Two non-conventional inequalities

Jean Dolbeault

http://www.ceremade.dauphine.fr/~dolbeaul

Ceremade, Université Paris-Dauphine

October 1st, 2021

Singularity Formation in Nonlinear PDEs

Birs Workshop 21w5503 (September 27 – October 1st, 2021)

Outline

Hardy-Littlewood-Sobolev and related inequalities

- Reverse Hardy-Littlewood-Sobolev inequalities
  ▷ *an interpolation inequality with a kernel with a positive exponent*
  [José A. Carrillo, Matias G. Delgadino, Jean Dolbeault, Rupert L. Frank, and Franca Hoffmann. Reverse Hardy-Littlewood-Sobolev inequalities. Journal de Mathématiques Pures et Appliquées, 132:133-165, Dec 2019.]
- Two-dimensional logarithmic inequalities
  ▷ *in dimension two, logarithms play a special role for scaling reasons*
  [Jean Dolbeault, Rupert L. Frank, and Louis Jeanjean. Logarithmic estimates for mean-field models in dimension two and the Schrödinger-Poisson system. Preprint arXiv: 2107.00610 & hal-03276199, to appear in C.R. Mathématiques]

Reverse Hardy-Littlewood-Sobolev inequality

Outline

The reverse HLS inequality

For any $\lambda > 0$ and any measurable function $\rho \geq 0$ on $\mathbb{R}^N$, let

$$I_\lambda[\rho] := \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \, \rho(x) \, \rho(y) \, dx \, dy$$

$$N \geq 1 \,, \quad 0 < q < 1 \,, \quad \alpha := \frac{2N - q(2N+\lambda)}{N(1-q)}$$

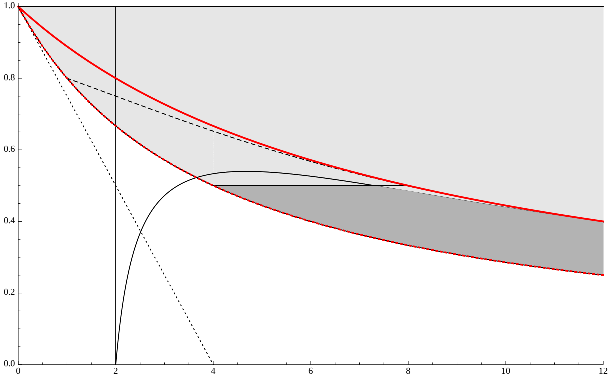
Convention: $\rho \in \mathrm{L}^p(\mathbb{R}^N)$ if $\int_{\mathbb{R}^N} |\rho(x)|^p \, dx$ for any $p > 0$

Theorem

The inequality

$$I_\lambda[\rho] \geq \mathscr{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho \, dx \right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q} \tag{1}$$

holds for any $\rho \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ with $\mathscr{C}_{N,\lambda,q} > 0$ if and only if $q > N/(N+\lambda)$
If either $N = 1$, 2 or if $N \geq 3$ and $q \geq \min\{1 - 2/N, 2N/(2N+\lambda)\}$, then there is a radial nonnegative optimizer $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$



$N = 4$, region of the parameters (λ, q) for which $\mathscr{C}_{N,\lambda,q} > 0$
Optimal functions exist in the light grey area

The conformally invariant case $q = 2N/(2N+\lambda)$

$$I_\lambda[\rho] = \iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^\lambda \rho(x)\,\rho(y)\,dx\,dy \geq \mathscr{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{2/q}$$

$$q = 2N/(2N+\lambda) \iff \alpha = 0$$

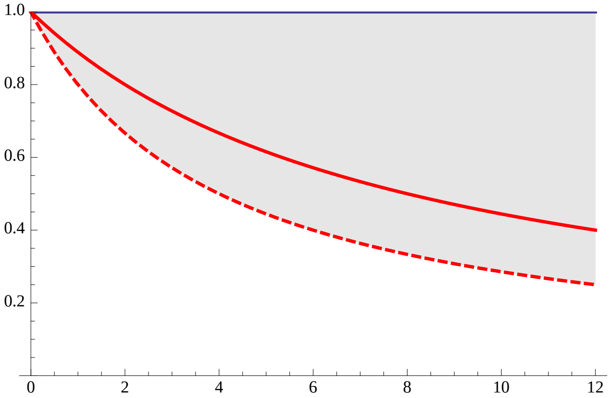
[Dou, Zhu 2015] [Ngô, Nguyen 2017]

