Entropy methods and sharp functional inequalities: new results

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Outline

- Entropy and the fast diffusion equation: from functional inequalities and characterization of optimal rates to *best matching Barenblatt* functions and improved inequalities
- Fast diffusion equations on manifolds and sharp functional inequalities: rigidity results, the *carré du champ* or Bakry-Emery method, and the use of nonlinear diffusion equations
- An equivalent point of view: optimal *Keller-Lieb-Thirring estimates* on manifolds
- Symmetry and *symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities*; an introduction to the lecture of Michael Loss

Entropy and the fast diffusion equation

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A summary

▷ *Relative entropy, linearization, functional inequalities, improvements, improved rates of convergence, delays*

The fast diffusion equation

The fast diffusion equation corresponds to $m < 1$

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

Self-similar (Barenblatt) functions attract all solutions as $t \to +\infty$ [Friedmann, Kamin]

▷ *Entropy methods allow to measure the speed of convergence of any solution to $\mathcal{U}$ in norms which are adapted to the equation*
▷ *Entropy methods provide explicit constants*

- The *Bakry-Emery method* [Carrillo, Toscani], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]
- The variational approach and Gagliardo-Nirenberg inequalities: [del Pino, JD]
- Mass transportation and gradient flow issues: [Otto et al.]
- Large time asymptotics and the spectral approach: [Blanchet, Bonforte, JD, Grillo, Vázquez], [Denzler, Koch, McCann], [Seis]
- Refined relative entropy methods

Time-dependent rescaling, free energy

- *Time-dependent rescaling*: Take $u(\tau,y) = R^{-d}(\tau)\, v\,(t, y/R(\tau))$ where

$$\frac{dR}{d\tau} = R^{d(1-m)-1}\,, \quad R(0) = 1\,, \quad t = \log R$$

- The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x\, v)\,, \quad v_{|\tau=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{F}[v] := \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}\,|x|^2 v \right) dx - \mathcal{F}_0$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{F}[v] = -\mathcal{I}[v]\,, \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$

Relative entropy and entropy production

- *Stationary solution:* choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) := \left(C + \frac{1-m}{2\,m}\,|x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{F}_0$ so that $\mathcal{F}[v_\infty] = 0$

- *Entropy – entropy production inequality*

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[v] \geq 2\,\mathcal{F}[v]$$

Corollary

A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[u_0]\,e^{-\,2\,t}$

An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\mathcal{F}[v] = \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \mathcal{F}_0 \le \frac{1}{2} \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2}\,\mathcal{I}[v]$$

Rewrite it with $p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \ge 0$$

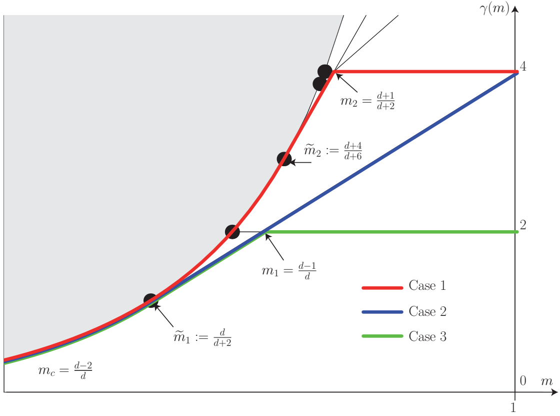
Theorem

[Del Pino, J.D.] *With $1 < p \le \frac{d}{d-2}$ (fast diffusion case) and $d \ge 3$*

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le \mathcal{C}^{\mathrm{GN}}_{p,d}\, \|\nabla w\|^{\theta}_{L^2(\mathbb{R}^d)}\, \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

Improved asymptotic rates

[Denzler, McCann], [Denzler, Koch, McCann], [Seis]
[Blanchet, Bonforte, J.D., Grillo, Vázquez], [Bonforte, J.D., Grillo, Vázquez], [J.D., Toscani]



Fast diffusion equations: some recent results

1. improved inequalities and scalings
2. scalings and a concavity property
3. improved rates and best matching

Improved inequalities and scalings

Gagliardo-Nirenberg inequalities and the FDE

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\vartheta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\vartheta}\geq \mathsf{C}_{\rm GN}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

With the right choice of the constants, the functional

$$\mathsf{J}[w]:=\tfrac14\,(q^2-1)\int_{\mathbb{R}^d}|\nabla w|^2\,dx+\beta\int_{\mathbb{R}^d}|w|^{q+1}\,dx-\mathcal{K}\,\mathsf{C}_{\rm GN}^{\alpha}\left(\int_{\mathbb{R}^d}|w|^{2q}\,dx\right)^{\frac{\alpha}{2q}}$$

is nonnegative and $\mathsf{J}[w]\geq\mathsf{J}[w_*]=0$

Theorem

[Dolbeault-Toscani] *For some nonnegative, convex, increasing* φ

$$\mathsf{J}[w]\geq\varphi\left[\beta\left(\int_{\mathbb{R}^d}|w_*|^{q+1}\,dx-\int_{\mathbb{R}^d}|w|^{q+1}\,dx\right)\right]$$

for any $w\in\mathrm{L}^{q+1}(\mathbb{R}^d)$ *such that* $\int_{\mathbb{R}^d}|\nabla w|^2\,dx<\infty$ *and* $\int_{\mathbb{R}^d}|w|^{2q}\,|x|^2\,dx=\int_{\mathbb{R}^d}w_*^{2q}\,|x|^2\,dx$

Consequence for decay rates of relative Rényi entropies: *faster* rates of convergence in intermediate asymptotics for $\frac{\partial u}{\partial t}=\Delta u^p$

Scalings and a concavity property

▷ Rényi entropies, the entropy approach without rescaling: [Savaré, Toscani]

▷ faster rates of convergence: [Carrillo, Toscani], [JD, Toscani]

The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial u}{\partial t} = \Delta u^m$$

with initial datum $u(x,t=0) = u_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} u_0\,dx = 1$ and $\int_{\mathbb{R}^d} |x|^2\,u_0\,dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t,x) := \frac{1}{\left(\kappa\,t^{1/\mu}\right)^d}\,\mathcal{B}_\star\!\left(\frac{x}{\kappa\,t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The entropy

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} u^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} u \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, u^{m-1}$$

p is the *pressure variable*. If u solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1) \, \mathsf{p} \, \Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m} \left(\frac{1}{d} + m - 1 \right) = \frac{2}{d} \, \frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The concavity property

Theorem

[Toscani-Savaré] *Assume that $m \geq 1 - \frac{1}{d}$ if $d > 1$ and $m > 0$ if $d = 1$. Then $F(t)$ is increasing, $(1-m)\,\mathsf{F}''(t) \leq 0$ and*

$$\lim_{t \to +\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t \to +\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

[Dolbeault-Toscani] The inequality

$$\sigma^{-1}\,\mathsf{F}' = \mathsf{E}^{\sigma-1}\,\mathsf{I} \geq \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

if $1 - \frac{1}{d} \leq m < 1$. Hint: $u^{m-1/2} = \frac{w}{\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}}$, $q = \frac{1}{2\,m-1}$

The proof

Lemma

If u solves $\frac{\partial u}{\partial t} = \Delta u^m$ with $\frac{1}{d} \le m < 1$, then

$$\mathsf{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} u \, |\nabla \mathsf{p}|^2 \, dx = -2 \int_{\mathbb{R}^d} u^m \left(\|\mathrm{D}^2 \mathsf{p}\|^2 + (m-1)\,(\Delta \mathsf{p})^2 \right) dx$$

$$\|\mathrm{D}^2 \mathsf{p}\|^2 = \frac{1}{d}\,(\Delta \mathsf{p})^2 + \left\| \mathrm{D}^2 \mathsf{p} - \frac{1}{d}\,\Delta \mathsf{p}\,\mathrm{Id} \right\|^2$$

$$\begin{aligned}
\frac{1}{\sigma\,(1-m)}\,\mathsf{E}^{2-\sigma}\,(\mathsf{E}^\sigma)'' &= (1-m)\,(\sigma-1) \left(\int_{\mathbb{R}^d} u \, |\nabla \mathsf{p}|^2 \, dx \right)^2 \\
&\quad - 2 \left(\frac{1}{d} + m - 1 \right) \int_{\mathbb{R}^d} u^m \, dx \int_{\mathbb{R}^d} u^m \,(\Delta \mathsf{p})^2 \, dx \\
&\quad - 2 \int_{\mathbb{R}^d} u^m \, dx \int_{\mathbb{R}^d} u^m \left\| \mathrm{D}^2 \mathsf{p} - \frac{1}{d}\,\Delta \mathsf{p}\,\mathrm{Id} \right\|^2 dx
\end{aligned}$$

Best matching

Relative entropy and best matching

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_\star + \frac{1}{\sigma} |x|^2 \right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d$$

The Barenblatt profile B_σ plays the role of a *local Gibbs state* if $C_\star$ is chosen so that $\int_{\mathbb{R}^d} B_\sigma\, dx = \int_{\mathbb{R}^d} v\, dx$
The relative entropy is defined by

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1} (v - B_\sigma) \right] dx$$

