Hypocoercivity

Jean Dolbeault

http://www.ceremade.dauphine.fr/∼dolbeaul

Ceremade, Université Paris-Dauphine

October 7, 2019

IIT Bombay

Outline

- **From φ-entropies to H^1 hypocoercivity**
  ▷ φ-entropies and diffusions
  ▷ φ-hypocoercivity (H^1 framework)
- **An L^2 abstract result and mode-by-mode hypocoercivity**
  ▷ Abstract statement, toy model, global L^2 hypocoercivity result
  ▷ Diffusion limit, application to the torus and a more numerical point of view
  ▷ Decay rates in the Euclidean space without confinement
- **Diffusion and kinetic transport with very weak confinement**
- **The Vlasov-Poisson-Fokker-Planck system: linearization and hypocoercivity**

From φ-entropies to H^1 hypocoercivity

▷ Some references of related works
(Chafaï 2004), (Bolley, Gentil 2010)
(Baudoin 2017)
(Monmarché), (Evans, 2017)
(Arnold, Erb, 2014), (Arnold, Stürzer), (Achleitner, Arnold, Stürzer, 2016), (Achleitner, Arnold, Carlen, 2017), (Arnold, Einav, Wöhrer, 2017)

▷ In collaboration with X. Li

Definition of the φ-entropies

$$\mathcal{E}[w] := \int_{\mathbb{R}^d} \varphi(w)\, d\gamma$$

φ is a nonnegative convex continuous function on $\mathbb{R}^+$ such that $\varphi(1) = 0$ and $1/\varphi''$ is concave on $(0, +\infty)$:

$$\varphi'' \geq 0\,, \quad \varphi \geq \varphi(1) = 0 \quad \text{and} \quad (1/\varphi'')'' \leq 0$$

Classical examples

$$\varphi_p(w) := \tfrac{1}{p-1}\left(w^p - 1 - p\,(w-1)\right) \quad p \in (1, 2]$$

$$\varphi_1(w) := w \log w - (w-1)$$

The invariant measure

$$d\gamma = e^{-\psi}\, dx$$

where ψ is a *potential* such that $e^{-\psi}$ is in $L^1(\mathbb{R}^d, dx)$
$d\gamma$ is a probability measure

Diffusions

Ornstein-Uhlenbeck equation or *backward Kolmogorov equation*

$$\frac{\partial w}{\partial t} = \mathsf{L}\, w := \Delta w - \nabla\psi \cdot \nabla w$$

- $-\int_{\mathbb{R}^d} (\mathsf{L}\, w_1)\, w_2\, d\gamma = \int_{\mathbb{R}^d} \nabla w_1 \cdot \nabla w_2\, d\gamma \quad \forall\, w_1,\, w_2 \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$
- $1 = \int_{\mathbb{R}^d} w_0\, d\gamma = \int_{\mathbb{R}^d} w(t,\cdot)\, d\gamma$ and $\lim_{t\to+\infty} w(t,\cdot) = 1$
- $\frac{d}{dt}\mathcal{E}[w] = -\int_{\mathbb{R}^d} \varphi''(w)\, |\nabla_x w|^2\, d\gamma =: -\mathcal{I}[w]$ *(Fisher information)*

If for some $\Lambda > 0$: *entropy – entropy production* inequality

$$\mathcal{I}[w] \ge \Lambda\, \mathcal{E}[w] \quad \forall\, w \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$$

$$\mathcal{E}[w(t,\cdot)] \le \mathcal{E}[w_0]\, e^{-\Lambda\, t} \quad \forall\, t \ge 0$$

Fokker-Planck equation : $u = w\,\gamma$ converges to $u_\star = \gamma$

$$\frac{\partial u}{\partial t} = \Delta u + \nabla_x \cdot (u\, \nabla_x \psi)$$

Generalized Csiszár-Kullback-Pinsker inequality

(Pinsker), (Csiszár 1967), (Kullback 1967), (Cáceres, Carrillo, JD, 2002)

Proposition

Let $p \in [1,2]$, $w \in \mathrm{L}^1 \cap \mathrm{L}^p(\mathbb{R}^d, d\gamma)$ *be a nonnegative function, and assume that* $\varphi \in C^2(0,+\infty)$ *is a nonnegative strictly convex function such that* $\varphi(1) = \varphi'(1) = 0$*. If* $A := \inf_{s\in(0,\infty)} s^{2-p}\,\varphi''(s) > 0$*, then*

$$\mathcal{E}[w] \geq 2^{-\frac{2}{p}}\, A\, \min\left\{1, \|w\|^{p-2}_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}\right\} \|w-1\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}$$

Convexity, tensorization and sub-additivity

$$\int_{\mathbb{R}^{d_i}} \varphi''(w)\,|\nabla w|^2\,d\gamma_i =: \mathcal{I}_{\gamma_i}[w] \ge \Lambda_i\,\mathcal{E}_{\gamma_i}[w] \quad \forall\, w \in \mathrm{H}^1(\mathbb{R}^{d_i}, d\gamma_i)$$

Theorem

If $d\gamma_1$ and $d\gamma_2$ are two probability measures on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, then

$$\mathcal{I}_{\gamma_1\otimes\gamma_2}[w] = \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \varphi''(w)\,|\nabla w|^2\,d\gamma_1\,d\gamma_2$$
$$\ge \min\{\Lambda_1, \Lambda_2\}\,\mathcal{E}_{\gamma_1\otimes\gamma_2}[w] \quad \forall\, w \in \mathrm{H}^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, d\gamma)$$

$$\mathcal{I}\gamma_1 \otimes \gamma_2[w] = \int_{\mathbb{R}^{d_2}} \mathcal{I}\gamma_1[w]\,d\gamma_2 + \int_{\mathbb{R}^{d_1}} \mathcal{I}\gamma_2[w]\,d\gamma_1$$

$$\mathcal{E}_{\gamma_1\otimes\gamma_2}[w] \le \int_{\mathbb{R}^{d_2}} \mathcal{E}_{\gamma_1}[w]\,d\gamma_2 + \int_{\mathbb{R}^{d_1}} \mathcal{E}_{\gamma_2}[w]\,d\gamma_1 \quad \forall\, w \in \mathrm{L}^1(d\gamma_1 \otimes \gamma_2)$$

Perturbation (Holley-Stroock type) results

With $\overline{w} := \int_{\mathbb{R}^d} w\, d\gamma$, assume that

$$\Lambda\left[\int_{\mathbb{R}^d} \varphi(w)\, d\gamma - \varphi(\overline{w})\right] \le \int_{\mathbb{R}^d} \varphi''(w)|\nabla w|^2\, d\gamma \quad \forall\, w \in \mathrm{H}^1(d\gamma)$$

and, for some constants $a,\ b \in \mathbb{R}$,

$$e^{-b}\, d\gamma \le d\mu \le e^{-a}\, d\gamma$$

Lemma

If φ is a C^2 function such that $\varphi'' > 0$ and $\widetilde{w} := \int_{\mathbb{R}^d} w\, d\mu \,/ \int_{\mathbb{R}^d} d\mu$, then

$$e^{a-b}\,\Lambda \int_{\mathbb{R}^d} \left[\varphi(w) - \varphi(\widetilde{w}) - \varphi'(\widetilde{w})(w-\widetilde{w})\right] d\mu \le \int_{\mathbb{R}^d} \varphi''(w)\,|\nabla w|^2\, d\mu$$

Entropy – entropy production inequalities, linear flows

On a smooth convex bounded domain Ω, consider

$$\frac{\partial w}{\partial t} = \mathsf{L}\, w := \Delta w - \nabla\psi \cdot \nabla w\,, \quad \nabla w \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

$$\frac{d}{dt}\int_\Omega \frac{w^p - 1}{p-1}\, d\gamma = -\frac{4}{p}\int_\Omega |\nabla z|^2\, d\gamma \quad \text{and} \quad z = w^{p/2}$$

$$\frac{d}{dt}\int_\Omega |\nabla z|^2\, d\gamma \le -\,2\,\Lambda(p)\int_\Omega |\nabla z|^2\, d\gamma$$

where $\Lambda(p) > 0$ is the best constant in the inequality

$$\frac{2}{p}\,(p-1)\int_\Omega |\nabla X|^2\, d\gamma + \int_\Omega \mathrm{Hess}\,\psi : X \otimes X\, d\gamma \ge \Lambda(p)\int_\Omega |X|^2\, d\gamma$$

Proposition

$$\int_\Omega \frac{w^p - 1}{p-1}\, d\gamma \le \frac{4}{p\,\Lambda}\int_\Omega |\nabla w^{p/2}|^2\, d\gamma \quad \textit{for any } w \textit{ s.t.} \quad \int_\Omega w\, d\gamma = 1$$

An interpolation inequality

Corollary

Assume that $q \in [1,2)$. *With* $\Lambda = \Lambda(2/q)$, *we have*

$$\frac{\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \|f\|^2_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)}}{2-q} \le \frac{1}{\Lambda} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma \quad \forall\, f \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$$

Improved entropy – entropy production inequalities

In the special case $\psi(x)=|x|^2/2$, with $z=w^{p/2}$, we obtain that

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla z|^2\,d\gamma+\int_{\mathbb{R}^d}|\nabla z|^2\,d\gamma\le-\,\frac{2}{p}\,\kappa_p\int_{\mathbb{R}^d}\frac{|\nabla z|^4}{z^2}\,d\gamma$$

with $\kappa_p=(p-1)\,(2-p)/p$

Cauchy-Schwarz: $\left(\int_{\mathbb{R}^d}|\nabla z|^2\,d\gamma\right)^2\le\int_{\mathbb{R}^d}\frac{|\nabla z|^4}{z^2}\,d\gamma\int_{\mathbb{R}^d}z^2\,d\gamma$

$$\frac{d}{dt}\mathcal{I}[w]+2\,\mathcal{I}[w]\le-\,\kappa_p\,\frac{\mathcal{I}[w]^2}{1+(p-1)\,\mathcal{E}[w]}$$

Proposition

Assume that $q\in(1,2)$ and $d\gamma=(2\pi)^{-d/2}\,e^{-|x|^2/2}\,dx$. There exists a strictly convex function F such that $F(0)=0$ and $F'(0)=1$ and

$$F\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}-1\right)\le\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}\quad\textit{if }\|f\|_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)}=1$$