The optimizers are given, up to translations, dilations and multiplications by constants, by

$$\rho(x) = \left(1+|x|^2\right)^{-N/q} \quad \forall\, x \in \mathbb{R}^N$$

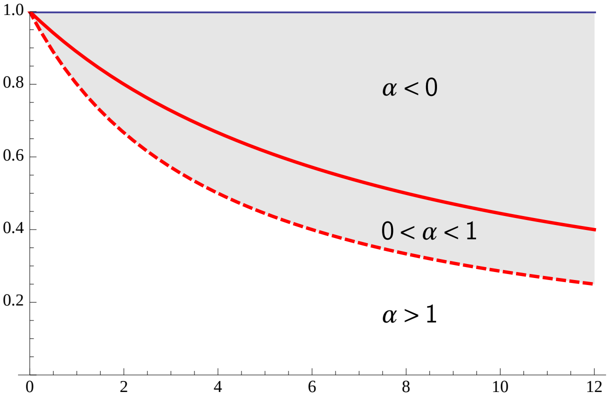
and the value of the optimal constant is

$$\mathscr{C}_{N,\lambda,q(\lambda)} = \frac{1}{\pi^{\frac{\lambda}{2}}}\,\frac{\Gamma\left(\frac{N}{2}+\frac{\lambda}{2}\right)}{\Gamma\left(N+\frac{\lambda}{2}\right)}\left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right)^{1+\frac{\lambda}{N}}$$



$N = 4$, region of the parameters (λ, q) for which $\mathscr{C}_{N,\lambda,q} > 0$
The plain, red curve is the conformally invariant case $\alpha = 0$

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}|x-y|^\lambda\,\rho(x)\,\rho(y)\,dx\,dy\geq\mathscr{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho\,dx\right)^\alpha\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{(2-\alpha)/q}$$



A Carlson type inequality

Lemma

Let $\lambda > 0$ *and* $N/(N+\lambda) < q < 1$

$$\left(\int_{\mathbb{R}^N} \rho\, dx\right)^{1-\frac{N(1-q)}{\lambda q}} \left(\int_{\mathbb{R}^N} |x|^\lambda \rho\, dx\right)^{\frac{N(1-q)}{\lambda q}} \geq c_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho^q\, dx\right)^{\frac{1}{q}}$$

$$c_{N,\lambda,q} = \frac{1}{\lambda}\left(\frac{(N+\lambda)\,q-N}{q}\right)^{\frac{1}{q}} \left(\frac{N\,(1-q)}{(N+\lambda)\,q-N}\right)^{\frac{N}{\lambda}\,\frac{1-q}{q}} \left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{1}{1-q}\right)}{2\,\pi^{\frac{N}{2}}\,\Gamma\left(\frac{1}{1-q}-\frac{N}{\lambda}\right)\Gamma\left(\frac{N}{\lambda}\right)}\right)^{\frac{1-q}{q}}$$

Equality is achieved if and only if

$$\rho(x) = \left(1+|x|^\lambda\right)^{-\frac{1}{1-q}}$$

up to translations, dilations and constant multiples

[Carlson 1934] [Levine 1948]

Proposition

Let $\lambda > 0$. If $N/(N+\lambda) < q < 1$, then $\mathscr{C}_{N,\lambda,q} > 0$

By rearrangement inequalities: prove the reverse HLS inequality for symmetric non-increasing ρ's so that

$$\int_{\mathbb{R}^N} |x-y|^\lambda \rho(y)\,dx \ge \int_{\mathbb{R}^N} |x|^\lambda \rho\,dx \quad \text{for all} \quad x \in \mathbb{R}^N$$

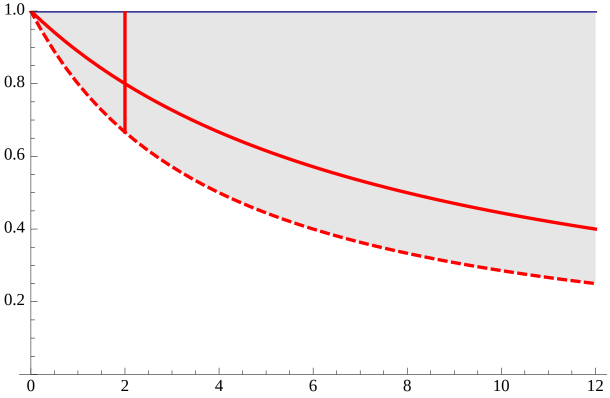
implies

$$I_\lambda[\rho] \ge \int_{\mathbb{R}^N} |x|^\lambda \rho\,dx \int_{\mathbb{R}^N} \rho\,dx$$

In the range $\frac{N}{N+\lambda} < q < 1$

$$\frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N} \rho(x)\,dx\right)^\alpha} \ge \left(\int_{\mathbb{R}^N} \rho\,dx\,dx\right)^{1-\alpha} \int_{\mathbb{R}^N} |x|^\lambda \rho\,dx \ge c_{N,\lambda,q}^{2-\alpha} \left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{\frac{2-\alpha}{q}}$$

and conclude with Carlson's inequality



$N=4$, region of the parameters (λ, q) for which $\mathscr{C}_{N,\lambda,q}>0$. The dashed, red curve is the threshold case $q=N/(N+\lambda)$