To minimize $\mathcal{F}_\sigma[v]$ with respect to σ is equivalent to fix σ such that

$$\sigma \int_{\mathbb{R}^d} |x|^2\, B_1\, dx = \int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx = \int_{\mathbb{R}^d} |x|^2\, v\, dx$$

A Csiszár-Kullback(-Pinsker) inequality

Let $m \in (\frac{d}{d+2}, 1)$ and consider the relative entropy

$$\mathcal{F}_\sigma[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - B_\sigma^m - m\, B_\sigma^{m-1} \left(u - B_\sigma \right) \right]\, dx$$

Theorem

[J.D., Toscani] *Assume that u is a nonnegative function in $\mathrm{L}^1(\mathbb{R}^d)$ such that u^m and $x \mapsto |x|^2\, u$ are both integrable on $\mathbb{R}^d$. If $\|u\|_{\mathrm{L}^1(\mathbb{R}^d)} = M$ and $\int_{\mathbb{R}^d} |x|^2\, u\, dx = \int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx$, then*

$$\frac{\mathcal{F}_\sigma[u]}{\sigma^{\frac{d}{2}(1-m)}} \geq \frac{m}{8 \int_{\mathbb{R}^d} B_1^m\, dx} \left(C_\star \|u - B_\sigma\|_{\mathrm{L}^1(\mathbb{R}^d)} + \frac{1}{\sigma} \int_{\mathbb{R}^d} |x|^2\, |u - B_\sigma|\, dx \right)^2$$

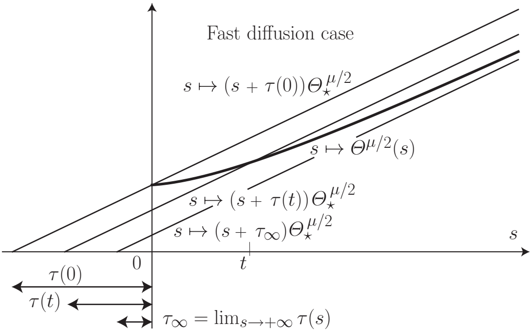
Temperature (fast diffusion case)

The *second moment functional* (temperature) is defined by

$$\Theta(t) := \frac{1}{d} \int_{\mathbb{R}^d} |x|^2\, u(t,x)\, dx$$

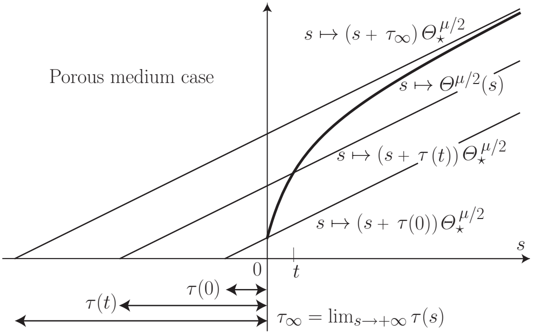
and such that

$$\Theta' = 2\, \mathsf{E}$$



Temperature (porous medium case)

Let $\mathcal{U}_\star^s$ be the *best matching Barenblatt* function, in the sense of relative entropy $\mathcal{F}[u\,|\,\mathcal{U}_\star^s]$, among all Barenblatt functions $(\mathcal{U}_\star^s)_{s>0}$.



A result on delays

Theorem

Assume that $m \geq 1-\frac{1}{d}$ and $m \neq 1$. The best matching Barenblatt function of a solution u is $(t,x) \mapsto \mathcal{U}_\star(t+\tau(t),x)$ and the function $t \mapsto \tau(t)$ is nondecreasing if $m>1$ and nonincreasing if $1-\frac{1}{d} \leq m < 1$

With $\mathsf{G} := \Theta^{1-\frac{\eta}{2}}$, $\eta = d\,(1-m) = 2-\mu$, the *Rényi entropy power* functional $\mathsf{H} := \Theta^{-\frac{\eta}{2}}\,\mathsf{E}$ is such that

$$\mathsf{G}' = \mu\,\mathsf{H} \quad \text{with} \quad \mathsf{H} := \Theta^{-\frac{\eta}{2}}\,\mathsf{E}$$

$$\frac{\mathsf{H}'}{1-m} = \Theta^{-1-\frac{\eta}{2}}\left(\Theta\,\mathsf{I} - d\,\mathsf{E}^2\right) = \frac{d\,\mathsf{E}^2}{\Theta^{\frac{\eta}{2}+1}}\,(\mathsf{q}-1) \quad \text{with} \quad \mathsf{q} := \frac{\Theta\,\mathsf{I}}{d\,\mathsf{E}^2} \geq 1$$

$$d\,\mathsf{E}^2 = \frac{1}{d}\left(-\int_{\mathbb{R}^d} x\cdot\nabla(u^m)\,dx\right)^2 = \frac{1}{d}\left(\int_{\mathbb{R}^d} x\cdot u\,\nabla\mathsf{p}\,dx\right)^2$$
$$\leq \frac{1}{d}\int_{\mathbb{R}^d} u\,|x|^2\,dx \int_{\mathbb{R}^d} u\,|\nabla\mathsf{p}|^2\,dx = \Theta\,\mathsf{I}$$

Fast diffusion equations on manifolds and sharp functional inequalities

1. The sphere
2. The line
3. Compact Riemannian manifolds
4. The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Interpolation inequalities on the sphere

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss

A family of interpolation inequalities on the sphere

The following interpolation inequality holds on the sphere

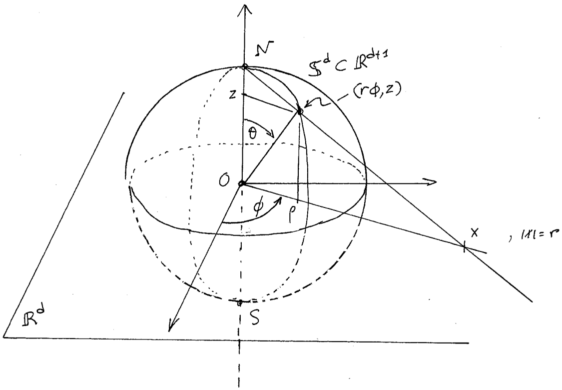
$$\frac{p-2}{d}\int_{\mathbb{S}^d}|\nabla u|^2\,d\,v_g+\int_{\mathbb{S}^d}|u|^2\,d\,v_g\geq\left(\int_{\mathbb{S}^d}|u|^p\,d\,v_g\right)^{2/p}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d,dv_g)$$

- for any $p\in(2,2^*]$ with $2^*=\frac{2\,d}{d-2}$ if $d\geq 3$
- for any $p\in(2,\infty)$ if $d=2$

Here dv_g is the uniform probability measure: $v_g(\mathbb{S}^d)=1$

- 1 is the optimal constant, equality achieved by constants
- $p=2^*$ corresponds to Sobolev's inequality...

Stereographic projection



Sobolev's inequality

The stereographic projection of $\mathbb{S}^d \subset \mathbb{R}^d \times \mathbb{R} \ni (\rho\,\phi, z)$ onto $\mathbb{R}^d$: to $\rho^2 + z^2 = 1$, $z \in [-1,1]$, $\rho \geq 0$, $\phi \in \mathbb{S}^{d-1}$ we associate $x \in \mathbb{R}^d$ such that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}\,, \quad \rho = \frac{2\,r}{r^2+1}$$

and transform any function u on $\mathbb{S}^d$ into a function v on $\mathbb{R}^d$ using

$$u(y) = \left(\tfrac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\tfrac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1-z)^{-\frac{d-2}{2}}\, v(x)$$

- $p = 2^*$, $\mathsf{S}_d = \frac{1}{4}\,d\,(d-2)\,|\mathbb{S}^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2\,dx \geq \mathsf{S}_d \left[\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\,dx\right]^{\frac{d-2}{d}} \quad \forall\, v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

Schwarz symmetrization and the ultraspherical setting

$(\xi_0,\,\xi_1,\ldots\xi_d)\in\mathbb{S}^d$, $\xi_d=z$, $\sum_{i=0}^d|\xi_i|^2=1$ [Smets-Willem]

Lemma

Up to a rotation, any minimizer of $\mathcal{Q}$ depends only on $\xi_d=z$

- Let $d\sigma(\theta):=\frac{(\sin\theta)^{d-1}}{Z_d}\,d\theta$, $Z_d:=\sqrt{\pi}\,\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}$: $\forall\,v\in\mathrm{H}^1([0,\pi],d\sigma)$

$$\frac{p-2}{d}\int_0^\pi|v'(\theta)|^2\,d\sigma+\int_0^\pi|v(\theta)|^2\,d\sigma\ge\left(\int_0^\pi|v(\theta)|^p\,d\sigma\right)^{\frac{2}{p}}$$

- Change of variables $z=\cos\theta$, $v(\theta)=f(z)$

$$\frac{p-2}{d}\int_{-1}^1|f'|^2\,\nu\,d\nu_d+\int_{-1}^1|f|^2\,d\nu_d\ge\left(\int_{-1}^1|f|^p\,d\nu_d\right)^{\frac{2}{p}}$$

where $\nu_d(z)\,dz=d\nu_d(z):=Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z):=1-z^2$