φ-hypocoercivity (H^1 framework)

▷ adapt the strategy of φ-entropies to kinetic equations

▷ Villani's strategy: derive H^1 estimates (using a twisted Fisher information) and then use standard interpolation inequalities to establish entropy decay rates

The twisted Fisher information *is not* the derivative of the φ-entropy

The *kinetic Fokker-Planck equation*, or *Vlasov-Fokker-Planck equation*:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v\, f) \tag{1}$$

with $\psi(x) = |x|^2/2$ and $\|f\|_{\mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$ has a unique nonnegative stationary solution

$$f_\star(x,v) = (2\,\pi)^{-d}\, e^{-\frac{1}{2}(|x|^2+|v|^2)}$$

and the function $g = f/f_\star$ solves the *kinetic Ornstein-Uhlenbeck equation*

$$\frac{\partial g}{\partial t} + \mathsf{T} g = \mathsf{L}\, g$$

with transport operator T and Ornstein-Uhlenbeck operator L given by

$$\mathsf{T} g := v \cdot \nabla_x g - x \cdot \nabla_v g \quad \text{and} \quad \mathsf{L}\, g := \Delta_v g - v \cdot \nabla_v g$$

Sharp rates for the kinetic Fokker-Planck equation

Let $\psi(x) = |x|^2/2$, $d\mu := f_\star\, dx\, dv$, $\mathcal{E}[g] := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_p(g)\, d\mu$

Proposition

Let $p \in [1,2]$ and consider a nonnegative solution $g \in \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)$ of the kinetic Fokker-Planck equation. There is a constant $\mathcal{C} > 0$ such that

$$\mathcal{E}[g(t,\cdot,\cdot)] \le \mathcal{C}\, e^{-t} \quad \forall\, t \ge 0$$

and the rate e^{-t} is sharp as $t \to +\infty$

(Villani), (Arnold, Erb): a twisted *Fisher information* functional

$$\mathcal{J}_\lambda[h] = (1-\lambda)\int_{\mathbb{R}^d} |\nabla_v h|^2\, d\mu + (1-\lambda)\int_{\mathbb{R}^d} |\nabla_x h|^2\, d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2\, d\mu$$

(Arnold, Erb) relies on $\lambda = 1/2$ and $\frac{d}{dt}\mathcal{J}_{1/2}[h(t,\cdot)] \le -\mathcal{J}_{1/2}[h(t,\cdot)]$

Improved rates (in the large entropy regime)

Rewrite the decay of the *Fisher information* functional as

$$-\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^d} X^\perp\cdot\mathfrak{M}_0\,X\,d\mu=\int_{\mathbb{R}^d} X^\perp\cdot\mathfrak{M}_1\,X\,d\mu+\int_{\mathbb{R}^d} Y^\perp\cdot\mathfrak{M}_2\,Y\,d\mu$$

where $X=(\nabla_v h,\nabla_x h)\;,\quad Y=(\mathsf{H}_{vv},\mathsf{H}_{xv},\mathsf{M}_{vv},\mathsf{M}_{xv})$

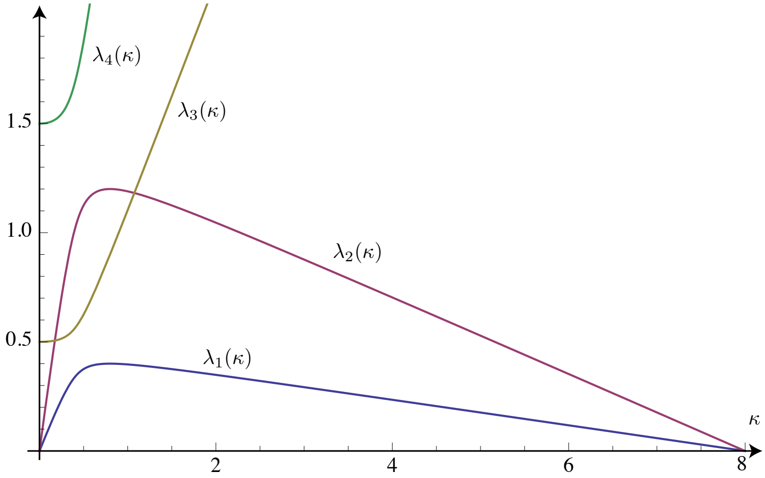
$$\mathfrak{M}_0=\begin{pmatrix}1&\lambda\\ \lambda&\nu\end{pmatrix}\otimes\mathrm{Id}_{\mathbb{R}^d}\,,\quad \mathfrak{M}_1=\begin{pmatrix}1-\lambda&\frac{1+\lambda-\nu}{2}\\ \frac{1+\lambda-\nu}{2}&\lambda\end{pmatrix}\otimes\mathrm{Id}_{\mathbb{R}^d}$$

$$\mathfrak{M}_2=\begin{pmatrix}1&\lambda&-\frac{\kappa}{2}&-\frac{\kappa\,\lambda}{2}\\ \lambda&\nu&-\frac{\kappa\,\lambda}{2}&-\frac{\kappa\,\nu}{2}\\ -\frac{\kappa}{2}&-\frac{\kappa\,\lambda}{2}&2\,\kappa&2\,\kappa\,\lambda\\ -\frac{\kappa\,\lambda}{2}&-\frac{\kappa\,\nu}{2}&2\,\kappa\,\lambda&2\,\kappa\nu\end{pmatrix}\otimes\mathrm{Id}_{\mathbb{R}^d\times\mathbb{R}^d}$$

With constant coefficients

$$\lambda_\star(\lambda,\nu)=\max\left\{\min_X\frac{X^\perp\cdot\mathfrak{M}_1\,X}{X^\perp\cdot\mathfrak{M}_0\,X}\;:\;(\lambda,\nu)\in\mathbb{R}^2\text{ s.t. }\mathfrak{M}_2\geq 0\right\}$$

For $(\lambda, \nu) = (1/2)$, $\lambda_\star = 1/2$ and the eigenvalues of $\mathfrak{M}_2(\frac{1}{2}, 1)$ are given as a function of $\kappa = 8\,(2-p)/p \in [0,8]$ are all nonnegative



We know that

$$Y^\perp \cdot \mathfrak{M}_2 \, Y \geq \lambda_1(p,\lambda) \, |Y|^2$$

for some $\lambda_1(p,\lambda) > 0$ and $|Y|^2 \geq \|\mathsf{M}_{vv}\|^2$ so that, by Cauchy-Schwarz,

$$\left(\int_{\mathbb{R}^d} |\nabla_v h|^2 \, d\mu \right)^2 \leq \int_{\mathbb{R}^d} h^2 \, d\mu \int_{\mathbb{R}^d} \|\mathsf{M}_{vv}\|^2 \, d\mu \leq c_0 \int_{\mathbb{R}^d} \|\mathsf{M}_{vv}\|^2 \, d\mu$$

Theorem

Let $p \in (1,2)$ and h be a solution of the kinetic Ornstein-Uhlenbeck equation. Then there exists a function $\lambda : \mathbb{R}^+ \to [1/2, 1)$ such that $\lambda(0) = \lim_{t\to+\infty} \lambda(t) = 1/2$ and a function $\rho > 1/2$ s.t.

$$\frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t,\cdot)] \leq -\,2\,\rho(t)\,\mathcal{J}_{\lambda(t)}[h(t,\cdot)]$$

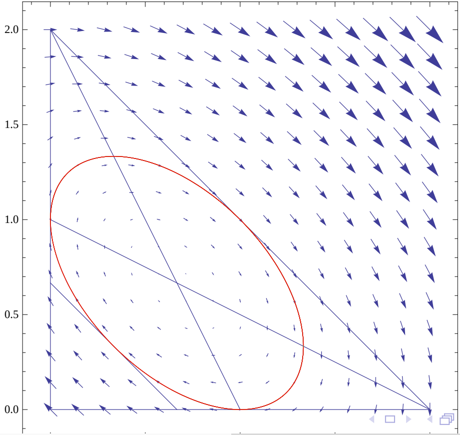
As a consequence, for any $t \geq 0$ we have the global estimate

$$\mathcal{J}_{\lambda(t)}[h(t,\cdot)] \leq \mathcal{J}_{1/2}[h_0] \, \exp\left(-\,2 \int_0^t \rho(s) \, ds \right)$$

Let us define $\mathsf{a} := e^t \int_{\mathbb{R}^d} |\nabla_v h|^2\, d\mu$, $\mathsf{b} := e^t \int_{\mathbb{R}^d} \nabla_v h \cdot \nabla_x h\, d\mu$, $\mathsf{c} := e^t \int_{\mathbb{R}^d} |\nabla_x h|^2\, d\mu$ and $\mathsf{j} := \mathsf{a} + \mathsf{b} + \mathsf{c}$

$$\frac{d\,\mathsf{a}}{dt} \le \mathsf{a} - 2\,(\mathsf{j} - \mathsf{c})\,, \quad \frac{d\,\mathsf{c}}{dt} \le 2\,(\mathsf{j} - \mathsf{a}) - \mathsf{c} \quad \text{and} \quad \frac{d\,\mathsf{j}}{dt} \le 0$$

with the constraints $\mathsf{a} \ge 0$, $\mathsf{c} \ge 0$ and $\mathsf{b}^2 \le \mathsf{a}\,\mathsf{c}$



An abstract hypocoercivity result and mode-by-mode hypocoercivity

▷ Abstract statement, toy model and a global L^2 hypocoercivity result

▷ Mode-by-mode hypocoercivity

▷ Application to the torus and numerics

▷ Decay rates in the whole space

Collaboration with E. Bouin, S. Mischler, C. Mouhot, C. Schmeiser

An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F \tag{2}$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$

$$\mathsf{A} := \Big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\Big)^{-1}(\mathsf{T}\Pi)^*$$

* denotes the adjoint with respect to $\langle\cdot,\cdot\rangle$

Π is the orthogonal projection onto the null space of L

The assumptions

λ_m, λ_M, and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity:*

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \, \|(1-\Pi)F\|^2 \tag{H1}$$

▷ *macroscopic coercivity:*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \, \|\Pi F\|^2 \tag{H2}$$