Rearrangement inequalities: ρ is symmetric non-increasing, $\int_{\mathbb{R}^N} x\,\rho\,dx=0$

$$I_2[\rho]=2\int_{\mathbb{R}^N}\rho\,dx\int_{\mathbb{R}^N}|x|^2\rho\,dx$$

The threshold case $q = N/(N+\lambda)$ and below

Proposition

If $0 < q \le N/(N+\lambda)$, *then* $\mathscr{C}_{N,\lambda,q} = 0 = \lim_{q \to N/(N+\lambda)_+} \mathscr{C}_{N,\lambda,q}$

Let $\rho, \sigma \ge 0$ such that $\int_{\mathbb{R}^N} \sigma\, dx = 1$, smooth (+ compact support)

$$\rho_\varepsilon(x) := \rho(x) + M\,\varepsilon^{-N}\sigma(x/\varepsilon)$$

Then $\int_{\mathbb{R}^N} \rho_\varepsilon\, dx = \int_{\mathbb{R}^N} \rho\, dx + M$ and, as $\varepsilon \to 0_+$

$$\int_{\mathbb{R}^N} \rho_\varepsilon^q\, dx \to \int_{\mathbb{R}^N} \rho^q\, dx +$$

$$I_\lambda[\rho_\varepsilon] \to I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\, \rho\, dx$$

If $0 < q < N/(N+\lambda)$, *i.e.*, $\alpha > 1$, take ρ_ε as a trial function,

$$\mathscr{C}_{N,\lambda,q} \le \frac{I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\, \rho\, dx}{\left(\int_{\mathbb{R}^N} \rho\, dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q\, dx\right)^{(2-\alpha)/q}} =: \mathscr{Q}[\rho, M]$$

and let $M \to +\infty$...

The threshold case: $\rho_R(x) := |x|^{-(N+\lambda)}\,\mathbb{1}_{1 \le |x| \le R}(x)$

A relaxed inequality

$$I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho \, dx \geq \mathscr{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho \, dx + M \right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q} \tag{2}$$

Proposition

If $q > N/(N+\lambda)$, the relaxed inequality (2) holds with the same optimal constant $\mathscr{C}_{N,\lambda,q}$ as (1) and admits an optimizer (ρ, M)

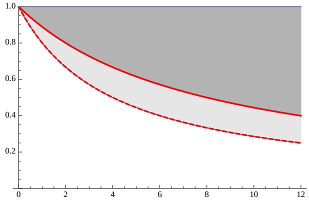
Heuristically, this is the extension of the reverse HLS inequality (1)

$$I_\lambda[\rho] \geq \mathscr{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho \, dx \right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q}$$

to measures of the form $\rho + M\delta$

Existence of minimizers and relaxation

Existence of a minimizer: first case



The $\alpha < 0$ case: dark grey region

Proposition

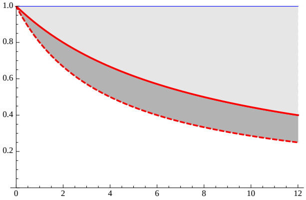
If $\lambda > 0$ and $\frac{2N}{2N+\lambda} < q < 1$, there is a minimizer ρ for $\mathscr{C}_{N,\lambda,q}$

The limit case $\alpha = 0$, $q = \frac{2N}{2N+\lambda}$ is the *conformally invariant* case: see [Dou, Zhu 2015] and [Ngô, Nguyen 2017]
Tools: radial functions, Helly's selection theorem, dominated convergence

Existence of a minimizer: second case

If $N/(N+\lambda) < q < 2N/(2N+\lambda)$ we consider the *relaxed inequality*

$$I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho\,dx \geq \mathscr{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho\,dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{(2-\alpha)/q}$$



The $0 < \alpha < 1$ case: dark grey region

Proposition

If $q > N/(N+\lambda)$, the relaxed inequality holds with the same optimal constant $\mathscr{C}_{N,\lambda,q}$ as (1) and admits an optimizer (ρ, M)

Optimizers are positive

$$\mathcal{Q}[\rho,M] := \frac{I_\lambda[\rho] + 2M\int_{\mathbb{R}^N}|x|^\lambda\,\rho\,dx}{\left(\int_{\mathbb{R}^N}\rho\,dx + M\right)^\alpha\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{(2-\alpha)/q}}$$

Lemma

Let $\lambda > 0$ and $N/(N+\lambda) < q < 1$. If $\rho \geq 0$ is an optimal function for some $M > 0$, then ρ is radial (up to a translation), monotone non-increasing and positive a.e. on $\mathbb{R}^N$

If ρ vanishes on a set $E \subset \mathbb{R}^N$ of finite, positive measure, then

$$\mathcal{Q}[\rho, M + \varepsilon\,\mathbb{1}_E] = \mathcal{Q}[\rho,M]\left(1 - \frac{2-\alpha}{q}\,\frac{|E|}{\int_{\mathbb{R}^N}\rho(x)^q\,dx}\,\varepsilon^q + o(\varepsilon^q)\right)$$

as $\varepsilon \to 0_+$, a contradiction if (ρ, M) is a minimizer of $\mathcal{Q}$