The ultraspherical operator

With $d\nu_d = Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1 - z^2$, consider the space $\mathrm{L}^2((-1,1),d\nu_d)$ with scalar product

$$\langle f_1, f_2\rangle = \int_{-1}^{1} f_1\, f_2\, d\nu_d\,, \quad \|f\|_p = \left(\int_{-1}^{1} f^p\, d\nu_d\right)^{\frac{1}{p}}$$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\,f := (1-z^2)\,f'' - d\,z\,f' = \nu\,f'' + \frac{d}{2}\,\nu'\,f'$$

which satisfies $\langle f_1, \mathcal{L}\,f_2\rangle = -\int_{-1}^{1} f_1'\,f_2'\,\nu\,d\nu_d$

Proposition

Let $p \in [1,2)\cup(2,2^*]$, $d \geq 1$

$$-\langle f, \mathcal{L}\,f\rangle = \int_{-1}^{1} |f'|^2\,\nu\,d\nu_d \geq d\,\frac{\|f\|_p^2 - \|f\|_2^2}{p-2} \quad \forall\, f \in \mathrm{H}^1([-1,1],d\nu_d)$$

Flows on the sphere

- Heat flow and the Bakry-Emery method
- Fast diffusion (porous media) flow and the choice of the exponents

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss

Heat flow and the Bakry-Emery method

With $g = f^p$, *i.e.* $f = g^\alpha$ with $\alpha = 1/p$

(Ineq.) $$-\langle f, \mathcal{L}\, f\rangle = -\langle g^\alpha, \mathcal{L}\, g^\alpha\rangle =: \mathcal{I}[g] \ge d\, \frac{\|g\|_1^{2\,\alpha} - \|g^{2\,\alpha}\|_1}{p-2} =: \mathcal{F}[g]$$

Heat flow

$$\frac{\partial g}{\partial t} = \mathcal{L}\, g$$

$$\frac{d}{dt}\,\|g\|_1 = 0\,, \quad \frac{d}{dt}\,\|g^{2\,\alpha}\|_1 = -\,2\,(p-2)\,\langle f, \mathcal{L}\, f\rangle = 2\,(p-2)\int_{-1}^{1} |f'|^2\,\nu\,d\nu_d$$

which finally gives

$$\frac{d}{dt}\mathcal{F}[g(t,\cdot)] = -\frac{d}{p-2}\,\frac{d}{dt}\,\|g^{2\,\alpha}\|_1 = -\,2\,d\,\mathcal{I}[g(t,\cdot)]$$

Ineq. $\Longleftrightarrow$ $\frac{d}{dt}\mathcal{F}[g(t,\cdot)] \le -\,2\,d\,\mathcal{F}[g(t,\cdot)]$ $\Longleftarrow$ $\frac{d}{dt}\mathcal{I}[g(t,\cdot)] \le -\,2\,d\,\mathcal{I}[g(t,\cdot)]$

The equation for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f,\mathcal{L}\,f\rangle = \langle\mathcal{L}\,f,\mathcal{L}\,f\rangle + (p-1)\,\langle\frac{|f'|^2}{f}\,\nu,\mathcal{L}\,f\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\,d\,\mathcal{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$

$$= -\,2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} = 2^\# < \frac{2\,d}{d-2} = 2^*$$

... up to the critical exponent: a proof in two slides

$$\left[\frac{d}{dz},\mathcal{L}\right]u=(\mathcal{L}\,u)'-\mathcal{L}\,u'=-2\,z\,u''-d\,u'$$

$$\int_{-1}^1(\mathcal{L}\,u)^2\,d\nu_d=\int_{-1}^1|u''|^2\,\nu^2\,d\nu_d+d\int_{-1}^1|u'|^2\,\nu\,d\nu_d$$

$$\int_{-1}^1(\mathcal{L}\,u)\,\frac{|u'|^2}{u}\,\nu\,d\nu_d=\frac{d}{d+2}\int_{-1}^1\frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d-2\,\frac{d-1}{d+2}\int_{-1}^1\frac{|u'|^2\,u''}{u}\,\nu^2\,d\nu_d$$

On $(-1,1)$, let us consider the *porous medium (fast diffusion)* flow

$$u_t=u^{2-2\beta}\left(\mathcal{L}\,u+\kappa\,\frac{|u'|^2}{u}\,\nu\right)$$

If $\kappa=\beta\,(p-2)+1$, the L^p norm is conserved

$$\frac{d}{dt}\int_{-1}^1u^{\beta p}\,d\nu_d=\beta\,p\,(\kappa-\beta\,(p-2)-1)\int_{-1}^1u^{\beta(p-2)}\,|u'|^2\,\nu\,d\nu_d=0$$

$f = u^\beta$, $\|f'\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right) \geq 0$?

$$\begin{aligned}\mathcal{A} := \int_{-1}^{1} |u''|^2\,\nu^2\,d\nu_d - 2\,\frac{d-1}{d+2}\,(\kappa+\beta-1)\int_{-1}^{1} u''\,\frac{|u'|^2}{u}\,\nu^2\,d\nu_d \\ + \left[\kappa\,(\beta-1) + \frac{d}{d+2}\,(\kappa+\beta-1)\right]\int_{-1}^{1}\frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d\end{aligned}$$

$\mathcal{A}$ is nonnegative for some β if

$$\frac{8\,d^2}{(d+2)^2}\,(p-1)\,(2^*-p) \geq 0$$

$\mathcal{A}$ is a sum of squares if $p \in (2, 2^*)$ for an arbitrary choice of β in a certain interval (depending on p and d)

$$\mathcal{A} = \int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\,d\nu_d \geq 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

The rigidity point of view

Which computation have we done ? $u_t = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$

$$-\,\mathcal{L}\,u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\,d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\,d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\,d\nu_d$$

The two terms cancel and we are left only with the two-homogenous terms

Improvements of the inequalities (subcritical range)

- as long as the exponent is either in the range $(1,2)$ or in the range $(2,2^*)$, on can establish *improved inequalities*
- An improvement automatically gives an explicit stability result of the optimal functions in the (non-improved) inequality
- By duality, this provides a stability result for Keller-Lieb-Tirring inequalities

What does "improvement" mean ?

An *improved* inequality is

$$d\,\Phi\,(\mathsf{e}) \le \mathsf{i} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$$

for some function Φ such that $\Phi(0)=0$, $\Phi'(0)=1$, $\Phi'>0$ and $\Phi(s)>s$ for any s. With $\Psi(s) := s - \Phi^{-1}(s)$

$$\mathsf{i} - d\,\mathsf{e} \ge d\,(\Psi\circ\Phi)(\mathsf{e}) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$$

Lemma (Generalized Csiszár-Kullback inequalities)

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{d}{p-2}\left[\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right]$$
$$\ge d\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,(\Psi\circ\Phi)\left(\mathsf{C}\,\frac{\|u\|^{2\,(1-r)}_{\mathrm{L}^s(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\,\|u^r - \bar{u}^r\|^2_{\mathrm{L}^q(\mathbb{S}^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

$s(p) := \max\{2,p\}$ and $p\in(1,2)$: $q(p) := 2/p$, $r(p) := p$; $p\in(2,4)$: $q = p/2$, $r=2$; $p\ge 4$: $q = p/(p-2)$, $r = p-2$

Linear flow: improved Bakry-Emery method

Cf. [Arnold, JD]

$$w_t = \mathcal{L}\, w + \kappa\, \frac{|w'|^2}{w}\, \nu$$

With $2^\sharp := \frac{2\,d^2+1}{(d-1)^2}$

$$\gamma_1 := \left(\frac{d-1}{d+2}\right)^2 (p-1)\,(2^\# - p) \quad \text{if} \quad d > 1\,, \quad \gamma_1 := \frac{p-1}{3} \quad \text{if} \quad d = 1$$

If $p \in [1,2) \cup (2, 2^\sharp]$ and w is a solution, then

$$\frac{d}{dt}\,(\mathsf{i} - \,d\,\mathsf{e}) \leq -\,\gamma_1 \int_{-1}^{1} \frac{|w'|^4}{w^2}\, d\nu_d \leq -\,\gamma_1\, \frac{|\mathsf{e}'|^2}{1 - \,(p-2)\,\mathsf{e}}$$

Recalling that $\mathsf{e}' = -\,\mathsf{i}$, we get a differential inequality

$$\mathsf{e}'' + \,d\,\mathsf{e}' \geq \gamma_1\, \frac{|\mathsf{e}'|^2}{1 - \,(p-2)\,\mathsf{e}}$$

After integration: $d\,\Phi(\mathsf{e}(0)) \leq \mathsf{i}(0)$

Nonlinear flow: the Hölder estimate of J. Demange

$$w_t = w^{2-2\beta}\left(\mathcal{L}\,w + \kappa\,\frac{|w'|^2}{w}\right)$$

For all $p \in [1, 2^*]$, $\kappa = \beta\,(p-2)+1$, $\frac{d}{dt}\int_{-1}^{1} w^{\beta p}\,d\nu_d = 0$

$$-\frac{1}{2\beta^2}\,\frac{d}{dt}\int_{-1}^{1}\left(|(w^\beta)'|^2\,\nu + \frac{d}{p-2}\,\left(w^{2\beta} - \overline{w}^{2\beta}\right)\right)d\nu_d \ge \gamma\int_{-1}^{1}\frac{|w'|^4}{w^2}\,\nu^2\,d\nu_d$$

