi(t,x) = -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} F(t,x,v) \, dv$$

Rotation at constant angular speed

Reduced problem in $\mathbb{R}^2$

$$(x,v) \mapsto (x\,e^{i\omega t}, (v + i\omega x)\,e^{i\omega t}) =: \mathcal{R}_{\omega,t}(x,v)$$

$$F(t,x,v) =: f(t, x\,e^{i\omega t}, (v + i\omega x)\,e^{i\omega t})) = f \circ \mathcal{R}_{\omega,t}(x,v)\ .$$

The equation satisfied by f can be written as

$$\partial_t f + v\cdot\nabla_x f + \omega^2\, x\cdot\nabla_v f + 2\,\mathrm{Re}\left(i\,\omega\,v\,\overline{\nabla_v f}\right) - \nabla_x\phi\cdot\nabla_v f = Q(f)$$

where the collision kernel Q is defined by
$Q(f) := \mathcal{Q}_\omega(F)\circ\mathcal{R}_{\omega,t}^{-1}$ and the potential ϕ is given by

$$\phi(t,x) = -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f(t,x,v)\,dv$$

in the rotating reference frame...

Written in cartesian coordinates, the equation satisfied by f is

$$\partial_t f + v \cdot \nabla_x f + \omega^2 \, x \cdot \nabla_v f + 2\,\omega\, v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f = Q(f)$$
$$\phi = -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f \; dv$$

where $a \wedge b := a^\perp \cdot b = (-a_2, a_1) \cdot (b_1, b_2) = a_1 b_2 - a_2 b_1 = \mathrm{Re}(i(a_1 + i\, a_2)(b_1 - i\, b_2))$

Local Gibbs state and collision kernel:

$$G_f(t,x,v) = \gamma \left(\frac{1}{2}\,|v|^2 + \phi(t,x) - \frac{1}{2}\,\omega^2\,|x|^2 + \mu_f(t,x) \right)$$
$$\int_{\mathbb{R}^2} G_f(t,x,v)\; dv = \int_{\mathbb{R}^2} f(t,x,v)\; dv \;, \qquad Q(f) = G_f - f$$

Polytropes

$$\partial_t f + v \cdot \nabla_x f + \omega^2 \, x \cdot \nabla_v f + 2\,\omega\, v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f = G_f - f$$

$$\phi = -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f \; dv$$

For simplicity: case of the *polytropic gases,* or *polytropes:*

$$\gamma(s) := \left(\frac{-s}{k+1}\right)_+^k \quad \text{and} \quad \bar{\mu}(\rho) = -(k+1)\left(\frac{\rho}{2\pi}\right)^{\frac{1}{k+1}}$$

$$G(s) := \int_{\mathbb{R}^2} \gamma\Big(\frac{1}{2}\,|v|^2 - s\Big)\, ds = 2\pi\left(\frac{-s}{k+1}\right)_+^{k+1}$$

A priori estimates

Mass:

$$M = \iint_{\mathbb{R}^2\times\mathbb{R}^2} f\,dx\,dv > 0$$

Free energy functional: with $\beta(s) = \int_s^0 \gamma^{-1}(\sigma)\,d\sigma$

$$\mathcal{F}[f] := \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left[f\left(\frac{1}{2}\,|v|^2 - \frac{1}{2}\,\omega^2\,|x|^2 + \frac{1}{2}\,\phi\right) + \beta(f)\right] dx\,dv$$

is such that

$$\frac{d}{dt}\mathcal{F}[f(t,\cdot,\cdot)] := \iint_{\mathbb{R}^2\times\mathbb{R}^2} \Big(G_f - f\Big)\Big(\gamma^{-1}(G_f) - \gamma^{-1}(f)\Big)\,dx\,dv$$

