Entropy methods and sharp functional inequalities: new results

Jean Dolbeault

http://www.ceremade.dauphine.fr/~dolbeaul

Ceremade, Université Paris-Dauphine

September 8, 2015

Nonlinear PDEs
Brussels, September 7-11, 2015
Outline

- Entropy and the fast diffusion equation: from functional inequalities and characterization of optimal rates to best matching Barenblatt functions and improved inequalities
- Fast diffusion equations on manifolds and sharp functional inequalities: rigidity results, the carré du champ or Bakry-Emery method, and the use of nonlinear diffusion equations
- An equivalent point of view: optimal Keller-Lieb-Thirring estimates on manifolds
- Symmetry and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities; an introduction to the lecture of Michael Loss
Entropy and the fast diffusion equation

A summary

▷ Relative entropy, linearization, functional inequalities, improvements, improved rates of convergence, delays
The fast diffusion equation

The fast diffusion equation corresponds to $m < 1$

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d, \ t > 0$$

Self-similar (Barenblatt) functions attract all solutions as $t \to +\infty$

[Friedmann, Kamin]

- Entropy methods allow to measure the speed of convergence of any solution to $U$ in norms which are adapted to the equation
- Entropy methods provide explicit constants

- The Bakry-Emery method [Carrillo, Toscani], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]
- The variational approach and Gagliardo-Nirenberg inequalities: [del Pino, JD]
- Mass transportation and gradient flow issues: [Otto et al.]
- Large time asymptotics and the spectral approach: [Blanchet, Bonforte, JD, Grillo, Vázquez], [Denzler, Koch, McCann], [Seis]
- Refined relative entropy methods
Time-dependent rescaling, free energy


**Time-dependent rescaling:** Take $u(\tau, y) = R^{-d}(\tau) v(t, y/R(\tau))$

where

$$
\frac{dR}{d\tau} = R^{d(1-m)-1}, \quad R(0) = 1, \quad t = \log R
$$

The function $v$ solves a Fokker-Planck type equation

$$
\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x v), \quad v|_{\tau=0} = u_0
$$

[Ralston, Newman, 1984] Lyapunov functional:

**Generalized entropy or Free energy**

$$
\mathcal{F}[v] := \int_{\mathbb{R}^d} \left( \frac{v^m}{m-1} + \frac{1}{2} |x|^2 v \right) dx - \mathcal{F}_0
$$

Entropy production is measured by the **Generalized Fisher information**

$$
\frac{d}{dt} \mathcal{F}[v] = -\mathcal{I}[v], \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx
$$
Relative entropy and entropy production

Stationary solution: choose \( C \) such that \( \|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0 \)

\[
v_\infty(x) := \left( C + \frac{1 - m}{2 m} |x|^2 \right)^{-1/(1-m)}
\]

Relative entropy: Fix \( \mathcal{F}_0 \) so that \( \mathcal{F}[v_\infty] = 0 \)

Entropy – entropy production inequality

Theorem

\( d \geq 3, \ m \in \left[ \frac{d-1}{d}, +\infty \right), \ m > \frac{1}{2}, \ m \neq 1 \)

\[
\mathcal{I}[v] \geq 2 \mathcal{F}[v]
\]

Corollary

A solution \( v \) with initial data \( u_0 \in L^1_+(\mathbb{R}^d) \) such that \( |x|^2 u_0 \in L^1(\mathbb{R}^d) \), \( u_0^m \in L^1(\mathbb{R}^d) \) satisfies

\[
\mathcal{F}[v(t, \cdot)] \leq \mathcal{F}[u_0] e^{-2 t}
\]
An equivalent formulation: Gagliardo-Nirenberg inequalities

\[ \mathcal{F}[v] = \int_{\mathbb{R}^d} \left( \frac{v^m}{m-1} + \frac{1}{2} |x|^2 v \right) dx - \mathcal{F}_0 \leq \frac{1}{2} \int_{\mathbb{R}^d} v \left| \nabla \frac{v^m}{v} + x \right|^2 dx = \frac{1}{2} \mathcal{I}[v] \]

Rewrite it with \( p = \frac{1}{2m-1} \), \( v = w^{2p} \), \( v^m = w^{p+1} \) as

\[ \frac{1}{2} \left( \frac{2m}{2m-1} \right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left( \frac{1}{1-m} - d \right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \geq 0 \]

**Theorem**

[Del Pino, J.D.] *With* \( 1 < p \leq \frac{d}{d-2} \) (fast diffusion case) and \( d \geq 3 \)

\[ \| w \|_{L^{2p}(\mathbb{R}^d)} \leq C_{p,d}^{GN} \| \nabla w \|_{L^2(\mathbb{R}^d)}^{\theta} \| w \|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta} \]
Improved asymptotic rates