▷ *parabolic macroscopic dynamics:*

$$\Pi \mathsf{T} \Pi \, F = 0 \tag{H3}$$

▷ *bounded auxiliary operators:*

$$\|\mathsf{AT}(1-\Pi)F\| + \|\mathsf{AL}F\| \leq C_M \, \|(1-\Pi)F\| \tag{H4}$$

The estimate

$$\frac{1}{2}\,\frac{d}{dt}\|F\|^2 = \langle \mathsf{L}F, F\rangle \leq -\lambda_m \, \|(1-\Pi)F\|^2$$

is not enough to conclude that $\|F(t,\cdot)\|^2$ decays exponentially

Equivalence and entropy decay

For some $\delta > 0$ to be determined later, the L^2 entropy / Lyapunov functional is defined by

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

as in (J.D.-Mouhot-Schmeiser) so that $\langle \mathsf{AT}\Pi F, F\rangle \sim \|\Pi F\|^2$ and

$$\begin{aligned}-\frac{d}{dt}\mathsf{H}[F] &=: \mathsf{D}[F]\\ &= -\,\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT}\Pi F, F\rangle\\ &\quad -\,\delta\,\mathrm{Re}\langle \mathsf{TA}F, F\rangle + \delta\,\mathrm{Re}\langle \mathsf{AT}(1-\Pi)F, F\rangle - \delta\,\mathrm{Re}\langle \mathsf{AL}F, F\rangle\end{aligned}$$

▷ *entropy decay rate:* for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$\lambda\,\mathsf{H}[F] \le \mathsf{D}[F]$$

▷ *norm equivalence* of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \frac{2+\delta}{4}\,\|F\|^2$$

Exponential decay of the entropy

$$\lambda = \frac{\lambda_M}{3\,(1+\lambda_M)}\,\min\left\{1,\lambda_m,\frac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\},\ \delta = \frac{1}{2}\,\min\left\{1,\lambda_m,\frac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\}$$

$$h_1(\delta,\lambda) := (\delta\,C_M)^2 - 4\left(\lambda_m - \delta - \frac{2+\delta}{4}\,\lambda\right)\left(\frac{\delta\,\lambda_M}{1+\lambda_M} - \frac{2+\delta}{4}\,\lambda\right)$$

Theorem

Let L *and* T *be closed linear operators (respectively Hermitian and anti-Hermitian) on* $\mathcal{H}$. *Under* (H1)–(H4), *for any* $t \geq 0$

$$\mathsf{H}[F(t,\cdot)] \leq \mathsf{H}[F_0]\,e^{-\lambda_\star\,t}$$

where $\lambda_\star$ *is characterized by*

$$\lambda_\star := \sup\left\{\lambda>0\,:\,\exists\,\delta>0\ s.t.\ h_1(\delta,\lambda)=0\,,\ \lambda_m-\delta-\tfrac{1}{4}\,(2+\delta)\,\lambda>0\right\}$$

Sketch of the proof

- Since $\mathsf{AT\Pi} = \big(1 + (\mathsf{T\Pi})^* \mathsf{T\Pi}\big)^{-1} (\mathsf{T\Pi})^* \mathsf{T\Pi}$, from (H1) and (H2)

$$-\langle \mathsf{L} F, F\rangle + \delta \langle \mathsf{AT\Pi} F, F\rangle \geq \lambda_m \, \|(1-\Pi) F\|^2 + \frac{\delta\,\lambda_M}{1+\lambda_M} \, \|\Pi F\|^2$$

- By (H4), we know that

$$|\mathrm{Re}\langle \mathsf{AT}(1-\Pi)F, F\rangle + \mathrm{Re}\langle \mathsf{AL} F, F\rangle| \leq C_M \, \|\Pi F\| \, \|(1-\Pi)F\|$$

- The equation $G = \mathsf{A}F$ is equivalent to $(\mathsf{T\Pi})^* F = G + (\mathsf{T\Pi})^* \, \mathsf{T\Pi}\, G$

$$\langle \mathsf{TA} F, F\rangle = \langle G, (\mathsf{T\Pi})^* \, F\rangle = \|G\|^2 + \|\mathsf{T\Pi} G\|^2 = \|\mathsf{A}F\|^2 + \|\mathsf{TA}F\|^2$$

$$\langle G, (\mathsf{T\Pi})^* \, F\rangle \leq \|\mathsf{TA}F\| \, \|(1-\Pi)F\| \leq \frac{1}{2\,\mu} \, \|\mathsf{TA}F\|^2 + \frac{\mu}{2} \, \|(1-\Pi)F\|^2$$

$$\|\mathsf{A}F\| \leq \frac{1}{2} \, \|(1-\Pi)F\| \,, \; \|\mathsf{TA}F\| \leq \|(1-\Pi)F\| \,, \; |\langle \mathsf{TA}F, F\rangle| \leq \|(1-\Pi)F\|^2$$

- With $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$

$$\mathsf{D}[F] - \lambda\,\mathsf{H}[F] \geq (\lambda_m - \delta)\, X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M} \, Y^2 - \delta\, C_M \, X\, Y - \frac{2+\delta}{4} \, \lambda\, (X^2 + Y^2)$$

Hypocoercivity

Corollary

For any $\delta\in(0,2)$, if $\lambda(\delta)$ is the largest positive root of $h_1(\delta,\lambda)=0$ for which $\lambda_m-\delta-\frac14\,(2+\delta)\,\lambda>0$, then for any solution F of (2)

$$\|F(t)\|^2\le\frac{2+\delta}{2-\,\delta}\,e^{-\lambda(\delta)\,t}\,\|F(0)\|^2\quad\forall\,t\ge0$$

From the norm equivalence of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\,\delta}{4}\,\|F\|^2\le\mathsf{H}[F]\le\frac{2+\delta}{4}\,\|F\|^2$$

We use $\frac{2-\,\delta}{4}\,\|F_0\|^2\le\mathsf{H}[F_0]$ so that $\lambda_\star\ge\sup_{\delta\in(0,2)}\lambda(\delta)$

Formal macroscopic (diffusion) limit

Scaled evolution equation

$$\varepsilon \frac{dF}{dt} + \mathsf{T}F = \frac{1}{\varepsilon}\mathsf{L}F$$

on the Hilbert space $\mathcal{H}$. $F_\varepsilon = F_0 + \varepsilon\, F_1 + \varepsilon^2\, F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$

$$\begin{aligned} \varepsilon^{-1}: &\qquad \mathsf{L}F_0 = 0\,, \\ \varepsilon^{0}: &\qquad \mathsf{T}F_0 = \mathsf{L}F_1\,, \\ \varepsilon^{1}: &\qquad \tfrac{dF_0}{dt} + \mathsf{T}F_1 = \mathsf{L}F_2 \end{aligned}$$

The first equation reads as $F_0 = \Pi F_0$
The second equation is simply solved by $F_1 = -\,(\mathsf{T}\Pi)\,F_0$
After projection, the third equation is

$$\tfrac{d}{dt}\,(\Pi F_0) - \,\Pi\mathsf{T}\,(\mathsf{T}\Pi)\,F_0 = \Pi\mathsf{L}F_2 = 0$$

$$\partial_t u + (\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)\,u = 0$$

is such that $\tfrac{d}{dt}\|u\|^2 = -\,2\,\|(\mathsf{T}\Pi)\,u\|^2 \le -\,2\,\lambda_M\,\|u\|^2$

A toy problem

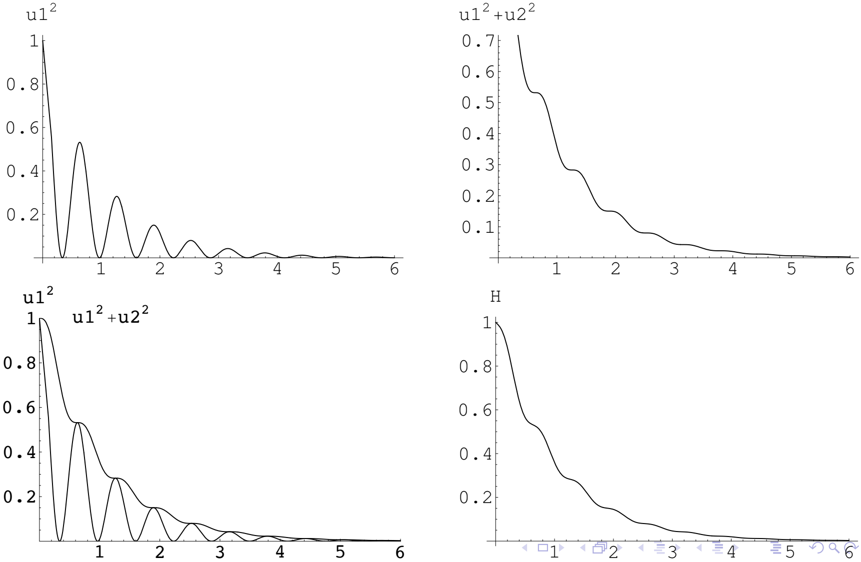
$$\frac{du}{dt} = (\mathsf{L}-\mathsf{T})\,u\,,\quad \mathsf{L} = \left(\begin{array}{cc} 0 & 0 \\ 0 & -1 \end{array}\right),\quad \mathsf{T} = \left(\begin{array}{cc} 0 & -k \\ k & 0 \end{array}\right),\quad k^2 \geq \Lambda > 0$$

Non-monotone decay, a well known picture:
see for instance (Filbet, Mouhot, Pareschi, 2006)

- H-theorem: $\frac{d}{dt}|u|^2 = -\,2\,u_2^2$
- macroscopic limit: $\frac{du_1}{dt} = -\,k^2\,u_1$
- generalized entropy: $\mathsf{H}(u) = |u|^2 - \frac{\delta\,k}{1+k^2}\,u_1\,u_2$

$$\begin{aligned}\frac{d\mathsf{H}}{dt} &= -\left(2-\frac{\delta\,k^2}{1+k^2}\right)u_2^2 - \frac{\delta\,k^2}{1+k^2}\,u_1^2 + \frac{\delta\,k}{1+k^2}\,u_1\,u_2 \\ &\leq -(2-\delta)\,u_2^2 - \frac{\delta\Lambda}{1+\Lambda}\,u_1^2 + \frac{\delta}{2}\,u_1\,u_2\end{aligned}$$