Critical points

Polytropes: $\gamma(s) := \left(\frac{-s}{k+1}\right)_+^k$. Local Lagrange multiplier

$$\mu_f(t,x) = \frac{1}{2}\,\phi - \frac{1}{2}\,\omega^2\,|x|^2 - \bar{\mu}(\rho)$$

“Global” Gibbs state (on a ball)

$$f^\infty(x,v) := \gamma\left(\frac{1}{2}\,|v|^2 + \phi^\infty(x) - \frac{1}{2}\,\omega^2\,|x|^2 - C\right)$$

with $\phi^\infty(x) := -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f^\infty(x,v)\,dv$

f^∞ is a critical point of $\mathcal{F}$ under the constraint $\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f^\infty(x,v)\,dx\,dv = M$

Stationary solutions

$$\beta(f) = \frac{f^q}{q-1} \quad \text{with} \quad k = \frac{1}{q-1} \quad \Longleftrightarrow \quad q = 1 + \frac{1}{k}$$

$$\phi = -\frac{1}{4\pi|x|} * \rho \quad \text{with} \quad \rho = G\left(\phi - \frac{1}{2}\,\omega^2\,|x|^2 - C\right)$$

C is determined by the condition: $\int_{\mathbb{R}^2} \rho\, dx = M$

$$G(s) = 2\pi \left(-\frac{q-1}{q}\,s\right)^{\frac{q}{q-1}} \quad \Longleftrightarrow \quad -\frac{q}{q-1}\,\rho^{q-1} + \phi_{\text{eff}} - C = 0$$

on the support of ρ, where the effective potential is

$$\phi_{\text{eff}}(x) := -\frac{1}{4\pi|x|} * \rho - \frac{1}{2}\,\omega^2\,|x|^2\,.$$

Reduced variational problem

Free energy of a local Gibbs state

$$\mathcal{F}[G_\rho] =: \mathcal{G}[\rho] \quad \text{with} \quad G_\rho(x,v) := \gamma\left(\frac{1}{2}|v|^2 + \bar{\mu}(\rho)\right)$$

Reduced variational problem takes the form

$$\mathcal{G}[\rho] = \int_{\mathbb{R}^2} \left[h(\rho) + \left(\phi(x) - \frac{1}{2}\omega^2 |x|^2\right)\rho\right] dx$$

$$h(\rho) := \int_{\mathbb{R}^2} \left[(\beta\circ\gamma)\left(\frac{1}{2}|v|^2 + \bar{\mu}(\rho)\right) + \frac{1}{2}|v|^2\,\gamma\left(\frac{1}{2}|v|^2 + \bar{\mu}(\rho)\right)\right] dv$$

$$= 2\pi \int_0^\infty \left[(\beta\circ\gamma)(s+\bar{\mu}(\rho)) + s\,\gamma(s+\bar{\mu}(\rho))\right] ds$$

Polytropes: $h(\rho) = \frac{\kappa}{m-1}\rho^m$ with $m = 2 - \frac{1}{q}$

Results

$\omega = 0$: [Rein] Under the mass constraint, both functionals $\mathcal{F}$ and $\mathcal{G}$ have a radial minimizer
[Schaeffer]: the radial minimizer is unique
[J.D., Ben Abdallah,...], [J.D., J. Fernández]: dynamical stability holds for both models

$\omega \neq 0$: [J.D., J. Fernández] (work in progress)

Theorem 2. *For any $M > 0$, there exists an angular velocity $\tilde{\omega}(M)$ such that for any $\omega \in (0, \tilde{\omega}(M))$, there is a stationary solution, which is a minimizer of the localized energy. This solution is never radially symmetric*

Schwarz foliated symmetry
For any $M > 0$, there exists an angular velocity $\hat{\omega}(M)$ such that for any $\omega \in (0, \hat{\omega}(M))$, there is a radial stationary solution

Open questions

Systematic construction of stationary solutions with higher Morse indices ?

Dynamical stability of these solutions (at the diffusion level and at the kinetic level) for $\omega \neq 0$?