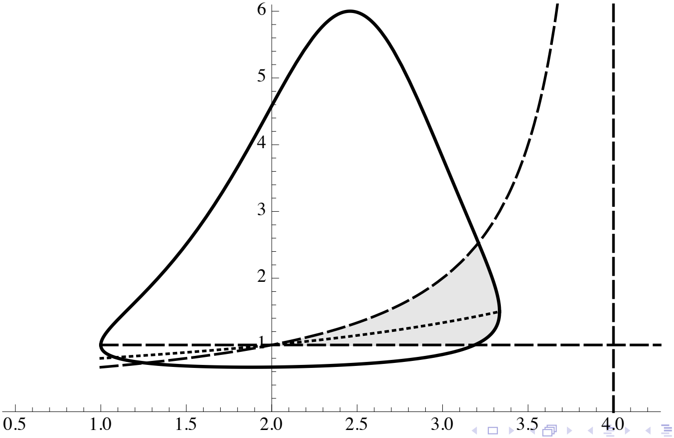
Lemma (Demange)

For all $w \in \mathrm{H}^1\big((-1,1), d\nu_d\big)$, such that $\int_{-1}^{1} w^{\beta p}\,d\nu_d = 1$

$$\int_{-1}^{1}\frac{|w'|^4}{w^2}\,\nu^2\,d\nu_d \ge \frac{1}{\beta^2}\,\frac{\int_{-1}^{1}|(w^\beta)'|^2\,\nu\,d\nu_d\,\int_{-1}^{1}|w'|^2\,\nu\,d\nu_d}{\left(\int_{-1}^{1} w^{2\beta}\,d\nu_d\right)^\delta}$$

.... but there are conditions on β

Admissible (p, β) for $d = 5$



The line

- A first example of a non-compact manifold

Joint work with M.J. Esteban, A. Laptev and M. Loss

One-dimensional Gagliardo-Nirenberg-Sobolev inequalities

$$\|f\|_{\mathrm{L}^p(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|f'\|^{\theta}_{\mathrm{L}^2(\mathbb{R})}\,\|f\|^{1-\theta}_{\mathrm{L}^2(\mathbb{R})} \quad \text{if} \quad p \in (2,\infty)$$

$$\|f\|_{\mathrm{L}^2(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|f'\|^{\eta}_{\mathrm{L}^2(\mathbb{R})}\,\|f\|^{1-\eta}_{\mathrm{L}^p(\mathbb{R})} \quad \text{if} \quad p \in (1,2)$$

with $\theta = \frac{p-2}{2\,p}$ and $\eta = \frac{2-p}{2+p}$

The threshold case corresponding to the limit as $p \to 2$ is the logarithmic Sobolev inequality

$$\int_{\mathbb{R}} u^2 \,\log\left(\frac{u^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R})}}\right) dx \le \frac{1}{2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{R})}\,\log\left(\frac{2}{\pi\,e}\,\frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{R})}}{\|u\|^2_{\mathrm{L}^2(\mathbb{R})}}\right)$$

If $p > 2$, $u_\star(x) = (\cosh x)^{-\frac{2}{p-2}}$ solves

$$-(p-2)^2\,u'' + 4\,u - 2\,p\,|u|^{p-2}\,u = 0$$

If $p \in (1,2)$ consider $u_*(x) = (\cos x)^{\frac{2}{2-p}}$, $x \in (-\pi/2, \pi/2)$

Flow

Let us define on $\mathrm{H}^1(\mathbb{R})$ the functional

$$\mathcal{F}[v] := \|v'\|^2_{\mathrm{L}^2(\mathbb{R})} + \frac{4}{(p-2)^2}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R})} - \mathsf{C}\,\|v\|^2_{\mathrm{L}^p(\mathbb{R})} \quad \text{s.t. } \mathcal{F}[u_\star] = 0$$

With $z(x) := \tanh x$, consider the *flow*

$$v_t = \frac{v^{1-\frac{p}{2}}}{\sqrt{1-z^2}}\left[v'' + \frac{2\,p}{p-2}\,z\,v' + \frac{p}{2}\,\frac{|v'|^2}{v} + \frac{2}{p-2}\,v\right]$$

Theorem (Dolbeault-Esteban-Laptev-Loss)

Let $p \in (2,\infty)$. Then

$$\frac{d}{dt}\mathcal{F}[v(t)] \le 0 \quad \textit{and} \quad \lim_{t\to\infty}\mathcal{F}[v(t)] = 0$$

$$\tfrac{d}{dt}\mathcal{F}[v(t)] = 0 \quad \Longleftrightarrow \quad v_0(x) = u_\star(x - x_0)$$

Similar results for $p \in (1,2)$

The inequality ($p > 2$) and the ultraspherical operator

- *The problem on the line is equivalent to the critical problem for the ultraspherical operator*

$$\int_{\mathbb{R}} |v'|^2 \, dx + \frac{4}{(p-2)^2} \int_{\mathbb{R}} |v|^2 \, dx \ge \mathsf{C} \left(\int_{\mathbb{R}} |v|^p \, dx \right)^{\frac{2}{p}}$$

With

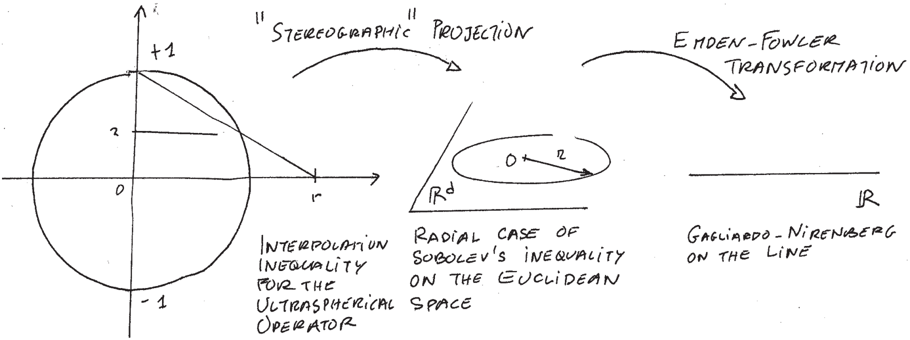
$$z(x) = \tanh x \,, \quad v_\star = (1 - z^2)^{\frac{1}{p-2}} \quad \text{and} \quad v(x) = v_\star(x)\, f(z(x))$$

equality is achieved for $f = 1$ and, if we let $\nu(z) := 1 - z^2$, then

$$\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{2\,p}{(p-2)^2} \int_{-1}^{1} |f|^2 \, d\nu_d \ge \frac{2\,p}{(p-2)^2} \left(\int_{-1}^{1} |f|^p \, d\nu_d \right)^{\frac{2}{p}}$$

where $d\nu_p$ denotes the probability measure $d\nu_p(z) := \frac{1}{\zeta_p}\, \nu^{\frac{2}{p-2}}\, dz$

$$d = \frac{2\,p}{p-2} \quad \Longleftrightarrow \quad p = \frac{2\,d}{d-2}$$



Change of variables = stereographic projection + Emden-Fowler

Compact Riemannian manifolds

- no sign is required on the Ricci tensor and an improved integral criterion is established
- the flow explores the energy landscape... and shows the non-optimality of the improved criterion

Riemannian manifolds with positive curvature

$(\mathfrak{M}, g)$ is a smooth closed compact connected Riemannian manifold dimension d, no boundary, Δ_g is the Laplace-Beltrami operator $\mathrm{vol}(\mathfrak{M}) = 1$, $\mathfrak{R}$ is the Ricci tensor, $\lambda_1 = \lambda_1(-\Delta_g)$

$$\rho := \inf_{\mathfrak{M}} \inf_{\xi \in \mathbb{S}^{d-1}} \mathfrak{R}(\xi\,,\xi)$$

Theorem (Licois-Véron, Bakry-Ledoux)

Assume $d \geq 2$ and $\rho > 0$. If

$$\lambda \leq (1-\theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} \quad \textit{where} \quad \theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} > 0$$

then for any $p \in (2, 2^*)$, the equation

$$-\Delta_g v + \frac{\lambda}{p-2}\,\left(v - v^{p-1}\right) = 0$$

has a unique positive solution $v \in C^2(\mathfrak{M})$: $v \equiv 1$

Riemannian manifolds: first improvement

Theorem (Dolbeault-Esteban-Loss)