Euler–Lagrange equation and regularity

Euler–Lagrange equation for a minimizer (ρ_*, M_*)

$$\frac{2\int_{\mathbb{R}^N} |x-y|^\lambda \rho_*(y)\,dy + M_* |x|^\lambda}{I_\lambda[\rho_*] + 2M_* \int_{\mathbb{R}^N} |y|^\lambda \rho_*\,dy} - \frac{\alpha}{\int_{\mathbb{R}^N} \rho_*\,dy + M_*} - \frac{(2-\alpha)\,\rho_*(x)^{-1+q}}{\int_{\mathbb{R}^N} \rho_*(y)^q\,dy} = 0$$

We can reformulate the question of the optimizers of (1) as: when is it true that $M_* = 0$? We already know that $M_* = 0$ if

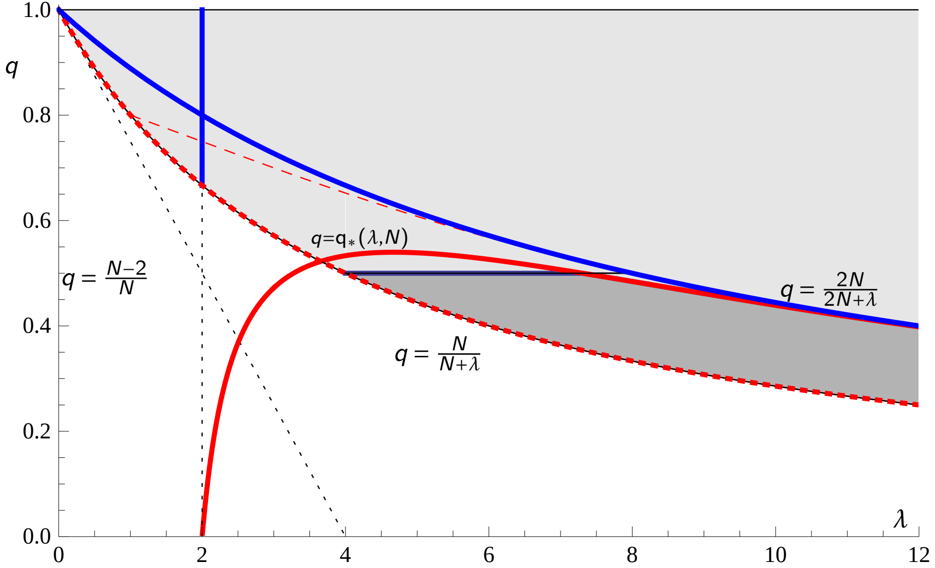
$$\frac{2N}{2N+\lambda} < q < 1$$

Proposition (regularity)

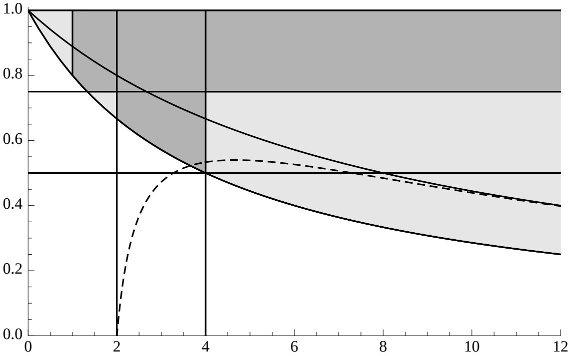
If $N \ge 3$, $\lambda > 2N/(N-2)$ and

$$\frac{N}{N+\lambda} < q < \min\left\{\frac{N-2}{N}, \frac{2N}{2N+\lambda}\right\},$$

and $(\rho_*, M_*) \in \mathrm{L}^{N(1-q)/2}(\mathbb{R}^N) \times [0,+\infty)$ is a minimizer, then $M_* = 0$

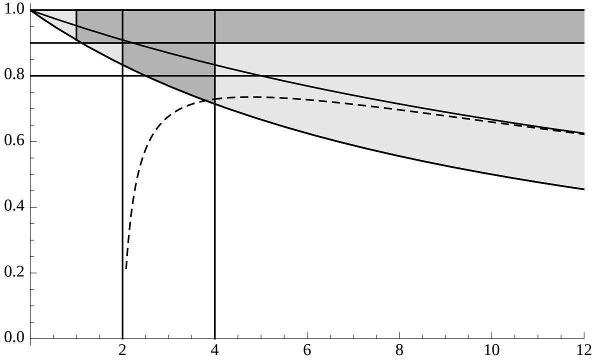

1.0
0.8
0.6
0.4
0.2
0.0
q
0
2
4
6
8
10
12
λ
q=q∗(λ,N)
q = (N−2)/N
q = 2N/(2N+λ)
q = N/(N+λ)

Uniqueness $N = 4$



[Lopes, 2017] $I_\lambda[h] \geq 0$ *if* $2 \leq \lambda \leq 4$, *for all h such that* $\int_{\mathbb{R}^N} \left(1 + |x|^\lambda\right) |h|\, dx < \infty$ *with* $\int_{\mathbb{R}^N} h\, dx = 0$ *and* $\int_{\mathbb{R}^N} x\, h\, dx = 0$...