[Denzler, McCann], [Denzler, Koch, McCann], [Seis]
[Blanchet, Bonforte, J.D., Grillo, Vázquez], [Bonforte, J.D., Grillo, Vázquez], [J.D., Toscani]
Fast diffusion equations: some recent results

- improved inequalities and scalings
- scalings and a concavity property
- improved rates and best matching
Improved inequalities and scalings
Gagliardo-Nirenberg inequalities and the FDE

\[ \| \nabla w \|_{L^2(\mathbb{R}^d)}^{q} \| w \|_{L^{q+1}(\mathbb{R}^d)}^{1-q} \geq C_{\text{GN}} \| w \|_{L^2(\mathbb{R}^d)} \]

With the right choice of the constants, the functional

\[ J[w] := \frac{1}{4} (q^2 - 1) \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \beta \int_{\mathbb{R}^d} |w|^{q+1} \, dx - K C_{\text{GN}}^{\alpha} (\int_{\mathbb{R}^d} |w|^{2q} \, dx)^{\frac{\alpha}{2q}} \]

is nonnegative and \( J[w] \geq J[w^*] = 0 \)

**Theorem**

[Dolbeault-Toscani] For some nonnegative, convex, increasing \( \varphi \)

\[ J[w] \geq \varphi \left[ \beta \left( \int_{\mathbb{R}^d} |w^*|^{q+1} \, dx - \int_{\mathbb{R}^d} |w|^{q+1} \, dx \right) \right] \]

for any \( w \in L^{q+1}(\mathbb{R}^d) \) such that \( \int_{\mathbb{R}^d} |\nabla w|^2 \, dx < \infty \) and

\[ \int_{\mathbb{R}^d} |w|^{2q} |x|^2 \, dx = \int_{\mathbb{R}^d} w_{*}^{2q} |x|^2 \, dx \]

Consequence for decay rates of relative Rényi entropies: faster rates of convergence in intermediate asymptotics for \( \frac{\partial u}{\partial t} = \Delta u^p \)
Scalings and a concavity property

- Rényi entropies, the entropy approach without rescaling: [Savaré, Toscani]

- faster rates of convergence: [Carrillo, Toscani], [JD, Toscani]
The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial u}{\partial t} = \Delta u^m$$

with initial datum $u(x, t = 0) = u_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} u_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, u_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$U_*(t, x) := \frac{1}{(\kappa t^{1/\mu})^d} B_*(\frac{x}{\kappa t^{1/\mu}})$$

where

$$\mu := 2 + d(m - 1), \quad \kappa := \left| \frac{2 \mu m}{m - 1} \right|^{1/\mu}$$

and $B_*$ is the Barenblatt profile

$$B_*(x) := \begin{cases} 
(C_* - |x|^2)^{1/(m-1)} & \text{if } m > 1 \\
(C_* + |x|^2)^{1/(m-1)} & \text{if } m < 1 
\end{cases}$$
The entropy

The entropy is defined by

\[ E := \int_{\mathbb{R}^d} u^m \, dx \]

and the Fisher information by

\[ I := \int_{\mathbb{R}^d} u |\nabla p|^2 \, dx \quad \text{with} \quad p = \frac{m}{m-1} u^{m-1} \]

\( p \) is the pressure variable. If \( u \) solves the fast diffusion equation, then

\[ E' = (1 - m) I \]

To compute \( I' \), we will use the fact that

\[ \frac{\partial p}{\partial t} = (m - 1) p \Delta p + |\nabla p|^2 \]

\[ F := E^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d (1 - m)} = 1 + \frac{2}{1 - m} \left( \frac{1}{d} + m - 1 \right) = \frac{2}{d} \frac{1}{1 - m} - 1 \]

has a linear growth asymptotically as \( t \to +\infty \).
The concavity property

**Theorem**

**[Toscani-Savaré]** Assume that \( m \geq 1 - \frac{1}{d} \) if \( d > 1 \) and \( m > 0 \) if \( d = 1 \).

Then \( F(t) \) is increasing, \((1 - m) F''(t) \leq 0 \) and

\[
\lim_{t \to +\infty} \frac{1}{t} F(t) = (1 - m) \sigma \quad \text{and} \quad \lim_{t \to +\infty} E^{\sigma-1} I = (1 - m) \sigma E^{\sigma-1} l_*.
\]