For any $p \in (1,2) \cup (2,2^*)$

$$0 < \lambda < \lambda_\star = \inf_{u \in \mathrm{H}^2(\mathfrak{M})} \frac{\int_{\mathfrak{M}} \left[(1-\theta)\,(\Delta_g u)^2 + \frac{\theta\, d}{d-1}\,\mathfrak{R}(\nabla u, \nabla u)\right] d\,v_g}{\int_{\mathfrak{M}} |\nabla u|^2\, d\,v_g}$$

there is a unique positive solution in $C^2(\mathfrak{M})$*:* $u \equiv 1$

$\lim_{p \to 1_+} \theta(p) = 0 \Longrightarrow \lim_{p \to 1_+} \lambda_\star(p) = \lambda_1$ if ρ is bounded
$\lambda_\star = \lambda_1 = d\,\rho/(d-1) = d$ if $\mathfrak{M} = \mathbb{S}^d$ since $\rho = d-1$

$$(1-\theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} \le \lambda_\star \le \lambda_1$$

Riemannian manifolds: second improvement

$\mathrm{H}_g u$ denotes Hessian of u and $\theta = \dfrac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1}$

$$\mathrm{Q}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u - \frac{(d-1)\,(p-1)}{\theta\,(d+3-p)}\left[\frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right]$$

$$\Lambda_\star := \inf_{u\in\mathrm{H}^2(\mathfrak{M})\setminus\{0\}} \frac{(1-\theta)\displaystyle\int_{\mathfrak{M}} (\Delta_g u)^2\,d v_g + \frac{\theta\,d}{d-1}\int_{\mathfrak{M}}\left[\|\mathrm{Q}_g u\|^2 + \mathfrak{R}(\nabla u,\nabla u)\right]}{\displaystyle\int_{\mathfrak{M}} |\nabla u|^2\,d v_g}$$

Theorem (Dolbeault-Esteban-Loss)

Assume that $\Lambda_\star > 0$. For any $p \in (1,2)\cup(2,2^*)$, the equation has a unique positive solution in $C^2(\mathfrak{M})$ if $\lambda \in (0,\Lambda_\star)$: $u \equiv 1$

Optimal interpolation inequality

For any $p \in (1,2) \cup (2,2^*)$ or $p = 2^*$ if $d \geq 3$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})}\right] \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M})$$

Theorem (Dolbeault-Esteban-Loss)

Assume $\Lambda_\star > 0$. The above inequality holds for some $\lambda = \Lambda \in [\Lambda_\star, \lambda_1]$
If $\Lambda_\star < \lambda_1$, then the optimal constant Λ is such that

$$\Lambda_\star < \Lambda \leq \lambda_1$$

If $p = 1$, then $\Lambda = \lambda_1$

Using $u = 1 + \varepsilon\,\varphi$ as a test function where φ we get $\lambda \leq \lambda_1$
A minimum of

$$v \mapsto \|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} - \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})}\right]$$

under the constraint $\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 1$ is negative if $\lambda > \lambda_1$

The flow

The key tool is the flow

$$u_t = u^{2-2\,\beta}\left(\Delta_g u + \kappa\,\frac{|\nabla u|^2}{u}\right)\,,\quad \kappa = 1+\beta\,(p-2)$$

If $v = u^\beta$, then $\frac{d}{dt}\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 0$ and the functional

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2\,d\,v_g + \frac{\lambda}{p-2}\left[\int_{\mathfrak{M}} u^{2\,\beta}\,d\,v_g - \left(\int_{\mathfrak{M}} u^{\beta\,p}\,d\,v_g\right)^{2/p}\right]$$

is monotone decaying

Elementary observations (1/2)

Let $d \geq 2$, $u \in C^2(\mathfrak{M})$, and consider the trace free Hessian

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d} \Delta_g u$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, d v_g = \frac{d}{d-1} \int_{\mathfrak{M}} \| \mathrm{L}_g u \|^2 \, d v_g + \frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) \, d v_g$$

Based on the Bochner-Lichnerovicz-Weitzenböck formula

$$\frac{1}{2} \Delta |\nabla u|^2 = \|\mathrm{H}_g u\|^2 + \nabla(\Delta_g u) \cdot \nabla u + \mathfrak{R}(\nabla u, \nabla u)$$

Elementary observations (2/2)

Lemma

$$\int_{\mathfrak{M}} \Delta_g u \, \frac{|\nabla u|^2}{u} \, d\,v_g = \frac{d}{d+2} \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, d\,v_g - \frac{2\,d}{d+2} \int_{\mathfrak{M}} [\mathrm{L}_g u] : \left[\frac{\nabla u \otimes \nabla u}{u}\right] d\,v_g$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, d\,v_g \geq \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 \, d\,v_g \quad \forall\, u \in \mathrm{H}^2(\mathfrak{M})$$

and λ_1 is the optimal constant in the above inequality

The key estimates

$$\mathcal{G}[u] := \int_{\mathfrak{M}} \left[\theta\,(\Delta_g u)^2 + (\kappa+\beta-1)\,\Delta_g u\,\frac{|\nabla u|^2}{u} + \kappa\,(\beta-1)\,\frac{|\nabla u|^4}{u^2}\right] d\,v_g$$

Lemma

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] = -\,(1-\theta)\int_{\mathfrak{M}} (\Delta_g u)^2\,d\,v_g - \mathcal{G}[u] + \lambda\int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g$$

$$\mathrm{Q}_g^\theta u := \mathrm{L}_g u - \tfrac{1}{\theta}\,\tfrac{d-1}{d+2}\,(\kappa+\beta-1)\left[\frac{\nabla u\otimes\nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right]$$

Lemma

$$\mathcal{G}[u] = \frac{\theta\,d}{d-1}\left[\int_{\mathfrak{M}} \|\mathrm{Q}_g^\theta u\|^2\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u,\nabla u)\,d\,v_g\right] - \mu\int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2}\,d\,v_g$$

with $\mu := \dfrac{1}{\theta}\left(\dfrac{d-1}{d+2}\right)^2(\kappa+\beta-1)^2 - \kappa\,(\beta-1) - (\kappa+\beta-1)\,\dfrac{d}{d+2}$

The end of the proof

Assume that $d \geq 2$. If $\theta = 1$, then μ is nonpositive if

$$\beta_-(p) \leq \beta \leq \beta_+(p) \quad \forall\, p \in (1, 2^*)$$

where $\beta_\pm := \frac{\mathsf{b} \pm \sqrt{\mathsf{b}^2 - \mathsf{a}}}{2\,\mathsf{a}}$ with $\mathsf{a} = 2 - p + \left[\frac{(d-1)\,(p-1)}{d+2}\right]^2$ and $\mathsf{b} = \frac{d+3-p}{d+2}$
Notice that $\beta_-(p) < \beta_+(p)$ if $p \in (1, 2^*)$ and $\beta_-(2^*) = \beta_+(2^*)$

$$\theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} \quad \text{and} \quad \beta = \frac{d+2}{d+3-p}$$

Proposition

Let $d \geq 2$, $p \in (1,2) \cup (2, 2^*)$ ($p \neq 5$ or $d \neq 2$)

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] \leq (\lambda - \Lambda_\star) \int_{\mathfrak{M}} |\nabla u|^2\, d\,v_g$$

The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban

- Extension to compact Riemannian manifolds of dimension 2...

We shall also denote by $\mathfrak{R}$ the Ricci tensor, by $\mathrm{H}_g u$ the Hessian of u and by

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

the trace free Hessian. Let us denote by $\mathrm{M}_g u$ the trace free tensor

$$\mathrm{M}_g u := \nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2$$

We define

$$\lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M}) \setminus \{0\}} \frac{\int_{\mathfrak{M}} \left[\, \|\, \mathrm{L}_g u - \frac{1}{2}\,\mathrm{M}_g u\,\|^2 + \mathfrak{R}(\nabla u, \nabla u)\right] e^{-u/2}\,d\,v_g}{\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\,d\,v_g}$$

Theorem

Assume that $d = 2$ and $\lambda_\star > 0$. If u is a smooth solution to

$$-\frac{1}{2}\,\Delta_g u + \lambda = e^u$$

then u is a constant function if $\lambda \in (0, \lambda_\star)$

The Moser-Trudinger-Onofri inequality on $\mathfrak{M}$

$$\frac{1}{4}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \lambda \int_{\mathfrak{M}} u\,d\,v_g \ge \lambda \log\left(\int_{\mathfrak{M}} e^u\,d\,v_g\right) \quad \forall\, u \in \mathrm{H}^1(\mathfrak{M})$$

for some constant $\lambda > 0$. Let us denote by λ_1 the first positive eigenvalue of $-\,\Delta_g$

Corollary

If $d = 2$, then the MTO inequality holds with $\lambda = \Lambda := \min\{4\,\pi, \lambda_\star\}$. Moreover, if Λ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than Λ

The flow

$$\frac{\partial f}{\partial t} = \Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}$$

$$\mathcal{G}_\lambda[f] := \int_{\mathfrak{M}} \|\,\mathrm{L}_g f - \tfrac{1}{2}\,\mathrm{M}_g f\,\|^2\,e^{-f/2}\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla f,\nabla f)\,e^{-f/2}\,d\,v_g - \lambda\int_{\mathfrak{M}} |\nabla f|^2\,e^{-f/2}\,d\,v_g$$