Uniqueness $N = 10$



... or geodesic convexity in the Wasserstein-p metric for $p \in (1,2)$

Free energy point of view

A toy model

Assume that u solves the *fast diffusion with external drift* V given by

$$\frac{\partial u}{\partial t} = \Delta u^q + \nabla \cdot (u \nabla V)$$

To fix ideas: $V(x) = 1 + \frac{1}{2}|x|^2 + \frac{1}{\lambda}|x|^\lambda$. *Free energy* functional

$$\mathscr{F}[u] := \int_{\mathbb{R}^N} V\, u\, dx - \frac{1}{1-q}\int_{\mathbb{R}^N} u^q\, dx$$

- Under the mass constraint $M = \int_{\mathbb{R}^N} u\, dx$, smooth minimizers are

$$u_\mu(x) = (\mu + V(x))^{-\frac{1}{1-q}}$$

- The equation can be seen as a gradient flow

$$\frac{d}{dt}\mathscr{F}[u(t,\cdot)] = -\int_{\mathbb{R}^N} u \left|\frac{q}{1-q}\nabla u^{q-1} - \nabla V\right|^2 dx$$

A toy model (continued)

If $\lambda = 2$, the so-called *Barenblatt profile* u_μ has finite mass if and only if

$$q > q_c := \frac{N-2}{N}$$

- For $\lambda > 2$, the integrability condition is $q > 1 - \lambda/N$ but $q = q_c$ is a threshold for the regularity: the mass of $u_\mu = (\mu + V)^{1/(1-q)}$ is

$$M(\mu) := \int_{\mathbb{R}^N} u_\mu \, dx \le M_\star = \int_{\mathbb{R}^N} \left(\frac{1}{2} |x|^2 + \frac{1}{\lambda} |x|^\lambda \right)^{-\frac{1}{1-q}} dx$$

- If one tries to minimize the free energy under the mass contraint $\int_{\mathbb{R}^N} u \, dx = M$ for an arbitrary $M > M_\star$, the limit of a minimizing sequence is the measure

$$(M - M_\star)\,\delta + u_{-1}$$

A model for nonlinear springs: heuristics

$$V=\rho*W_\lambda\,,\quad W_\lambda(x):=\tfrac{1}{\lambda}\,|x|^\lambda$$

is motivated by the study of the nonnegative solutions of the evolution equation

$$\frac{\partial\rho}{\partial t}=\Delta\rho^q+\nabla\cdot(\rho\,\nabla W_\lambda*\rho)$$

Optimal functions for (1) are energy minimizers (eventually measure valued) for the *free energy* functional

$$\mathscr{F}[\rho]:=\frac{1}{2}\int_{\mathbb{R}^N}\rho\,(W_\lambda*\rho)\,dx-\frac{1}{1-q}\int_{\mathbb{R}^N}\rho^q\,dx=\frac{1}{2\lambda}\,I_\lambda[\rho]-\frac{1}{1-q}\int_{\mathbb{R}^N}\rho^q\,dx$$

under a *mass* constraint $M=\int_{\mathbb{R}^N}\rho\,dx$ while smooth solutions obey to

$$\frac{d}{dt}\mathscr{F}[\rho(t,\cdot)]=-\int_{\mathbb{R}^N}\rho\left|\tfrac{q}{1-q}\nabla\rho^{q-1}-\nabla W_\lambda*\rho\right|^2dx$$

Further recent results

- Chuqi Cao, Xingyu Li. *Large Time Asymptotic Behaviors of Two Types of Fast Diffusion Equations.* arXiv:2011.02343
- J.A. Carrillo, M. Delgadino, R. Frank, M. Lewin. *Fast diffusion leads to partial mass concentration in Keller-Segel type stationary solutions.* arXiv:2012.08586

Logarithmic Hardy-Littlewood-Sobolev (HLS) inequality

A critical nonlinear Schrödinger-Poisson system on $\mathbb{R}^2$

$$i\,\frac{\partial\psi}{\partial t}=\Delta\psi+\alpha\,V\,\psi+\beta\,W\,\psi+\gamma\,\log|\psi|^2\,\psi$$
$$-\Delta W=|\psi|^2$$

The ***critical case***

▷ dimension $d=2$ so that W has a logarithmic growth as $|x|\to+\infty$
▷ the logarithmic nonlinearity $\log|\psi|^2$ (as *e.g.* a limit case of power law nonlinearities)... soliton-like solutions of Gaussian shape called *Gaussons*
▷ an external potential with critical growth

$$V(x)=2\,\log\left(1+|x|^2\right)\quad\forall\,x\in\mathbb{R}^2$$

- Energy

$$\mathcal{E}[\psi]:=\int_{\mathbb{R}^2}|\nabla\psi|^2\,dx+\alpha\int_{\mathbb{R}^2}V\,|\psi|^2\,dx+2\pi\,\beta\int_{\mathbb{R}^2}W\,|\psi|^2\,dx+\gamma\int_{\mathbb{R}^2}|\psi|^2\,\log|\psi|^2\,dx$$