Plots for the toy problem



Mode-by-mode hypocoercivity

- ▷ Fokker-Planck equation and scattering collision operators
- ▷ A mode-by-mode hypocoercivity result
- ▷ Enlargement of the space by factorization
- ▷ Application to the torus and some numerical results

(Bouin, J.D., Mischler, Mouhot, Schmeiser)

Fokker-Planck equation with general equilibria

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad f(0,x,v) = f_0(x,v) \tag{3}$$

for a distribution function $f(t,x,v)$, with *position* variable $x \in \mathbb{R}^d$ or $x \in \mathbb{T}^d$ the flat d-dimensional torus

Fokker-Planck collision operator with a general equilibrium M

$$\mathsf{L} f = \nabla_v \cdot \left[M\, \nabla_v \left(M^{-1} f \right) \right]$$

Notation and assumptions: an *admissible local equilibrium* M is positive, radially symmetric and

$$\int_{\mathbb{R}^d} M(v)\, dv = 1\,, \quad d\gamma = \gamma(v)\, dv := \frac{dv}{M(v)}$$

γ is an *exponential weight* if

$$\lim_{|v|\to\infty} \frac{|v|^k}{\gamma(v)} = \lim_{|v|\to\infty} M(v)\, |v|^k = 0 \quad \forall\, k \in (d,\infty)$$

Definitions

$$\Theta = \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 \, M(v) \, dv = \int_{\mathbb{R}^d} (v \cdot \mathsf{e})^2 \, M(v) \, dv$$

for an arbitrary $\mathsf{e} \in \mathbb{S}^{d-1}$

$$\int_{\mathbb{R}^d} v \otimes v \, M(v) \, dv = \Theta \, \mathrm{Id}$$

Then

$$\theta = \frac{1}{d} \, \|\nabla_v M\|^2_{\mathrm{L}^2(d\gamma)} = \frac{4}{d} \int_{\mathbb{R}^d} \left|\nabla_v \sqrt{M}\right|^2 dv < \infty$$

If $M(v) = \frac{e^{-\frac{1}{2}\,|v|^2}}{(2\pi)^{d/2}}$, then $\Theta = 1$ and $\theta = 1$

$$\overline{\sigma} := \frac{1}{2} \sqrt{\theta / \Theta}$$

Microscopic coercivity property (Poincaré inequality): for all $u = M^{-1} F \in \mathrm{H}^1(M \, dv)$

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, M \, dv \geq \lambda_m \int_{\mathbb{R}^d} \left(u - \int_{\mathbb{R}^d} u \, M \, dv \right)^2 M \, dv$$

Scattering collision operators

Scattering collision operator

$$\mathsf{L}f=\int_{\mathbb{R}^d}\sigma(\cdot,v')\,\big(f(v')\,M(\cdot)-f(\cdot)\,M(v')\big)\,dv'$$

Main assumption on the *scattering rate* σ: for some positive, finite $\overline{\sigma}$

$$1\le\sigma(v,v')\le\overline{\sigma}\quad\forall\,v,\;v'\in\mathbb{R}^d$$

Example: linear BGK operator

$$\mathsf{L}f=M\rho_f-f\,,\quad\rho_f(t,x)=\int_{\mathbb{R}^d}f(t,x,v)\,dv$$

Local mass conservation

$$\int_{\mathbb{R}^d}\mathsf{L}f\,dv=0$$

and we have

$$\int_{\mathbb{R}^d}|\mathsf{L}f|^2\,d\gamma\le4\,\overline{\sigma}^2\int_{\mathbb{R}^d}|M\rho_f-f|^2\,d\gamma$$

The symmetry condition

$$\int_{\mathbb{R}^d} \big(\sigma(v,v') - \sigma(v',v)\big)\, M(v')\, dv' = 0 \quad \forall\, v \in \mathbb{R}^d$$

implies the *local mass conservation* $\int_{\mathbb{R}^d} \mathsf{L} f\, dv = 0$

Micro-reversibility, *i.e.*, the symmetry of σ, is not required

The null space of L is spanned by the local equilibrium M
L only acts on the velocity variable

Microscopic coercivity property: for some $\lambda_m > 0$

$$\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \sigma(v,v')\, M(v)\, M(v')\, \big(u(v) - u(v')\big)^2\, dv'\, dv \geq \lambda_m \int_{\mathbb{R}^d} \big(u - \rho_{u\,M}\big)^2\, M\, dv$$

holds according to Proposition 2.2 of (Degond, Goudon, Poupaud, 2000) for all $u = M^{-1} F \in \mathrm{L}^2(M\, dv)$. If $\sigma \equiv 1$, then $\lambda_m = 1$

Fourier modes

In order to perform a *mode-by-mode hypocoercivity* analysis, we introduce the Fourier representation with respect to x,

$$f(t,x,v)=\int_{\mathbb{R}^d}\hat f(t,\xi,v)\,e^{-i\,x\cdot\xi}\,d\mu(\xi)$$

$d\mu(\xi)=(2\pi)^{-d}\,d\xi$ and $d\xi$ is the Lesbesgue measure if $x\in\mathbb{R}^d$
$d\mu(\xi)=(2\pi)^{-d}\sum_{z\in\mathbb{Z}^d}\delta(\xi-z)$ is discrete for $x\in\mathbb{T}^d$

Parseval's identity if $\xi\in\mathbb{Z}^d$ and Plancherel's formula if $x\in\mathbb{R}^d$ read

$$\|f(t,\cdot,v)\|_{\mathrm{L}^2(dx)}=\left\|\hat f(t,\cdot,v)\right\|_{\mathrm{L}^2(d\mu(\xi))}$$

The Cauchy problem is now decoupled in the ξ-direction

$$\partial_t\hat f+\mathsf{T}\hat f=\mathsf{L}\hat f\,,\quad \hat f(0,\xi,v)=\hat f_0(\xi,v)$$

$$\mathsf{T}\hat f=i\,(v\cdot\xi)\,\hat f$$

For any fixed $\xi \in \mathbb{R}^d$, let us apply the abstract result with

$$\mathcal{H} = \mathrm{L}^2(d\gamma)\,, \quad \|F\|^2 = \int_{\mathbb{R}^d} |F|^2\,d\gamma\,, \quad \Pi F = M \int_{\mathbb{R}^d} F\,dv = M\,\rho_F$$

and $\mathsf{T}\hat{f} = i\,(v\cdot\xi)\,\hat{f}$, $\mathsf{T}\Pi F = i\,(v\cdot\xi)\,\rho_F\,M$,

$$\|\mathsf{T}\Pi F\|^2 = |\rho_F|^2 \int_{\mathbb{R}^d} |v\cdot\xi|^2\,M(v)\,dv = \Theta\,|\xi|^2\,|\rho_F|^2 = \Theta\,|\xi|^2\,\|\Pi F\|^2$$

(H2) *Macroscopic coercivity* $\|\mathsf{T}\Pi F\|^2 \ge \lambda_M\,\|\Pi F\|^2 \,:\, \lambda_M = \Theta\,|\xi|^2$

(H3) $\int_{\mathbb{R}^d} v\,M(v)\,dv = 0$

The operator A is given by

$$\mathsf{A}F = \frac{-\,i\,\xi\cdot\int_{\mathbb{R}^d} v'\,F(v')\,dv'}{1+\Theta\,|\xi|^2}\,M$$

A mode-by-mode hypocoercivity result

$$\begin{aligned}\|\mathsf{A}F\| = \|\mathsf{A}(1-\Pi)F\| &\le \frac{1}{1+\Theta\,|\xi|^2}\int_{\mathbb{R}^d}\frac{|(1-\Pi)F|}{\sqrt{M}}\,|v\cdot\xi|\sqrt{M}\,dv\\ &\le \frac{1}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|\left(\int_{\mathbb{R}^d}(v\cdot\xi)^2\,M\,dv\right)^{1/2}\\ &= \frac{\sqrt{\Theta}\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|\end{aligned}$$

- Scattering operator $\|\mathsf{L}F\|^2 \le 4\,\overline{\sigma}^2\,\|(1-\Pi)F\|^2$
- Fokker-Planck (FP) operator

$$\|\mathsf{AL}F\| \le \frac{2}{1+\Theta\,|\xi|^2}\int_{\mathbb{R}^d}\frac{|(1-\Pi)F|}{\sqrt{M}}\,|\xi\cdot\nabla_v\sqrt{M}|\,dv \le \frac{\sqrt{\theta}\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

In both cases with $\kappa=\sqrt{\theta}$ (FP) or $\kappa = 2\,\overline{\sigma}\,\sqrt{\Theta}$ we obtain

$$\|\mathsf{AL}F\| \le \frac{\kappa\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

$$\mathsf{TA}F(v) = -\frac{(v\cdot\xi)\,M}{1+\Theta\,|\xi|^2}\int_{\mathbb{R}^d}(v'\cdot\xi)\,(1-\Pi)F(v')\,dv'$$

is estimated by

$$\|\mathsf{TA}F\| \le \frac{\Theta\,|\xi|^2}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

(H4) holds with $C_M = \frac{\kappa\,|\xi|+\Theta\,|\xi|^2}{1+\Theta\,|\xi|^2}$

Two elementary estimates

$$\frac{\Theta\,|\xi|^2}{1+\Theta\,|\xi|^2} \ge \frac{\Theta}{\max\{1,\Theta\}}\,\frac{|\xi|^2}{1+|\xi|^2}$$

$$\frac{\lambda_M}{(1+\lambda_M)\,C_M^2} = \frac{\Theta\,(1+\Theta\,|\xi|^2)}{(\kappa+\Theta\,|\xi|)^2} \ge \frac{\Theta}{\kappa^2+\Theta}$$

Mode-by-mode hypocoercivity with exponential weights

Theorem

Let us consider an admissible M and a collision operator L satisfying Assumption (H)*, and take $\xi \in \mathbb{R}^d$. If $\hat f$ is a solution such that $\hat f_0(\xi,\cdot) \in \mathrm{L}^2(d\gamma)$, then for any $t \geq 0$, we have*

$$\left\|\hat f(t,\xi,\cdot)\right\|^2_{\mathrm{L}^2(d\gamma)} \leq 3\, e^{-\,\mu_\xi\, t} \left\|\hat f_0(\xi,\cdot)\right\|^2_{\mathrm{L}^2(d\gamma)}$$

where

$$\mu_\xi := \frac{\Lambda\,|\xi|^2}{1+|\xi|^2} \quad \textit{and} \quad \Lambda = \frac{\Theta}{3\,\max\{1,\Theta\}}\,\min\left\{1,\frac{\lambda_m\,\Theta}{\kappa^2+\Theta}\right\}$$

with $\kappa = 2\,\overline{\sigma}\,\sqrt{\Theta}$ for scattering operators
and $\kappa = \sqrt{\theta}$ for (FP) operators