Then for any $\lambda \le \lambda_\star$ we have

$$\frac{d}{dt}\mathcal{F}_\lambda[f(t,\cdot)] = \int_{\mathfrak{M}} \left(-\tfrac{1}{2}\,\Delta_g f + \lambda\right)\left(\Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}\right)d\,v_g = -\,\mathcal{G}_\lambda[f(t,\cdot)]$$

Since $\mathcal{F}_\lambda$ is nonnegative and $\lim_{t\to\infty}\mathcal{F}_\lambda[f(t,\cdot)] = 0$, we obtain that

$$\mathcal{F}_\lambda[u] \ge \int_0^\infty \mathcal{G}_\lambda[f(t,\cdot)]\,dt$$

Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

On the Euclidean space $\mathbb{R}^2$, given a general probability measure μ does the inequality

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \ge \lambda\left[\log\left(\int_{\mathbb{R}^d} e^u\,d\mu\right)-\int_{\mathbb{R}^d} u\,d\mu\right]$$

hold for some $\lambda>0$? Let

$$\Lambda_\star := \inf_{x\in\mathbb{R}^2}\frac{-\,\Delta\log\mu}{8\,\pi\,\mu}$$

Theorem

Assume that μ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda<\Lambda_\star$ and the inequality holds with $\lambda=\Lambda_\star$ if equality is achieved among radial functions

Caffarelli-Kohn-Nirenberg inequalities

—

The symmetry issue

—

A brief introduction to the lecture of Michael Loss

▷ *Nonlinear flows (fast diffusion equation) can be used as a tool for the investigation of sharp functional inequalities*

Caffarelli-Kohn-Nirenberg inequalities and the symmetry breaking issue

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

▷ *With*

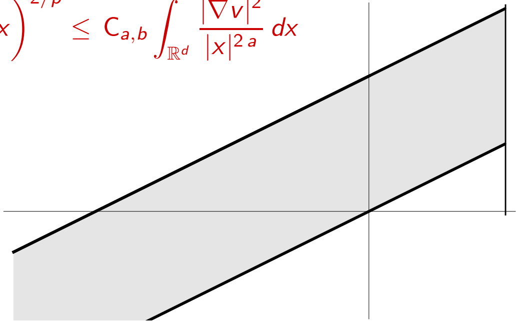
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

do we have $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry)*
or $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

CKN: range of the parameters

Figure: $d=3$

$$\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}\le C_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a\le b\le a+1$ if $d\ge 3$
$a<b\le a+1$ if $d=2$, $a+1/2<b\le a+1$ if $d=1$
and $a<a_c:=(d-2)/2$

$$p=\frac{2\,d}{d-2+2\,(b-a)}$$

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, Caffarelli-Kohn-Nirenberg inequalities on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega) = r^{a-a_c}\,\varphi(s,\omega) \quad \text{with} \quad r = |x|\,, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as

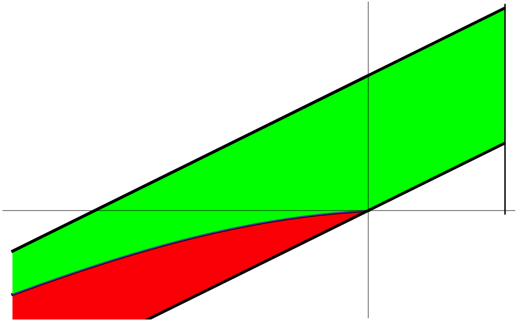
$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} \ge \mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})} \quad \forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda := (a_c - a)^2$, $\mathcal{C} = \mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda) = \frac{1}{\mathsf{C}_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda}$$

Symmetry vs. symmetry breaking: the sharp result

A result based on entropies and nonlinear flows



[J.D., Esteban, Loss, 2015]: http://arxiv.org/abs/1506.03664

Spectral estimates

1. Spectral estimates on the sphere
2. Spectral estimates on compact Riemannian manifolds
3. Spectral estimates on the cylinder

Spectral estimates on the sphere

- The Keller-Lieb-Tirring inequality is equivalent to an interpolation inequality of Gagliardo-Nirenberg-Sobolev type
- We measure a quantitative deviation with respect to the semi-classical regime due to finite size effects

Joint work with M.J. Esteban and A. Laptev

An introduction to Lieb-Thirring inequalities

Consider the Schrödinger operator $H = -\Delta - V$ on $\mathbb{R}^d$ and denote by $(\lambda_k)_{k\geq 1}$ its eigenvalues

- Euclidean case [Keller, 1961]

$$|\lambda_1|^\gamma \leq \mathrm{L}^1_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}$$

[Lieb-Thirring, 1976]

$$\sum_{k\geq 1} |\lambda_k|^\gamma \leq \mathrm{L}_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}$$

$\gamma \geq 1/2$ if $d=1$, $\gamma > 0$ if $d=2$ and $\gamma \geq 0$ if $d \geq 3$ [Weidl], [Cwikel], [Rosenbljum], [Aizenman], [Laptev-Weidl], [Helffer], [Robert], [Dolbeault-Felmer-Loss-Paturel]... [Dolbeault-Laptev-Loss 2008]

- Compact manifolds: log Sobolev case: [Federbusch], [Rothaus]; case $\gamma = 0$ (Rozenbljum-Lieb-Cwikel inequality): [Levin-Solomyak]; [Lieb], [Levin], [Ouabaz-Poupaud]... [Ilyin]

▷ *How does one take into account the finite size effects in the case of*

A Keller-Lieb-Thirring inequality on the sphere

Let $d \geq 1$, $p \in \big[\max\{1, d/2\}, +\infty\big)$ and

$$\mu_* := \tfrac{d}{2}\,(p-1)$$

Theorem (Dolbeault-Esteban-Laptev)

There exists a convex increasing function α s.t. $\alpha(\mu) = \mu$ if $\mu \in \big[0, \mu_*\big]$ and $\alpha(\mu) > \mu$ if $\mu \in \big(\mu_*, +\infty\big)$ and, for any $p < d/2$,

$$|\lambda_1(-\Delta - V)| \leq \alpha\big(\|V\|_{\mathrm{L}^p(\mathbb{R}^d)}\big) \quad \forall\, V \in \mathrm{L}^p(\mathbb{S}^d)$$

This estimate is optimal

For large values of μ, we have

$$\alpha(\mu)^{p-\frac{d}{2}} = \mathrm{L}^1_{p-\frac{d}{2},d}\,(\kappa_{q,d}\,\mu)^p\,(1+o(1))$$

If $p = d/2$ and $d \geq 3$, the inequality holds with $\alpha(\mu) = \mu$ iff $\mu \in [0, \mu_*]$

A Keller-Lieb-Thirring inequality: second formulation

Let $d \geq 1$, $\gamma = p - d/2$

Corollary (Dolbeault-Esteban-Laptev)

$$|\lambda_1(-\Delta - V)|^\gamma \lesssim \mathrm{L}^1_{\gamma,d} \int_{\mathbb{S}^d} V^{\gamma+\frac{d}{2}} \quad \textit{as} \quad \mu = \|V\|_{\mathrm{L}^{\gamma+\frac{d}{2}}(\mathbb{R}^d)} \to \infty$$

if either $\gamma > \max\{0, 1 - d/2\}$ *or* $\gamma = 1/2$ *and* $d = 1$

However, if $\mu = \|V\|_{\mathrm{L}^{\gamma+\frac{d}{2}}(\mathbb{R}^d)} \leq \mu_*$, *then we have*

$$|\lambda_1(-\Delta - V)|^{\gamma+\frac{d}{2}} \leq \int_{\mathbb{S}^d} V^{\gamma+\frac{d}{2}}$$

for any $\gamma \geq \max\{0, 1 - d/2\}$ *and this estimate is optimal*

$\mathrm{L}^1_{\gamma,d}$ is the optimal constant in the Euclidean one bound state ineq.

$$|\lambda_1(-\Delta - \phi)|^\gamma \leq \mathrm{L}^1_{\gamma,d} \int_{\mathbb{R}^d} \phi_+^{\gamma+\frac{d}{2}}\, dx$$

Hölder duality and link with interpolation inequalities

Consider the Schrödinger operator $-\Delta - V$ and the energy

$$\mathcal{E}[u] := \int_{\mathbb{S}^d} |\nabla u|^2 - \int_{\mathbb{S}^d} V\,|u|^2$$

$$\geq \int_{\mathbb{S}^d} |\nabla u|^2 - \mu\,\|u\|^2_{\mathrm{L}^q(\mathbb{R}^d)}$$

$$\geq -\alpha(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \text{if } \mu = \|V_+\|_{\mathrm{L}^p(\mathbb{R}^d)}$$

▷ *Is it true that*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mu(\alpha)\,\|u\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \quad ?$$

In other words, what are the properties of the minimum of

$$\mathcal{Q}_\alpha[u] := \frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|u\|^2_{\mathrm{L}^q(\mathbb{R}^d)}} \quad ?$$

An important convention (for the numerical value of the constants): we consider the uniform probability measure on the unit sphere $\mathbb{S}^d$

- $\mu_{\rm asymp}(\alpha) := \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\,\alpha^{1-\vartheta}$, $\vartheta := d\,\frac{q-2}{2\,q}$ corresponds to the *semi-classical regime* and $\mathsf{K}_{q,d}$ is the optimal constant in the *Euclidean* Gagliardo-Nirenberg-Sobolev inequality

$$\mathsf{K}_{q,d}\,\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \le \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, v \in \mathrm{H}^1(\mathbb{R}^d)$$

- Let φ be a non-trivial eigenfunction of the Laplace-Beltrami operator corresponding the first nonzero eigenvalue

$$-\Delta\varphi = d\,\varphi$$

Consider $u = 1 + \varepsilon\,\varphi$ as $\varepsilon \to 0$ Taylor expand $\mathcal{Q}_\alpha$ around $u = 1$

$$\mu(\alpha) \le \mathcal{Q}_\alpha[1+\varepsilon\,\varphi] = \alpha + \big[d + \alpha\,(2-q)\big]\,\varepsilon^2 \int_{\mathbb{S}^d} |\varphi|^2\,d\,v_g + o(\varepsilon^2)$$

By taking ε small enough, we get $\mu(\alpha) < \alpha$ for all $\alpha > d/(q-2)$
Optimizing on the value of $\varepsilon > 0$ (not necessarily small) provides an interesting test function...