- Standing waves

$$\psi(t,x)=e^{i\,E\,t}\,u(x)$$

Minimize

$$\mathcal{E}[u] := \int_{\mathbb{R}^2} |\nabla u|^2\,dx + \alpha \int_{\mathbb{R}^2} V\,|u|^2\,dx + 2\pi\,\beta \int_{\mathbb{R}^2} W\,|u|^2\,dx + \gamma \int_{\mathbb{R}^2} |u|^2 \log|u|^2\,dx$$

on

$$\mathcal{H}_M := \left\{ u \in \mathrm{H}^1(\mathbb{R}^2) \,:\, \|u\|_2^2 = M \right\}$$

▷ What are the conditions on α, β, $\gamma \in \mathbb{R}$ for which $\mathcal{E}$ is bounded from below ?

$$V(x) = 2 \log\left(1 + |x|^2\right) \quad \forall\, x \in \mathbb{R}^2$$

Poisson equation

$$-\,\Delta W = |u|^2$$

means that W is defined only up to an additive constant: choice $W = (-\Delta)^{-1}|u|^2$ with Green kernel

$$G(x,y) = -\frac{1}{2\pi} \log|x-y| \quad \forall\,(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2$$

so that

$$W(x) \sim -\frac{\|u\|_2^2}{2\pi} \log|x| \quad \text{as} \quad |x| \to +\infty$$

Logarithmic Hardy-Littlewood-Sobolev (HLS) inequality

For any $\rho \in L^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \rho\, dx = M > 0$

$$\int_{\mathbb{R}^2} \rho \log\left(\frac{\rho}{M}\right) dx + \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\,\rho(y) \log|x-y|\, dx\, dy + M\,(1+\log\pi) \ge 0$$

[E. Carlen and M. Loss, 1992] [W. Beckner, 1993]

Equality is achieved by

$$\rho_\star(x) := \frac{M}{\pi\,(1+|x|^2)^2} \quad \forall\, x \in \mathbb{R}^2$$

Invariances: homogeneity, scalings, translations

▷ fast diffusion flows [E. Carlen, J.A. Carrillo, M. Loss, 2010] [JD, G. Jankowiak, 2014]

▷ Duality and relations with Onofri type inequalities [Onofri, 1982] [Calvez, Corrias, 2008] [JD, M.J. Esteban, G. Jankowiak, 2015]

▷ rearrangement-free proof using reflection positivity [R. Frank, E. Lieb, 2011]

▷ a useful lower bound on the free energy in the Keller-Segel model [A. Blanchet, JD, B. Perthame, 2006]

Generalized log-HLS inequalities

With $V = \log \rho_\star + Const$, for any $\rho \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^2} \rho \, dx > 0$

$$\int_{\mathbb{R}^2} \rho \log\left(\frac{\rho}{M}\right) dx + 2\tau \int_{\mathbb{R}^2} \log\left(1+|x|^2\right) \rho \, dx + M\,(1-\tau+\log\pi) \geq \frac{2}{M}\,(\tau-1) \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\,\rho(y) \log|x-y| \, dx \, dy \quad \forall\,\tau \geq 0$$

[JD, X. Li, 2019]

- If $\tau \in [0,1]$: an interpolation between log-HLS and Jensen:

$$\int_{\mathbb{R}^2} \rho \log\left(\frac{\rho}{M\,\rho_\star}\right) dx \geq 0$$

▷ What happens in the limit as $\tau \to +\infty$

Lemma

For any function $\rho \in \mathrm{L}^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \rho \, dx = M$

$$2\int_{\mathbb{R}^2} \log\left(1+|x|^2\right) \rho \, dx - M \geq \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\,\rho(y) \log|x-y| \, dx \, dy$$

Free energy point of view

A free energy of Keller-Segel type

$$\mathscr{F}_{\mathrm{a,b}}[\rho] := \int_{\mathbb{R}^2} \rho \log\left(\frac{\rho}{M}\right) dx + \mathrm{a} \int_{\mathbb{R}^2} \log\left(1+|x|^2\right) \rho\, dx - \frac{\mathrm{b}}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\,\rho(y) \log|x-y|\, dx\, dy$$

for any $\rho \in \mathrm{L}^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \rho\, dx = M$

- Keller-Segel with an external potential of critical growth

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \nabla \cdot \left[\rho\left(\frac{\mathrm{a}}{2}\nabla V + 4\pi \frac{\mathrm{b}}{M} \nabla W\right)\right]$$