Exponential convergence to equilibrium in $\mathbb{T}^d$

The unique global equilibrium in the case $x \in \mathbb{T}^d$ is given by

$$f_\infty(x,v) = \rho_\infty \, M(v) \quad \text{with} \quad \rho_\infty = \frac{1}{|\mathbb{T}^d|} \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dx \, dv$$

Theorem

Assume that γ has an exponential growth. We consider an admissible M, a collision operator L satisfying Assumption (H). There exists a positive constant C such that the solution f of (3) on $\mathbb{T}^d \times \mathbb{R}^d$ with initial datum $f_0 \in \mathrm{L}^2(dx\, d\gamma)$ satisfies

$$\|f(t,\cdot,\cdot) - f_\infty\|_{\mathrm{L}^2(dx\,d\gamma)} \le C \, \|f_0 - f_\infty\|_{\mathrm{L}^2(dx\,d\gamma)} \, e^{-\frac{1}{4}\Lambda\, t} \quad \forall\, t \ge 0$$

Enlargement of the space by factorization

A simple case (factorization of order 1) of the *factorization method* of (Gualdani, Mischler, Mouhot)

Theorem

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be Banach spaces and let $\mathcal{B}_2$ be continuously imbedded in $\mathcal{B}_1$, i.e., $\|\cdot\|_1 \le c_1 \|\cdot\|_2$. Let $\mathfrak{B}$ and $\mathfrak{A}+\mathfrak{B}$ be the generators of the strongly continuous semigroups $e^{\mathfrak{B}\,t}$ and $e^{(\mathfrak{A}+\mathfrak{B})\,t}$ on $\mathcal{B}_1$. If for all $t \ge 0$,

$$\left\| e^{(\mathfrak{A}+\mathfrak{B})\,t} \right\|_{2\to2} \le c_2\, e^{-\lambda_2\,t}\,, \quad \left\| e^{\mathfrak{B}\,t} \right\|_{1\to1} \le c_3\, e^{-\lambda_1\,t}\,, \quad \|\mathfrak{A}\|_{1\to2} \le c_4$$

where $\|\cdot\|_{i\to j}$ denotes the operator norm for linear mappings from $\mathcal{B}_i$ to $\mathcal{B}_j$. Then there exists a positive constant $C = C(c_1, c_2, c_3, c_4)$ such that, for all $t \ge 0$,

$$\left\| e^{(\mathfrak{A}+\mathfrak{B})\,t} \right\|_{1\to1} \le \begin{cases} C\left(1+|\lambda_1-\lambda_2|^{-1}\right)\, e^{-\min\{\lambda_1,\lambda_2\}\,t} & \text{for } \lambda_1 \neq \lambda_2 \\ C\,(1+t)\, e^{-\lambda_1\,t} & \text{for } \lambda_1 = \lambda_2 \end{cases}$$

Integrating the identity $\frac{d}{ds}\left(e^{(\mathfrak{A}+\mathfrak{B})\,s}\,e^{\mathfrak{B}\,(t-s)}\right)=e^{(\mathfrak{A}+\mathfrak{B})\,s}\,\mathfrak{A}\,e^{\mathfrak{B}\,(t-s)}$ with respect to $s\in[0,t]$ gives

$$e^{(\mathfrak{A}+\mathfrak{B})\,t}=e^{\mathfrak{B}\,t}+\int_0^t e^{(\mathfrak{A}+\mathfrak{B})\,s}\,\mathfrak{A}\,e^{\mathfrak{B}\,(t-s)}\,ds$$

The proof is completed by the straightforward computation

$$\begin{aligned}\left\|e^{(\mathfrak{A}+\mathfrak{B})\,t}\right\|_{1\to1}&\le c_3\,e^{-\lambda_1\,t}+c_1\int_0^t\left\|e^{(\mathfrak{A}+\mathfrak{B})\,s}\,\mathfrak{A}\,e^{\mathfrak{B}\,(t-s)}\right\|_{1\to2}ds\\&\le c_3\,e^{-\lambda_1\,t}+c_1\,c_2\,c_3\,c_4\,e^{-\lambda_1\,t}\int_0^t e^{(\lambda_1-\lambda_2)\,s}\,ds\end{aligned}$$

Weights with polynomial growth

Let us consider the measure

$$d\gamma_k := \gamma_k(v)\,dv \quad \text{where} \quad \gamma_k(v) = \pi^{d/2}\,\frac{\Gamma((k-d)/2)}{\Gamma(k/2)}\,\left(1+|v|^2\right)^{k/2}$$

for an arbitrary $k \in (d,+\infty)$

We choose $\mathcal{B}_1 = \mathrm{L}^2(d\gamma_k)$ and $\mathcal{B}_2 = \mathrm{L}^2(d\gamma)$

Theorem

Let $\Lambda = \frac{\Theta}{3\,\max\{1,\Theta\}}\,\min\left\{1,\frac{\lambda_m\,\Theta}{\kappa^2+\Theta}\right\}$ *and* $k \in (d,\infty]$. *For any* $\xi \in \mathbb{R}^d$ *if* $\hat f$ *is a solution with initial datum* $\hat f_0(\xi,\cdot) \in \mathrm{L}^2(d\gamma_k)$, *then there exists a constant* $C = C(k,d,\overline{\sigma})$ *such that*

$$\left\|\hat f(t,\xi,\cdot)\right\|^2_{\mathrm{L}^2(d\gamma_k)} \le C\,e^{-\mu_\xi\,t}\,\left\|\hat f_0(\xi,\cdot)\right\|^2_{\mathrm{L}^2(d\gamma_k)} \quad \forall\,t \ge 0$$

- Fokker-Planck: $\mathfrak{A}F = N\,\chi_R F$ and $\mathfrak{B}F = -\,i\,(v\cdot\xi)\,F + \mathsf{L}F - \mathfrak{A}F$
N and R are two positive constants, χ is a smooth cut-off function and $\chi_R := \chi(\cdot/R)$
For any R and N large enough, according to Lemma 3.8 of (Mischler, Mouhot, 2016)

$$\int_{\mathbb{R}^d}(\mathsf{L}-\mathfrak{A})(F)F\,d\gamma_k \le -\,\lambda_1\int_{\mathbb{R}^d}F^2\,d\gamma_k$$

for some $\lambda_1>0$ if $k>d$, and $\lambda_2=\mu_\xi/2\le 1/4$

- Scattering operator:

$$\begin{aligned}\mathfrak{A}F(v) &= M(v)\int_{\mathbb{R}^d}\sigma(v,v')\,F(v')\,dv'\\ \mathfrak{B}F(v) &= -\left[i\,(v\cdot\xi)+\int_{\mathbb{R}^d}\sigma(v,v')\,M(v')\,dv'\right]F(v)\end{aligned}$$

Boundedness: $\|\mathfrak{A}F\|_{\mathrm{L}^2(d\gamma)}\le\overline{\sigma}\left(\int_{\mathbb{R}^d}\gamma_k^{-1}\,dv\right)^{1/2}\|F\|_{\mathrm{L}^2(d\gamma_k)}$
$\lambda_1=1$ and $\lambda_2=\mu_\xi/2\le 1/4$

Exponential convergence to equilibrium in $\mathbb{T}^d$

The unique global equilibrium in the case $x \in \mathbb{T}^d$ is given by

$$f_\infty(x,v) = \rho_\infty \, M(v) \quad \text{with} \quad \rho_\infty = \frac{1}{|\mathbb{T}^d|} \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dx \, dv$$

Theorem

Assume that $k \in (d, \infty]$ and γ has an exponential growth if $k = \infty$. We consider an admissible M, a collision operator L satisfying Assumption (H), and Λ given by (11)
There exists a positive constant C_k such that the solution f of (3) on $\mathbb{T}^d \times \mathbb{R}^d$ with initial datum $f_0 \in L^2(dx\, d\gamma_k)$ satisfies

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^2(dx\, d\gamma_k)} \leq C_k \, \|f_0 - f_\infty\|_{L^2(dx\, d\gamma_k)} \, e^{-\frac{1}{4}\Lambda\, t} \quad \forall\, t \geq 0$$

If we represent the flat torus $\mathbb{T}^d$ by the box $[0,2\pi)^d$ with periodic boundary conditions, the Fourier variable satisfies $\xi \in \mathbb{Z}^d$. For $\xi = 0$, the microscopic coercivity implies

$$\left\|\hat{f}(t,0,\cdot) - \hat{f}_\infty(0,\cdot)\right\|_{\mathrm{L}^2(d\gamma)} \le \left\|\hat{f}_0(0,\cdot) - \hat{f}_\infty(0,\cdot)\right\|_{\mathrm{L}^2(d\gamma)} e^{-t}$$

Otherwise $\mu_\xi \ge \Lambda/2$ for any $\xi \neq 0$

Parseval's identity applies, with measure $d\gamma(v)$ and $C_\infty = \sqrt{3}$
The result with weight γ_k follows from the factorization result for some $C_k > 0$

Computation of the constants

▷ A more numerical point of view

Two simple examples: L denotes either the *Fokker-Planck operator*

$$\mathsf{L}_1 f := \Delta_v f + \nabla_v \cdot (v\, f)$$

or the *linear BGK operator*

$$\mathsf{L}_2 f := \Pi f - f$$

$\Pi f = \rho_f\, M$ is the projection operator on the normalized Gaussian function

$$M(v) = \frac{e^{-\,\frac{1}{2}\,|v|^2}}{(2\,\pi)^{d/2}}$$

and $\rho_f := \int_{\mathbb{R}^d} f\, dv$ is the spatial density

Where do we have space for improvements ?