**[Dolbeault-Toscani]** The inequality

\[
\sigma^{-1} F' = E^{\sigma-1} I \geq E^{\sigma-1} l_*
\]

is equivalent to the Gagliardo-Nirenberg inequality

\[
\| \nabla w \|_{L^2(\mathbb{R}^d)}^\theta \| w \|_{L^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq C_{GN} \| w \|_{L^{2q}(\mathbb{R}^d)}
\]

if \( 1 - \frac{1}{d} \leq m < 1 \). Hint: \( u^{m-1/2} = \frac{w}{\| w \|_{L^{2q}(\mathbb{R}^d)}}, \quad q = \frac{1}{2(m-1)} \)
Lemma

If $u$ solves $\frac{\partial u}{\partial t} = \Delta u^m$ with $\frac{1}{d} \leq m < 1$, then

$$I' = \frac{d}{dt} \int_{\mathbb{R}^d} u |\nabla p|^2 \, dx = -2 \int_{\mathbb{R}^d} u^m \left( \|D^2 p\|^2 + (m - 1)(\Delta p)^2 \right) \, dx$$

$$\|D^2 p\|^2 = \frac{1}{d} (\Delta p)^2 + \left\| D^2 p - \frac{1}{d} \Delta p \text{Id} \right\|^2$$

$$\frac{1}{\sigma (1 - m)} E^{2 - \sigma} (E^\sigma)'' = (1 - m)(\sigma - 1) \left( \int_{\mathbb{R}^d} u |\nabla p|^2 \, dx \right)^2$$

$$- 2 \left( \frac{1}{d} + m - 1 \right) \int_{\mathbb{R}^d} u^m \, dx \int_{\mathbb{R}^d} u^m (\Delta p)^2 \, dx$$

$$- 2 \int_{\mathbb{R}^d} u^m \, dx \int_{\mathbb{R}^d} u^m \left\| D^2 p - \frac{1}{d} \Delta p \text{Id} \right\|^2 \, dx$$
Best matching
Relative entropy and best matching

Consider the family of the Barenblatt profiles

\[ B_\sigma(x) := \sigma^{-d/2} \left( C_\star + \frac{1}{\sigma} |x|^2 \right)^{\frac{1}{m-1}} \quad \forall \ x \in \mathbb{R}^d \]

The Barenblatt profile \( B_\sigma \) plays the role of a \textit{local Gibbs state} if \( C_\star \) is chosen so that \( \int_{\mathbb{R}^d} B_\sigma \, dx = \int_{\mathbb{R}^d} v \, dx \)

The relative entropy is defined by

\[ F_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[ v^m - B_\sigma^m - m B_\sigma^{m-1} (v - B_\sigma) \right] \, dx \]

To minimize \( F_\sigma[v] \) with respect to \( \sigma \) is equivalent to fix \( \sigma \) such that

\[ \sigma \int_{\mathbb{R}^d} |x|^2 \, B_1 \, dx = \int_{\mathbb{R}^d} |x|^2 \, B_\sigma \, dx = \int_{\mathbb{R}^d} |x|^2 \, v \, dx \]
Let $m \in (\frac{d}{d+2}, 1)$ and consider the relative entropy

$$\mathcal{F}_\sigma[u] := \frac{1}{m - 1} \int_{\mathbb{R}^d} \left[ u^m - B_\sigma^m - m B_\sigma^{m-1} (u - B_\sigma) \right] \, dx$$

**Theorem**

[J.D., Toscani] Assume that $u$ is a nonnegative function in $L^1(\mathbb{R}^d)$ such that $u^m$ and $x \mapsto |x|^2 u$ are both integrable on $\mathbb{R}^d$. If $\|u\|_{L^1(\mathbb{R}^d)} = M$ and $\int_{\mathbb{R}^d} |x|^2 u \, dx = \int_{\mathbb{R}^d} |x|^2 B_\sigma \, dx$, then

$$\frac{\mathcal{F}_\sigma[u]}{\sigma^{\frac{d}{2}(1-m)}} \geq \frac{m}{8 \int_{\mathbb{R}^d} B_1^m \, dx} \left( C_* \|u - B_\sigma\|_{L^1(\mathbb{R}^d)} + \frac{1}{\sigma} \int_{\mathbb{R}^d} |x|^2 |u - B_\sigma| \, dx \right)^2$$
Temperature (fast diffusion case)

The second moment functional (temperature) is defined by

$$\Theta(t) := \frac{1}{d} \int_{\mathbb{R}^d} |x|^2 u(t, x) \, dx$$

and such that

$$\Theta' = 2 \mathcal{E}$$
Let $U^s_\star$ be the best matching Barenblatt function, in the sense of relative entropy $F[u | U^s_\star]$, among all Barenblatt functions $(U^s_\star)_{s>0}$.
A result on delays

**Theorem**

Assume that $m \geq 1 - \frac{1}{d}$ and $m \neq 1$. The best matching Barenblatt function of a solution $u$ is $(t, x) \mapsto U_\ast(t + \tau(t), x)$ and the function $t \mapsto \tau(t)$ is nondecreasing if $m > 1$ and nonincreasing if $1 - \frac{1}{d} \leq m < 1$.