▷ Range of the parameters a and b such that

$$\mathscr{F}_{\mathrm{a,b}}[\rho] \geq \mathscr{C}(\mathrm{a,b})\, M \quad \forall\, \rho \in \mathrm{L}^1_+(\mathbb{R}^2) \quad \text{such that} \quad \|\rho\|_1 = M$$

Boundedness from below of the free energy

[JD, R. Frank, L. Jeanjean, 2021]

Theorem

$\mathscr{C}(a,b) > -\infty$ if either $a=0$ and $b=-2$, or

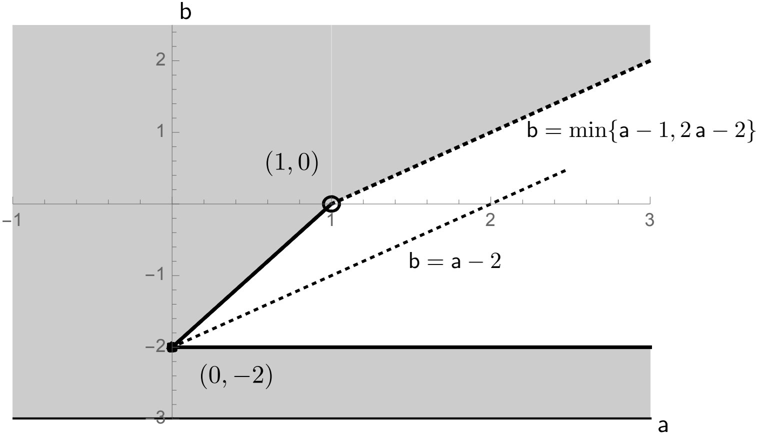
$$a>0\,,\quad -2\le b<a-1\quad \textit{and}\quad b\le 2a-2$$

If $0\le a<1$ and $b=2a-2$, there is no minimizer and

$$\mathscr{C}(a,2a-2) = -\log\left(\frac{e\,\pi}{1-a}\right)$$

If either $a<0$ or $b<-2$ or $b>\min\{a-1,2a-2\}$ or $(a,b)=(1,0)$, then

$$\inf_{\rho\in\mathscr{X}_1}\mathscr{F}_{a,b}[\rho] = -\infty$$



$$\mathscr{F}_{a,b}[\rho] = \int_{\mathbb{R}^2} \rho \log\left(\frac{\rho}{M}\right) dx + a \int_{\mathbb{R}^2} \log\left(1+|x|^2\right) \rho \, dx - \frac{b}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\,\rho(y) \log|x-y| \, dx\, dy$$

Proofs

- $\mathsf{a} < 0$: use translations

$$\int_{\mathbb{R}^2} \log\left(1+|x|^2\right)\rho_{x_0}(x)\,dx \sim 2\log|x_0|\int_{\mathbb{R}^2}\rho\,dx \quad \text{as} \quad |x_0| \to +\infty$$

- $\mathsf{b}+2-2\,\mathsf{a} > 0$: use scalings of $\rho_\lambda(x) = \lambda^2\rho(\lambda\,x)$ to get

$$\mathscr{F}_{\mathsf{a},\mathsf{b}}[\rho_\lambda] \sim (\mathsf{b}+2-2\,\mathsf{a})\log\lambda$$

- $\mathsf{b}+1-\mathsf{a} > 0$: take ρ such that that $\rho(x) = 0$ if $|x| \not\in [1,2]$, let

$$\rho_{\varepsilon,\lambda}(x) = (1-\varepsilon)\,\rho(x) + \lambda^2\,\varepsilon\,\rho\,(\lambda\,x)$$

Proofs (continued)

Lemma

If $0 \le \mathrm{a} < 1$ *and* $\mathrm{b} = 2\,\mathrm{a} - 2$,

$$\mathscr{C}(\mathrm{a},\mathrm{b}) = \inf_{\rho \in \mathrm{L}^1_+(\mathbb{R}^2),\, M=1} \left(\int_{\mathbb{R}^2} \rho \log \rho \, dx + 2\,\mathrm{a} \int_{\mathbb{R}^2} \log|x|\, \rho \, dx \right.$$
$$\left. + 2\,(\mathrm{a}-1) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \rho(x) \log \frac{1}{|x-y|}\, \rho(y)\, dx\, dy \right)$$

and there is no minimizer of the l.h.s.
The infimum of the r.h.s. is achieved if and only if, for some $\lambda > 0$,

$$\rho(x) = \frac{1-\mathrm{a}}{\pi}\, \frac{\lambda^2}{|x|^{2\mathrm{a}} \left(\lambda^2 + |x|^{2(1-\mathrm{a})}\right)^2}$$

Hints

1. consider $\rho_\lambda(x) = \lambda^{-2} \sigma(x/\lambda)$ with $\lambda \gg 1$
2. use symmetric decreasing rearrangement, set $\rho_0(x) := |x|^{2\mathrm{a}} \rho(x)$ and

A slightly more general free energy

$$\mathscr{F}^{\rm c}_{\rm a,b}[\rho] := {\rm a}\int_{\mathbb{R}^2}\log\left(1+|x|^2\right)\rho\,dx - \frac{\rm b}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\,\rho(y)\,\log|x-y|\,dx\,dy + {\rm c}\int_{\mathbb{R}^2}\rho\,\log\left(\frac{\rho}{M}\right)dx$$

By homogeneity: boundedness from below for any $\rm c > 0$ if

$$-2\,{\rm c} \le {\rm b} < \min\{{\rm a}-{\rm c},\, 2\,{\rm a}-2\,{\rm c}\}$$

▷ What about $\rm c < 0$?