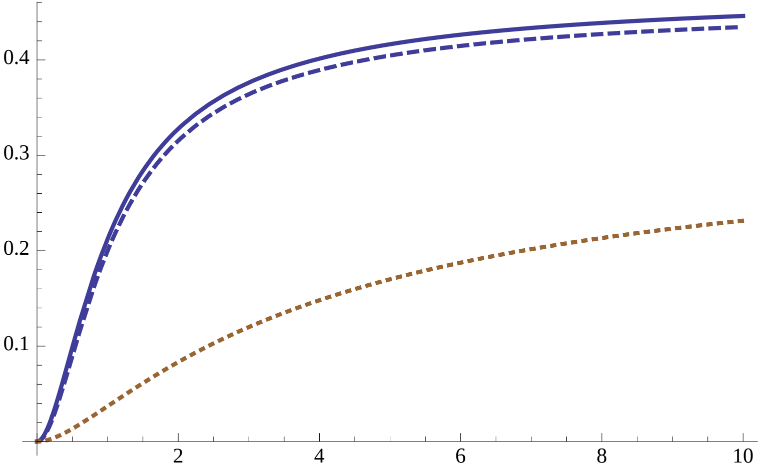
- With $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$, we wrote

$$\begin{aligned} &\mathsf{D}[F] - \lambda\,\mathsf{H}[F] \\ &\geq (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{\lambda}{2}\left(X^2 + Y^2 + \delta\,X\,Y\right) \\ &\quad \geq (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{2+\delta}{4}\,\lambda\,(X^2+Y^2) \end{aligned}$$

- We can directly study the positivity condition for the quadratic form

$$(\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{\lambda}{2}\left(X^2 + Y^2 + \delta\,X\,Y\right)$$

$\lambda_m = 1$, $\lambda_M = |\xi|^2$ and $C_M = |\xi|\,(1+|\xi|)/(1+|\xi|^2)$

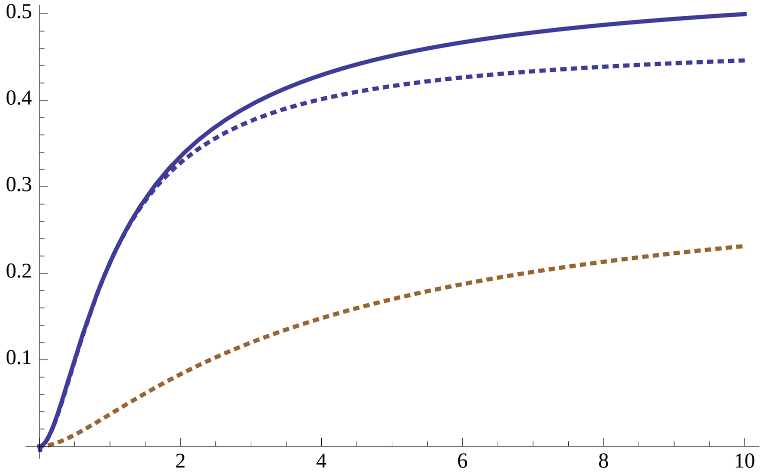

0.4
0.3
0.2
0.1
2
4
6
8
10

With $\lambda_m = 1$, $\lambda_M = |\xi|^2$ and $C_M = |\xi|\,(1+|\xi|)/(1+|\xi|^2)$, we optimize λ under the condition that the quadratic form

$$(\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{\lambda}{2}\left(X^2 + Y^2 + \delta\,X\,Y\right)$$

is positive, thus getting a $\lambda(\xi)$

- By taking also $\delta = \delta(\xi)$ where ξ is seen as a parameter, we get a better estimate of $\lambda(\xi)$


0.5
0.4
0.3
0.2
0.1
2
4
6
8
10

By taking $\delta = \delta(\xi)$, for each value of ξ we build a different Lyapunov function, namely

$$\mathsf{H}_\xi[F] := \tfrac{1}{2}\,\|F\|^2 + \delta(\xi)\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

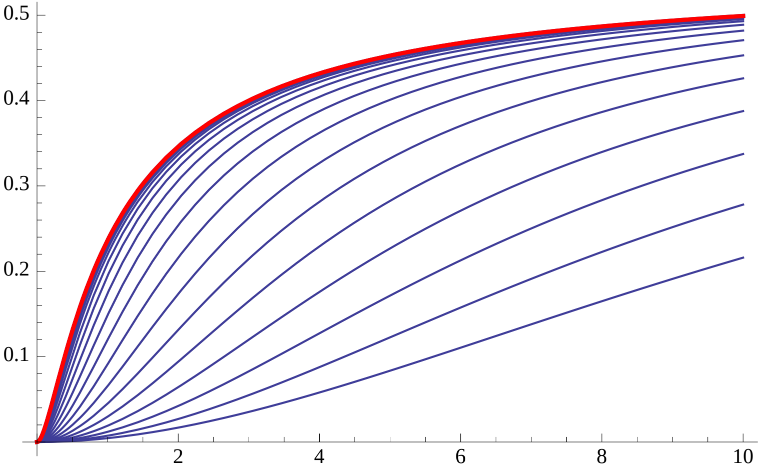
where the operator A is given by

$$\mathsf{A}F = \frac{-\,i\,\xi\cdot\int_{\mathbb{R}^d} v'\,F(v')\,dv'}{1+|\xi|^2}\,M$$

- We can consider

$$\mathsf{A}_\varepsilon F = \frac{-\,i\,\xi\cdot\int_{\mathbb{R}^d} v'\,F(v')\,dv'}{\varepsilon+|\xi|^2}\,M$$

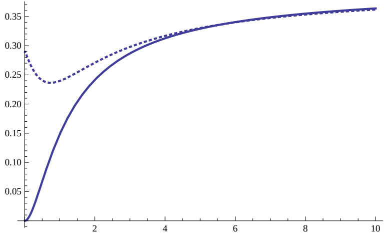
and look for the optimal value of ε...

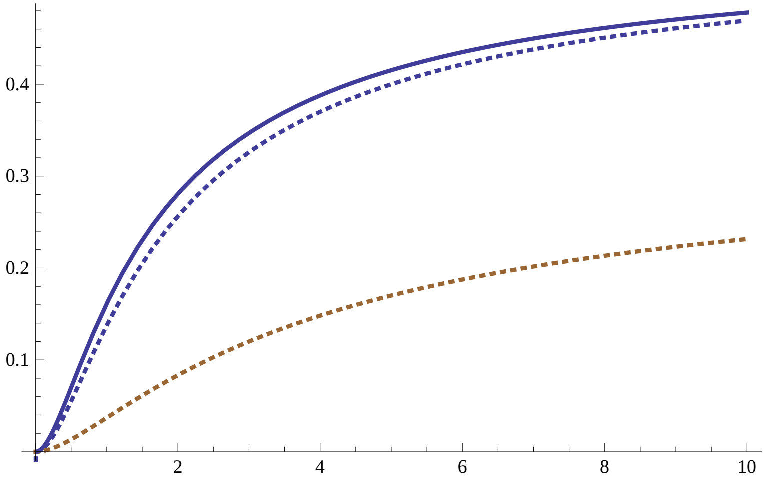

0.5
0.4
0.3
0.2
0.1
2
4
6
8
10

The dependence of λ in ε is monotone, and the limit as $\varepsilon \to 0_+$ gives the optimal estimate of λ. The operator

$$\mathsf{A}_0 F = \frac{-\, i\, \xi \cdot \int_{\mathbb{R}^d} v'\, F(v')\, dv'}{|\xi|^2}\, M$$

is not bounded anymore, but estimates still make sense and $\lim_{\xi \to 0} \delta(\xi) = 0$ (see below)




0.4
0.3
0.2
0.1
2
4
6
8
10

Theorem (Hypocoercivity on $\mathbb{T}^d$ with exponential weight)

Assume that $\mathsf{L} = \mathsf{L}_1$ *or* $\mathsf{L} = \mathsf{L}_2$*. If* f *is a solution, then*

$$\|f(t,\cdot,\cdot) - f_\infty\|^2_{\mathrm{L}^2(dx\,d\gamma)} \le \mathcal{C}_\star \, \|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma)} \, e^{-\lambda_\star t} \quad \forall\, t \ge 0$$

with $f_\infty(x,v) = M(v) \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x,v)\,dx\,dv$

$\mathcal{C}_\star \approx 1.75863$ *and* $\lambda_\star = \frac{2}{13}(5 - 2\sqrt{3}) \approx 0.236292$.

(work in progress)

Some comments on recent works

A more algebraic approach based on the spectral analysis of symmetric and non-symmetric operators

- On BGK models
(Achleitner, Arnold, Carlen)

- On Fokker-Planck models
(Arnold, Erb)
(Arnold, Stürzer)
(Arnold, Einav, Wöhrer)

Decay rates in the whole space

Algebraic decay rates in $\mathbb{R}^d$

On the whole Euclidean space, we can define the entropy

$$\mathsf{H}[f] := \frac{1}{2}\,\|f\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} + \delta\,\langle \mathsf{A}f, f\rangle_{dx\,d\gamma_k}$$

Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|^2_{\mathrm{L}^2(dx)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(dx)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(dx)}$$

proves that

$$\mathsf{H}[f] \le C\left(\mathsf{H}[f_0] + \|f_0\|^2_{\mathrm{L}^1(dx\,dv)}\right)(1+t)^{-\frac{d}{2}}$$

Theorem

Assume that γ_k has an exponential growth ($k=\infty$) or a polynomial growth of order $k>d$

There exists a constant $C>0$ such that, for any $t\ge 0$

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} \le C\left(\|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} + \|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(dx))}\right)(1+t)^{-\frac{d}{2}}$$

A direct proof... Recall that $\mu_\xi = \frac{\Lambda\,|\xi|^2}{1+|\xi|^2}$

By the Plancherel formula

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} \le C \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{-\,\mu_\xi\,t}\,|\hat{f}_0|^2\,d\xi \right) d\gamma_k$$

- if $|\xi|<1$, then $\mu_\xi \ge \frac{\Lambda}{2}\,|\xi|^2$

$$\int_{|\xi|\le 1} e^{-\,\mu_\xi\,t}\,|\hat{f}_0|^2\,d\xi \le C\,\|f_0(\cdot,v)\|^2_{\mathrm{L}^1(dx)} \int_{\mathbb{R}^d} e^{-\,\frac{\Lambda}{2}\,|\xi|^2\,t}\,d\xi \le C\,\|f_0(\cdot,v)\|^2_{\mathrm{L}^1(dx)}\;t^{-\,\frac{d}{2}}$$