Another inequality

Let $d \geq 1$ and $\gamma > d/2$ and assume that $\mathrm{L}^1_{-\gamma,d}$ is the optimal constant in

$$\lambda_1(-\Delta+\phi)^{-\gamma} \leq \mathrm{L}^1_{-\gamma,d} \int_{\mathbb{R}^d} \phi^{\frac{d}{2}-\gamma}\,dx$$

$$q = 2\,\frac{2\,\gamma-d}{2\,\gamma-d+2} \quad \text{and} \quad p = \frac{q}{2-q} = \gamma - \frac{d}{2}$$

Theorem (Dolbeault-Esteban-Laptev)

$$\big(\lambda_1(-\Delta+W)\big)^{-\gamma} \lesssim \mathrm{L}^1_{-\gamma,d} \int_{\mathbb{S}^d} W^{\frac{d}{2}-\gamma} \quad \textit{as} \quad \beta = \|W^{-1}\|^{-1}_{\mathrm{L}^{\gamma-\frac{d}{2}}(\mathbb{R}^d)} \to \infty$$

However, if $\gamma \geq \frac{d}{2}+1$ *and* $\beta = \|W^{-1}\|^{-1}_{\mathrm{L}^{\gamma-\frac{d}{2}}(\mathbb{R}^d)} \leq \frac{1}{4}\,d\,(2\,\gamma-d+2)$

$$\big(\lambda_1(-\Delta+W)\big)^{\frac{d}{2}-\gamma} \leq \int_{\mathbb{S}^d} W^{\frac{d}{2}-\gamma}$$

and this estimate is optimal

$\mathsf{K}^*_{q,d}$ is the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality

$$\mathsf{K}^*_{q,d}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \quad \forall\, v \in \mathrm{H}^1(\mathbb{R}^d)$$

and $\mathcal{L}^{\,1}_{-\gamma,d} := \left(\mathsf{K}^*_{q,d}\right)^{-\gamma}$ with $q = 2\,\frac{2\,\gamma-d}{2\,\gamma-d+2}$, $\delta := \frac{2\,q}{2\,d-q\,(d-2)}$

Lemma (Dolbeault-Esteban-Laptev)

Let $q \in (0,2)$ and $d \ge 1$. There exists a concave increasing function ν

$$\nu(\beta) \le \beta \quad \forall\,\beta > 0 \quad \textit{and} \quad \nu(\beta) < \beta \quad \forall\,\beta \in \left(\tfrac{d}{2-q},+\infty\right)$$

$$\nu(\beta) = \beta \quad \forall\,\beta \in \left[0,\tfrac{d}{2-q}\right] \quad \textit{if} \quad q \in [1,2)$$

$$\nu(\beta) = \mathsf{K}^*_{q,d}\,\left(\kappa_{q,d}\,\beta\right)^{\delta}\,(1+o(1)) \quad \textit{as} \quad \beta \to +\infty$$

such that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \beta\,\|u\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \ge \nu(\beta)\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

The threshold case: $q=2$

Lemma (Dolbeault-Esteban-Laptev)

Let $p>\max\{1,d/2\}$*. There exists a concave nondecreasing function* ξ

$$\xi(\alpha)=\alpha \quad \forall\,\alpha\in(0,\alpha_0) \quad \textit{and} \quad \xi(\alpha)<\alpha \quad \forall\,\alpha>\alpha_0$$

for some $\alpha_0\in\left[\frac d2\,(p-1),\frac d2\,p\right]$*, and* $\xi(\alpha)\sim\alpha^{1-\frac{d}{2\,p}}$ *as* $\alpha\to+\infty$

such that, for any $u\in\mathrm H^1(\mathbb S^d)$ *with* $\|u\|_{\mathrm L^2(\mathbb R^d)}=1$

$$\int_{\mathbb S^d}|u|^2\log|u|^2\,d\,v_g+p\,\log\left(\tfrac{\xi(\alpha)}{\alpha}\right)\le p\,\log\left(1+\tfrac1\alpha\,\|\nabla u\|^2_{\mathrm L^2(\mathbb R^d)}\right)$$

Corollary (Dolbeault-Esteban-Laptev)

$$e^{-\,\lambda_1(-\Delta-W)/\alpha}\le\frac{\alpha}{\xi(\alpha)}\,\left(\int_{\mathbb S^d}e^{-\,p\,W/\alpha}\,d\,v_g\right)^{1/p}$$

Spectral estimates on compact Riemannian manifolds

Joint work with M.J. Esteban, A. Laptev, and M. Loss

- The same kind of results as for the sphere. However, estimates are not, in general, sharp.

Manifolds: the first interpolation inequality

Let us define

$$\kappa := \mathrm{vol}_g(\mathfrak{M})^{1-2/q}$$

Proposition

Assume that $q \in (2, 2^*)$ if $d \geq 3$, or $q \in (2, \infty)$ if $d = 1$ or 2. There exists a concave increasing function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(\alpha) = \kappa\,\alpha$ for any $\alpha \leq \frac{\Lambda}{q-2}$, $\mu(\alpha) < \kappa\,\alpha$ for $\alpha > \frac{\Lambda}{q-2}$ and

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \mu(\alpha)\,\|u\|^2_{\mathrm{L}^q(\mathfrak{M})} \quad \forall\, u \in \mathrm{H}^1(\mathfrak{M})$$

The asymptotic behaviour of μ is given by $\mu(\alpha) \sim \mathsf{K}_{q,d}\,\alpha^{1-\vartheta}$ as $\alpha \to +\infty$, with $\vartheta = d\,\frac{q-2}{2\,q}$ and $\mathsf{K}_{q,d}$ defined by

$$\mathsf{K}_{q,d} := \inf_{v \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}$$

Manifolds: the first Keller-Lieb-Thirring estimate

We consider $\|V\|_{\mathrm{L}^p(\mathfrak{M})} = \mu \mapsto \alpha(\mu)$

$$\int_{\mathfrak{M}} |\nabla u|^2 \, d\,v_g - \int_{\mathfrak{M}} V\,|u|^2 \, d\,v_g + \alpha(\mu) \int_{\mathfrak{M}} |u|^2 \, d\,v_g \\ \geq \|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} - \mu\,\|u\|^2_{\mathrm{L}^q(\mathfrak{M})} + \alpha(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathfrak{M})}$$

p and $\frac{q}{2}$ are Hölder conjugate exponents

Theorem

Let $d \geq 1$, $p \in (1,+\infty)$ if $d=1$ and $p \in (\frac{d}{2},+\infty)$ if $d \geq 2$ and assume that $\Lambda_\star > 0$. With the above notations and definitions, for any nonnegative $V \in \mathrm{L}^p(\mathfrak{M})$, we have

$$|\lambda_1(-\Delta_g - V)| \leq \alpha\big(\|V\|_{\mathrm{L}^p(\mathfrak{M})}\big)$$

Moreover, we have $\alpha(\mu)^{p-\frac{d}{2}} = \mathrm{L}^1_{\gamma,d}\,\mu^p\,(1+o(1))$ as $\mu \to +\infty$ with $\mathrm{L}^1_{\gamma,d} := (\mathsf{K}_{q,d})^{-p}$, $\gamma = p - \frac{d}{2}$

Manifolds: the second Keller-Lieb-Thirring estimate

Theorem

Let $d \geq 1$, $p \in (0, +\infty)$. There exists an increasing concave function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\nu(\beta) = \beta/\kappa$, for any $\beta \in (0, \frac{p+1}{2}\,\kappa\,\Lambda)$ if $p > 1$, such that for any positive potential W we have

$$\lambda_1(-\Delta + W) \geq \nu(\beta) \quad \textit{with} \quad \beta = \left(\int_{\mathfrak{M}} W^{-p}\, d\,v_g\right)^{1/p}$$

Moreover, for large values of β, we have
$\nu(\beta)^{-\,(p+\frac{d}{2})} = \mathrm{L}^1_{-(p+\frac{d}{2}),d}\,\beta^{-p}\,(1 + o(1))$ *as* $\beta \to +\infty$