Proposition

For any $({\rm a},{\rm b})\in\mathbb{R}^2$ and $M>0$, with the above notations, if $\rm c<0$, then

$$\inf \mathscr{F}^{\rm c}_{\rm a,b}[\rho] = -\infty$$

$$R_{\varepsilon,n}(x) := \frac{1}{n^2}\sum_{k,\ell=1}^{n}\varepsilon^{-2}\,\rho\left(\varepsilon^{-1}\left(x-(k,\ell)\right)\right)$$

Back to Schrödinger energies

More interpolations

Euclidean logarithmic Sobolev inequality in scale invariant form

$$\|u\|_2^2 \log\left(\frac{1}{\pi\, e}\,\frac{\|\nabla u\|_2^2}{\|u\|_2^2}\right) \ge \int_{\mathbb{R}^2} |u|^2 \log\left(\frac{|u|^2}{\|u\|_2^2}\right) dx$$

Combined with log-HLS...

Proposition

For any function $u \in \mathrm{H}^1(\mathbb{R}^2)$, *we have*

$$2\pi \int_{\mathbb{R}^2} |u|^2 \,(-\Delta)^{-1} |u|^2 \,dx \le \|u\|_2^4 \log\left(\frac{\|\nabla u\|_2}{\|u\|_2}\right)$$

Bounds on the Schrödinger energy

Let $\gamma_+ := \max\{\gamma, 0\}$ and consider

$$\mathcal{E}[u] := \int_{\mathbb{R}^2} |\nabla u|^2\,dx + \alpha \int_{\mathbb{R}^2} V\,|u|^2\,dx + 2\pi\,\beta \int_{\mathbb{R}^2} W\,|u|^2\,dx + \gamma \int_{\mathbb{R}^2} |u|^2 \log|u|^2\,dx$$

Theorem

Let α, β, γ be real parameters and assume that $M > 0$. Then

(i) $\mathcal{E}$ is not bounded from below on $\mathcal{H}_M$ if one of the following conditions is satisfied:

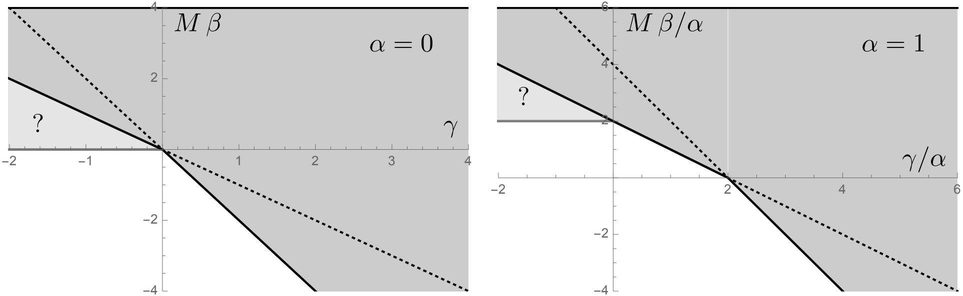
(a) $\alpha < 0$

(b) $\alpha \geq 0$ *and* $M\beta > \min\{2\alpha - \gamma, 4\alpha - 2\gamma\}$

(ii) $\mathcal{E}$ is bounded from below on $\mathcal{H}_M$ if either $\alpha = 0$, $\beta \leq 0$ and $M\beta + 2\gamma \leq 0$, or $\alpha > 0$ and one of the following conditions is satisfied:

(a) $\gamma \leq 0$ *and* $M\beta \leq 2\alpha$

(b) $\gamma > 0$, $M\beta \leq 4\alpha - 2\gamma$ *and* $M\beta < 2\alpha - \gamma$


$M\,\beta$
$\alpha = 0$
?
γ
$M\,\beta/\alpha$
$\alpha = 1$
?
γ/α

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/
▷ Lectures

The papers can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/list/
▷ Preprints and papers

For final versions, use `Dolbeault` as login and `Jean` as password

E-mail: dolbeault@ceremade.dauphine.fr

Thank you for your attention !

1. Reverse Hardy-Littlewood-Sobolev inequality
   - The reverse HLS inequality
   - Existence of minimizers and relaxation
   - Free energy point of view

2. Logarithmic Hardy-Littlewood-Sobolev (HLS) inequality
   - Logarithmic Schrödinger-Poisson system on $\mathbb{R}^2$
   - Free energy point of view
   - More interpolations and the Schrödinger energy