- if $|\xi|\ge 1$, then $\mu_\xi \ge \Lambda/2$ when $|\xi|\ge 1$

$$\int_{|\xi|>1} e^{-\,\mu_\xi\,t}\,|\hat{f}_0|^2\,d\xi \le C\,e^{-\frac{\Lambda}{2}\,t}\,\|f_0(\cdot,v)\|^2_{\mathrm{L}^2(dx)}$$

Improved decay rate for zero average solutions

Theorem

Assume that $f_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ *with* $\iint_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)\,dx\,dv = 0$ *and* $\mathcal{C}_0 := \|f_0\|^2_{\mathrm{L}^2(d\gamma_{k+2};\,\mathrm{L}^1(dx))} + \|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(|x|\,dx))} + \|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} < \infty$

Then there exists a constant $c_k > 0$ *such that*

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} \le c_k\,\mathcal{C}_0\,(1+t)^{-\left(1+\frac{d}{2}\right)}$$

Step 1: Decay of the average in space, factorization

- *x-average in space*

$$f_\bullet(t,v) := \int_{\mathbb{R}^d} f(t,x,v)\,dx$$

with $\int_{\mathbb{R}^d} f_\bullet(t,v)\,dv = 0$ and observe that $f_\bullet$ solves a *Fokker-Planck* equation

$$\partial_t f_\bullet = \mathsf{L} f_\bullet$$

From the *microscopic coercivity property*, we deduce that

$$\|f_\bullet(t,\cdot)\|^2_{\mathrm{L}^2(d\gamma)} \le \|f_\bullet(0,\cdot)\|^2_{\mathrm{L}^2(d\gamma)}\; e^{-\lambda_m\,t}$$

- *Factorisation*

$$\|f_\bullet(t,\cdot)\|^2_{\mathrm{L}^2(|v|^2\,d\gamma_k)} \le C\;\|f_0\|^2_{\mathrm{L}^2(|v|^2\,d\gamma_k\,;\mathrm{L}^1(dx))}\; e^{-\lambda\,t}$$

Step 2: Improved decay of f

Let us define $g := f - f_\bullet\,\varphi$, with $\varphi(x) := (2\pi)^{-d/2}\,e^{-|x|^2/2}$
The Fourier transform $\hat g$ solves

$$\partial_t \hat g + \mathsf{T}\hat g = \mathsf{L}\hat g - f_\bullet\,\mathsf{T}\hat\varphi \text{ with } \mathsf{T}\hat\varphi = i\,(v\cdot\xi)\,\hat\varphi$$

Duhamel's formula

$$\hat g = \underbrace{e^{i(\mathsf{L}-\mathsf{T})\,t}\hat g_0}_{C\,e^{-\frac12\,\mu_\xi\,t}\,\|\hat g_0(\xi,\cdot)\|_{\mathrm{L}^2(d\gamma_k)}} + \int_0^t \underbrace{e^{i(\mathsf{L}-\mathsf{T})\,(t-s)}\,\big(-f_\bullet(s,v)\,\mathsf{T}\hat\varphi(\xi)\big)}_{C\,e^{-\frac{\mu_\xi}{2}\,(t-s)}\,\|f_\bullet(s,\cdot)\|_{\mathrm{L}^2(|v|^2\,d\gamma_k)}\,|\xi|\,|\hat\varphi(\xi)|}\,ds$$

- $\hat g_0(\xi,v) = \underbrace{\hat g_0(0,v)}_{=0} + \int_0^{|\xi|}\frac{\xi}{|\xi|}\cdot\nabla_\xi\hat g_0\big(\eta\,\frac{\xi}{|\xi|},v\big)\,d\eta$ yields

$$|\hat g_0(\xi,v)| \le |\xi|\,\|\nabla_\xi\hat g_0(\cdot,v)\|_{\mathrm{L}^\infty(dv)} \le |\xi|\,\|g_0(\cdot,v)\|_{\mathrm{L}^1(|x|\,dx)}$$

- $\mu_\xi = \Lambda\,|\xi|^2/(1+|\xi|^2) \ge \Lambda/2$ if $|\xi|>1$ (contribution $O(e^{-\frac{\Lambda}{2}t})$) and

$$\int_{|\xi|\le1}\int_{\mathbb{R}^d}\left|e^{i(\mathsf{L}-\mathsf{T})\,t}\hat g_0\right|^2 d\gamma_k\,d\xi \le \underbrace{\int_{\mathbb{R}^d}|\xi|^2\,e^{-\frac{\Lambda}{2}\,|\xi|^2\,t}\,d\xi}\;\|g_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(|x|\,dx))}$$

Diffusion and kinetic transport with very weak confinement

In collaboration with Emeric Bouin and Christian Schmeiser

The macroscopic Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \Delta_x u + \nabla_x \cdot (\nabla_x V\, u) = \nabla_x \left(e^{-V} \nabla_x \left(e^{V}\, u \right) \right)$$

Here $x \in \mathbb{R}^d$, $d \geq 3$, and V is a potential such that $e^{-V} \notin \mathrm{L}^1(\mathbb{R}^d)$ corresponding to a *very weak confinement*

$e^{-V}\,dx$ is an *unbounded invariant measure*

Two examples

$$V_1(x) = \gamma \, \log|x| \quad \text{and} \quad V_2(x) = \gamma \, \log\langle x \rangle$$

with $\gamma < d$ and $\langle x \rangle := \sqrt{1 + |x|^2}$ for any $x \in \mathbb{R}^d$

A first decay result

Theorem

Assume that either $d \geq 3$, $\gamma < (d-2)/2$ and $V = V_1$ or $V = V_2$
F any solution u with initial datum $u_0 \in \mathrm{L}^1_+ \cap \mathrm{L}^2(\mathbb{R}^d)$,

$$\|u(t,\cdot)\|_2^2 \leq \frac{\|u_0\|_2^2}{(1+c\,t)^{\frac{d}{2}}} \quad \textit{with} \quad c := \frac{4}{d}\,\min\left\{1, 1-\tfrac{2\,\gamma}{d-2}\right\}\,\mathcal{C}_{\mathrm{Nash}}^{-1}\,\frac{\|u_0\|_2^{4/d}}{\|u_0\|_1^{4/d}}$$

Here $\mathcal{C}_{\mathrm{Nash}}$ denotes the optimal constant in Nash's inequality

$$\|u\|_2^{2+\frac{4}{d}} \leq \mathcal{C}_{\mathrm{Nash}}\,\|u\|_1^{\frac{4}{d}}\,\|\nabla u\|_2^2 \quad \forall\, u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)$$

An extended range of exponents: with moments

Theorem

Let $d \geq 1$, $0 < \gamma < d$, $V = V_1$ *or* $V = V_2$, *and* $u_0 \in L^1_+ \cap L^2(e^V)$ *with* $\big\| |x|^k u_0 \big\|_1 < \infty$ *for some* $k \geq \max\{2, \gamma/2\}$

$$\forall\, t \geq 0\,, \quad \|u(t,\cdot)\|^2_{L^2(e^V\,dx)} \leq \|u_0\|^2_{L^2(e^V\,dx)}\,(1 + c\,t)^{-\frac{d-\gamma}{2}}$$

for some c *depending on* d, γ, k, $\|u_0\|_{L^2(e^V\,dx)}$, $\|u_0\|_1$, *and* $\big\| |x|^k u_0 \big\|_1$

An extended range of exponents: in self-similar variables

$$u_\star(t,x)=\frac{c_\star}{(1+2\,t)^{\frac{d-\gamma}{2}}}\,|x|^{-\gamma}\exp\left(-\frac{|x|^2}{2\,(1+2\,t)}\right)$$

Here the initial data need have a sufficient decay...

$c_\star$ is chosen such that $\|u_\star\|_1=\|u_0\|_1$

Theorem

Let $d\ge1$, $\gamma\in(0,d)$, $V=V_1$ *assume that*

$$\forall\,x\in\mathbb R^d\,,\quad 0\le u_0(x)\le K\,u_\star(0,x)$$

for some constant $K>1$

$$\forall\,t\ge0\,,\quad \|u(t,\cdot)-u_\star(t,\cdot)\|_p\le K\,c_\star^{1-\frac1p}\,\|u_0\|_1^{\frac1p}\left(\frac{e}{2\,|\gamma|}\right)^{\frac\gamma2\,\left(1-\frac1p\right)}\,(1+2\,t)^{-\zeta_p}$$

for any $p\in[1,+\infty)$, *where* $\zeta_p:=\frac d2\left(1-\frac1p\right)+\frac1{2\,p}\,\min\left\{2,\frac{d}{d-\gamma}\right\}$

Proofs: basic case

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2\,dx = -\,2\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} \Delta V\,|u|^2\,dx$$

with either $V = V_1$ or $V = V_2$ and

$$\Delta V_1(x) = \gamma\,\frac{d-2}{|x|^2} \quad \text{and} \quad \Delta V_2(x) = \gamma\,\frac{d-2}{1+|x|^2} + \frac{2\,\gamma}{\left(1+|x|^2\right)^2}$$

For $\gamma \le 0$: apply Nash's inequality

$$\frac{d}{dt}\,\|u\|_2^2 \le -\,2\,\|\nabla u\|_2^2 \le -\,\frac{2}{\mathcal{C}_{\mathrm{Nash}}}\,\|u_0\|_1^{-4/d}\,\|u\|_2^{2+4/d}$$

For $0 < \gamma < (d-2)/2$: Hardy-Nash inequalities

Lemma

Let $d \ge 3$ and $\delta < (d-2)^2/4$

$$\|u\|_2^{2+\frac{4}{d}} \le \mathcal{C}_\delta\left(\|\nabla u\|_2^2 - \delta\int_{\mathbb{R}^d}\frac{u^2}{|x|^2}\,dx\right)\|u\|_1^{\frac{4}{d}} \quad \forall\, u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)$$