Spectral estimates on the cylinder

Joint work with M.J. Esteban and M. Loss

Spectral estimates and the symmetry breaking problem on the cylinder

Let $(\mathfrak{M}, g)$ be a smooth compact connected Riemannian manifold of dimension $d-1$ (no boundary) with $\mathrm{vol}_g(\mathfrak{M}) = 1$, and let

$$\mathcal{C} := \mathbb{R} \times \mathfrak{M} \ni x = (s, z)$$

be the *cylinder*. $\lambda_1^{\mathfrak{M}}$ is the lowest positive eigenvalue of the Laplace-Beltrami operator, $\kappa := \inf_{\mathfrak{M}} \inf_{\xi \in \mathbb{S}^{d-2}} \mathrm{Ric}(\xi\,,\xi)$

▷ *Is*

$$\Lambda(\mu) := \sup\left\{\lambda_1^{\mathcal{C}}[V] : V \in \mathrm{L}^q(\mathcal{C})\,,\ \|V\|_{\mathrm{L}^q(\mathcal{C})} = \mu\right\}$$

equal to

$$\Lambda_\star(\mu) := \sup\left\{\lambda_1^{\mathbb{R}}[V] : V \in \mathrm{L}^q(\mathbb{R}\,,\ \|V\|_{\mathrm{L}^q(\mathbb{R})} = \mu\right\} \quad ?$$

$-\lambda_1^{\mathcal{C}}[V]$ is the lowest eigenvalue of $-\partial_s^2 - \Delta_g - V$ and $-\partial_s^2 - V$ on $\mathcal{C}$

The Keller-Lieb-Thirring inequality on the line

Assume that $q \in (1,+\infty)$, $\beta = \frac{2\,q}{2\,q-1}$, $\mu_1 := q\,(q-1)\left(\frac{\sqrt{\pi}\,\Gamma(q)}{\Gamma(q+1/2)}\right)^{1/q}$.

$$\Lambda_\star(\mu) = (q-1)^2\,(\mu/\mu_1)^\beta \quad \forall\,\mu>0\,,$$

If V is a nonnegative real valued potential in $\mathrm{L}^q(\mathbb{R})$, then we have

$$\lambda_1^{\mathbb{R}}[V] \le \Lambda_\star(\|\,V\|_{\mathrm{L}^q(\mathbb{R})}) \quad \text{where} \quad \Lambda_\star(\mu) = (q-1)^2\left(\frac{\mu}{\mu_1}\right)^\beta \quad \forall\,\mu>0$$

and equality holds if and only if, up to scalings, translations and multiplications by a positive constant,

$$V(s) = \frac{q\,(q-1)}{(\cosh s)^2} =: V_1(s) \quad \forall\,s\in\mathbb{R}$$

where $\|\,V_1\|_{\mathrm{L}^q(\mathbb{R})} = \mu_1$, $\lambda_1^{\mathbb{R}}[V_1] = (q-1)^2$ and $\varphi(s) = (\cosh s)^{1-q}$

$$\lambda_\theta := \left(1+\delta\,\theta\,\tfrac{d-1}{d-2}\right)\kappa+\delta\,(1-\theta)\,\lambda_1^{\mathfrak{M}} \quad \text{with} \quad \delta=\frac{n-d}{(d-1)\,(n-1)}$$

$$\lambda_\star := \lambda_{\theta_\star} \quad \text{where} \quad \theta_\star := \frac{(d-2)\,(n-1)\,\big(3\,n+1-d\,(3\,n+5)\big)}{(d+1)\,\big(d\,(n^2-n-4)-n^2+3\,n+2\big)}$$

Theorem

Let $d\ge 2$ and $q\in(\min\{4,d/2\},+\infty)$. The function $\mu\mapsto\Lambda(\mu)$ is convex, positive and such that

$$\Lambda(\mu)^{q-d/2}\sim \mathrm{L}^1_{q-\frac d2,\,d}\,\mu^q \quad \textit{as} \quad \mu\to+\infty$$

Moreover, there exists a positive $\mu_\star$ with

$$\frac{\lambda_\star}{2\,(q-1)}\,\mu_1^\beta\le\mu_\star^\beta\le\frac{\lambda_1^{\mathfrak{M}}}{2\,q-1}\,\mu_1^\beta$$

such that

$$\Lambda(\mu)=\Lambda_\star(\mu)\quad\forall\,\mu\in(0,\mu_\star] \quad \textit{and} \quad \Lambda(\mu)>\Lambda_\star(\mu)\quad\forall\,\mu>\mu_\star$$

As a special case, if $\mathfrak{M}=\mathbb{S}^{d-1}$, inequalities are in fact equalities

The upper estimate

Lemma

If $\Lambda_\star(\mu) > \frac{4\,\lambda_1^{\mathfrak{M}}}{p^2-4}$, *then*

$$\sup\left\{\lambda_1^{\mathcal{C}}[V]\,:\,V\in\mathrm{L}^q(\mathcal{C})\,,\;\|V\|_{\mathrm{L}^q(\mathcal{C})}=\mu\right\}>\Lambda_\star(\mu)$$

$$\phi_\varepsilon(s,z):=\varphi_\mu(s)+\varepsilon\,\big(\varphi_\mu(s)\big)^{p/2}\,\psi_1(z)\quad\text{and}\quad V_\varepsilon(s,z):=\mu\,\frac{|\phi_\varepsilon(s,z)|^{p-2}}{\|\phi_\varepsilon\|_{\mathrm{L}^p(\mathcal{C})}^{p-2}}$$

where ψ_1 is an eigenfunction of $\lambda_1^{\mathfrak{M}}$ and φ_μ is optimal for $\Lambda_\star(\mu)$

$$-\,\lambda_1^{\mathcal{C}}[V_\varepsilon]+\Lambda_\star(\mu)\le\frac{4\,\varepsilon^2}{p+2}\left(\lambda_1^{\mathfrak{M}}-\tfrac14\,(p^2-4)\,\Lambda_\star(\mu)\right)+o(\varepsilon^2)$$

The lower estimate

$$\mathsf{J}[V] := \frac{\|V\|_{\mathrm{L}^q(\mathcal{C})}^q - \|\partial_s V^{(q-1)/2}\|_{\mathrm{L}^2(\mathcal{C})}^2 - \|\nabla_g V^{(q-1)/2}\|_{\mathrm{L}^2(\mathcal{C})}^2}{\|V^{(q-1)/2}\|_{\mathrm{L}^2(\mathcal{C})}^2}$$

Lemma

$$\Lambda(\mu) = \sup\left\{\mathsf{J}[V] : \|V\|_{\mathrm{L}^q(\mathcal{C})} = \mu\right\}$$

With $\alpha = \frac{1}{q-1}\sqrt{\Lambda_\star(\mu)}$, let us consider the operator $\mathfrak{L}$ such that

$$\mathfrak{L}\, u^m := -\frac{m}{m-1}\,\partial_s\Big(u\,e^{-2\alpha s}\,\partial_s\left(u^{m-1}\,e^{\alpha s}\right)\Big) + e^{-\alpha s}\,\Delta_g u^m$$

where $m = 1 - \frac{1}{n}$, $n = 2\,q$. To any potential $V \geq 0$ we associate the *pressure* function

$$\mathsf{p}_V(r) := r\,V(s)^{-\frac{q-1}{4\,q}} \quad \forall\, r = e^{-\alpha s}$$

$$\begin{aligned}
\mathsf{K}[\mathsf{p}] := \frac{n-1}{n}\,\alpha^4 \int_{\mathbb{R}^d} \left|\mathsf{p}'' - \frac{\mathsf{p}'}{r} - \frac{\Delta_g \mathsf{p}}{\alpha^2\,(n-1)\,r^2}\right|^2 \mathsf{p}^{1-n}\,d\mu \\
+ 2\,\alpha^2 \int_{\mathbb{R}^d} \frac{1}{r^2} \left|\nabla_g \mathsf{p}' - \frac{\nabla_g \mathsf{p}}{r}\right|^2 \mathsf{p}^{1-n}\,d\mu \\
+ \left(\lambda_\star - \frac{2}{q-1}\,\Lambda_\star(\mu)\right) \int_{\mathbb{R}^d} \frac{|\nabla_g \mathsf{p}|^2}{r^4}\,\mathsf{p}^{1-n}\,d\mu
\end{aligned}$$

where $d\mu$ is the measure on $\mathbb{R}^+ \times \mathfrak{M}$ with density r^{n-1}, and $'$ denotes the derivative with respect to r

Lemma

There exists a positive constant c *such that, if* V *is a critical point of* J *under the constraint* $\|V\|_{\mathrm{L}^q(\mathcal{C})} = \mu$ *and* $u_V = V^{(q-1)/2}$*, then we have*

$$\mathsf{J}[V + \varepsilon\,u_V^{-1}\,\mathfrak{L}\,u_V^m] - \mathsf{J}[V] \ge \mathsf{c}\,\varepsilon\,\mathsf{K}[\mathsf{p}_V] + o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0$$

These slides can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Conferences/
▷ Lectures

Thank you for your attention !