Proofs: moments

Growth of the moment

$$M_k(t) := \int_{\mathbb{R}^d} |x|^k u \, dx$$

From the equation

$$M_k' = k\,(d+k-2-\gamma)\int_{\mathbb{R}^d} u\,|x|^{k-2}\,dx \le k\,(d+k-2-\gamma)\,M_0^{\frac{2}{k}}\,M_k^{1-\frac{2}{k}}$$

then use the Caffarelli-Kohn-Nirenberg inequality

$$\int_{\mathbb{R}^d} |x|^\gamma\,u^2\,dx \le \mathcal{C}\left(\int_{\mathbb{R}^d} |x|^{-\gamma}\,\left|\nabla\left(|x|^\gamma u\right)\right|^2 dx\right)^a \left(\int_{\mathbb{R}^d} |x|^k\,|u|\,dx\right)^{2(1-a)}$$

+ similar estimates in the non-homogeneous case, based on a non-homogeneous Caffarelli-Kohn-Nirenberg inequality

Proofs: self-similar solutions

The proof relies on *uniform decay estimates* + Poincaré inequality in self-similar variables

Proposition

Let $\gamma \in (0, d)$ *and assume that*

$$0 \le u(0,x) \le c_\star \left(\sigma + |x|^2\right)^{-\gamma/2} \exp\left(-\frac{|x|^2}{2}\right) \quad \forall\, x \in \mathbb{R}^d$$

with $\sigma = 0$ *if* $V = V_1$ *and* $\sigma = 1$ *if* $V = V_2$*. Then*

$$0 \le u(t,x) \le \frac{c_\star}{(1+2\,t)^{\frac{d-\gamma}{2}}} \left(\sigma + |x|^2\right)^{-\gamma/2} \exp\left(-\frac{|x|^2}{2\,(1+2\,t)}\right)$$

for any $x \in \mathbb{R}^d$ *and* $t \ge 0$

The kinetic Fokker-Planck equation

Let us consider the kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \mathsf{L} f$$

where $\mathsf{L} f$ is one of the two following collision operators

(a) a Fokker-Planck operator

$$\mathsf{L} f = \nabla_v \cdot \Big(M \, \nabla_v \left(M^{-1} f \right) \Big)$$

(b) a scattering collision operator

$$\mathsf{L} f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \left(f(v') \, M(\cdot) - f(\cdot) \, M(v') \right) dv'$$

Decay rates

$$\forall\,(x,v)\in\mathbb{R}^d\times\mathbb{R}^d\,,\quad \mathcal{M}(x,v)=M(v)\,e^{-V(x)}\,,\quad M(v)=(2\pi)^{-\frac{d}{2}}\,e^{-\frac{1}{2}\,|v|^2}$$

$$\textbf{(H1)}\quad 1\le\sigma(v,v')\le\overline{\sigma}\,,\quad\forall\,v\,,\,v'\in\mathbb{R}^d\,,\quad\text{for some}\quad\overline{\sigma}\ge1$$

$$\textbf{(H2)}\quad \int_{\mathbb{R}^d}\big(\sigma(v,v')-\sigma(v',v)\big)\,M(v')\,dv'=0\quad\forall\,v\in\mathbb{R}^d$$

Theorem

Let $d\ge1$, $V=V_2$ *with* $\gamma\in[0,d)$, $k>\max\{2,\gamma/2\}$ *and* $f_0\in\mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)$ *such that*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^k\,f_0\,dx\,dv+\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^k\,f_0\,dx\,dv<+\infty$$

If (H1)–(H2) *hold, then there exists* $C>0$ *such that*

$$\forall\,t\ge0\,,\quad\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathcal{M}^{-1}\,dx\,dv)}\le C\,(1+\,t)^{-\frac{d-\gamma}{2}}$$

The Vlasov-Poisson-Fokker-Planck system: linearization and hypocoercivity

In collaboration with Lanoir Addala, Xingyu Li and Lazhar M. Tayeb

Linearized Vlasov-Poisson-Fokker-Planck system

The *Vlasov-Poisson-Fokker-Planck system* in presence of an external potential V is

$$\begin{gathered} \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v\, f) \\ -\Delta_x \phi = \rho_f = \int_{\mathbb{R}^d} f \, dv \end{gathered} \tag{VPFP}$$

Linearized problem around $f_\star$: $f = f_\star\,(1 + \eta\, h)$, $\iint_{\mathbb{R}^d \times \mathbb{R}^d} h\, f_\star \, dx\, dv = 0$

$$\begin{gathered} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h = \eta\, \nabla_x \psi_h \cdot \nabla_v h \\ -\Delta_x \psi_h = \int_{\mathbb{R}^d} h\, f_\star \, dv \end{gathered}$$

Drop the $\mathcal{O}(\eta)$ term : *linearized Vlasov-Poisson-Fokker-Planck* system

$$\begin{gathered} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h = 0 \\ -\Delta_x \psi_h = \int_{\mathbb{R}^d} h\, f_\star \, dv\,, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h\, f_\star \, dx\, dv = 0 \end{gathered} \tag{VPFPlin}$$

Hypocoercivity

Let us define the norm

$$\|h\|^2 := \iint_{\mathbb{R}^d\times\mathbb{R}^d} h^2\, f_\star\, dx\, dv + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2\, dx$$

Theorem

Let us assume that $d \geq 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. Then there exist two positive constants $\mathcal{C}$ and λ such that any solution h of (VPFPlin) with an initial datum h_0 of zero average with $\|h_0\|^2 < \infty$ is such that

$$\|h(t,\cdot,\cdot)\|^2 \leq \mathcal{C}\, \|h_0\|^2\, e^{-\lambda t} \quad \forall\, t \geq 0$$

Diffusion limit

Linearized problem in the parabolic scaling

$$\varepsilon\,\partial_t h + v\cdot\nabla_x h - (\nabla_x V + \nabla_x \phi_\star)\cdot\nabla_v h + v\cdot\nabla_x\psi_h - \frac{1}{\varepsilon}\left(\Delta_v h - v\cdot\nabla_v h\right) = 0$$

$$-\Delta_x\psi_h = \int_{\mathbb{R}^d} h\,f_\star\,dv\,,\quad \iint_{\mathbb{R}^d\times\mathbb{R}^d} h\,f_\star\,dx\,dv = 0 \tag{VPFPscal}$$

Expand $h_\varepsilon = h_0 + \varepsilon\,h_1 + \varepsilon^2\,h_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon\to 0_+$. With $W_\star = V + \phi_\star$

$$\begin{aligned}
\varepsilon^{-1}: &\quad \Delta_v h_0 - v\cdot\nabla_v h_0 = 0\\
\varepsilon^{0}: &\quad v\cdot\nabla_x h_0 - \nabla_x W_\star\cdot\nabla_v h_0 + v\cdot\nabla_x\psi_{h_0} = \Delta_v h_1 - v\cdot\nabla_v h_1\\
\varepsilon^{1}: &\quad \partial_t h_0 + v\cdot\nabla_x h_1 - \nabla_x W_\star\cdot\nabla_v h_1 = \Delta_v h_2 - v\cdot\nabla_v h_2
\end{aligned}$$

With $u = \Pi h_0$, $-\Delta\psi = u\,\rho_\star$, $w = u + \psi$, equations simply mean

$$u = h_0\,,\quad v\cdot\nabla_x w = \Delta_v h_1 - v\cdot\nabla_v h_1$$

from which we deduce that $h_1 = -\,v\cdot\nabla_x w$ and

$$\partial_t u - \Delta w + \nabla_x W_\star\cdot\nabla u = 0$$

Further results

Theorem

Let us assume that $d \geq 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. For any $\varepsilon > 0$ small enough, there exist two positive constants $\mathcal{C}$ and λ, which do not depend on ε, such that any solution h of (VPFPscal) *with an initial datum h_0 of zero average and such that $\|h_0\|^2 < \infty$ satisfies*

$$\|h(t,\cdot,\cdot)\|^2 \leq \mathcal{C}\,\|h_0\|^2\,e^{-\lambda t} \quad \forall\, t \geq 0$$

Corollary

Assume that $d = 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. If f solves (VPFP) *with initial datum $f_0 = (1 + h_0)\,f_\star$ such that h_0 has zero average, $\|h_0\|^2 < \infty$ and $(1 + h_0) \geq 0$, then*

$$\|h(t,\cdot,\cdot)\|^2 \leq \mathcal{C}\,\|h_0\|^2\,e^{-\lambda t} \quad \forall\, t \geq 0$$

holds with $h = f/f_\star - 1$ for some positive constants $\mathcal{C}$ and λ

References

- L. Addala, J. D., X. Li, M.L. Tayeb. L^2-Hypocoercivity and large time asymptotics of the linearized Vlasov-Poisson-Fokker-Planck . Preprint arXiv:1909.12762
- E. Bouin, J. D., C. Schmeiser. Diffusion with very weak confinement. Preprint arXiv:1901.08323, *to appear in Kinetic and Related Models*
- E. Bouin, J. D., C. Schmeiser. A variational proof of Nash's inequality, *to appear in Pure and Applied Analysis*. Preprint arXiv:1811.12770, *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Serie IX. Rendiconti Lincei. Matematica e Applicazioni, European Mathematical Society, In press*
- J. D., X. Li, φ-entropies for Fokker-Planck and kinetic Fokker-Planck equations. *Mathematical Models and Methods in Applied Sciences*, 28(13):2637–2666, 2018.

- E. Bouin, J. D., S. Mischler, C. Mouhot, C. Schmeiser. Hypocoercivity without confinement. Preprint arXiv:1708.06180
- E. Bouin, J. D., S. Mischler, C. Mouhot, C. Schmeiser. Two examples of accurate hypocoercivity estimates based on a mode-by-mode analysis in Fourier space. In preparation.
- J. D., C. Mouhot, and C. Schmeiser. Hypocoercivity for kinetic equations with linear relaxation terms. *Comptes Rendus Mathématique*, 347(9-10):511 – 516, 2009.
- J. D., A. Klar, C. Mouhot, and C. Schmeiser. Exponential rate of convergence to equilibrium for a model describing fiber lay-down processes. *Applied Mathematics Research eXpress*, 2012.
- J. D., C. Mouhot, and C. Schmeiser. Hypocoercivity for linear kinetic equations conserving mass. *Trans. Amer. Math. Soc.*, 367(6):3807–3828, 2015.

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/
▷ Lectures

The papers can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/
▷ Preprints / papers

For final versions, use `Dolbeault` as login and `Jean` as password

Thank you for your attention !