With $G := \Theta^{1 - \frac{n}{2}}$, $\eta = d(1 - m) = 2 - \mu$, the Rényi entropy power functional $H := \Theta^{-\frac{n}{2}} E$ is such that

$$G' = \mu H \quad \text{with} \quad H := \Theta^{-\frac{n}{2}} E$$

$$\frac{H'}{1 - m} = \Theta^{-1 - \frac{n}{2}} (\Theta I - dE^2) = \frac{dE^2}{\Theta_{\frac{n}{2} + 1}} (q - 1) \quad \text{with} \quad q := \frac{\Theta I}{dE^2} \geq 1$$

$$dE^2 = \frac{1}{d} \left( - \int_{\mathbb{R}^d} x \cdot \nabla (u^m) \, dx \right)^2 = \frac{1}{d} \left( \int_{\mathbb{R}^d} x \cdot u \nabla p \, dx \right)^2$$

$$\leq \frac{1}{d} \int_{\mathbb{R}^d} u |x|^2 \, dx \int_{\mathbb{R}^d} u |\nabla p|^2 \, dx = \Theta I$$
Fast diffusion equations on manifolds and sharp functional inequalities

- The sphere
- The line
- Compact Riemannian manifolds
- The Moser-Trudinger-Onofri inequality on Riemannian manifolds
Interpolation inequalities on the sphere

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss
A family of interpolation inequalities on the sphere

The following interpolation inequality holds on the sphere

$$\frac{p - 2}{d} \int_{S^d} |\nabla u|^2 \, d v_g + \int_{S^d} |u|^2 \, d v_g \geq \left( \int_{S^d} |u|^p \, d v_g \right)^{2/p} \quad \forall \, u \in H^1(S^d, d v_g)$$

1. for any $p \in (2, 2^*]$ with $2^* = \frac{2d}{d-2}$ if $d \geq 3$
2. for any $p \in (2, \infty)$ if $d = 2$

Here $d v_g$ is the uniform probability measure: $v_g(S^d) = 1$

- $1$ is the optimal constant, equality achieved by constants
- $p = 2^*$ corresponds to Sobolev’s inequality...
Stereographic projection
Sobolev’s inequality

The stereographic projection of $\mathbb{S}^d \subset \mathbb{R}^d \times \mathbb{R} \ni (\rho \phi, z)$ onto $\mathbb{R}^d$:
to $\rho^2 + z^2 = 1$, $z \in [-1, 1]$, $\rho \geq 0$, $\phi \in \mathbb{S}^{d-1}$ we associate $x \in \mathbb{R}^d$ such
that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2 - 1}{r^2 + 1} = 1 - \frac{2}{r^2 + 1}, \quad \rho = \frac{2r}{r^2 + 1}$$

and transform any function $u$ on $\mathbb{S}^d$ into a function $v$ on $\mathbb{R}^d$ using

$$u(y) = \left(\frac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\frac{r^2 + 1}{2}\right)^{\frac{d-2}{2}} v(x) = (1 - z)^{-\frac{d-2}{2}} v(x)$$

$p = 2^*$, $S_d = \frac{1}{4} d (d - 2) |\mathbb{S}^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2 \, dx \geq S_d \left[ \int_{\mathbb{R}^d} |v|^\frac{2d}{d-2} \, dx \right]^{\frac{d-2}{d}} \quad \forall \, v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$
Schwarz symmetrization and the ultraspherical setting

\[(\xi_0, \xi_1, \ldots, \xi_d) \in S^d, \xi_d = z, \sum_{i=0}^d |\xi_i|^2 = 1 \quad [\text{Smets-Willem}]\]

**Lemma**

*Up to a rotation, any minimizer of \(Q\) depends only on \(\xi_d = z\)*

- Let \(d\sigma(\theta) := \frac{(\sin \theta)^{d-1}}{Z_d} \, d\theta\), \(Z_d := \sqrt{\pi} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\): \(\forall \nu \in H^1([0, \pi], d\sigma)\)

\[
\frac{p-2}{d} \int_0^\pi |\nu'(\theta)|^2 \, d\sigma + \int_0^\pi |\nu(\theta)|^2 \, d\sigma \geq \left(\int_0^\pi |\nu(\theta)|^p \, d\sigma\right)^{\frac{2}{p}}
\]

- Change of variables \(z = \cos \theta\), \(\nu(\theta) = f(z)\)

\[
\frac{p-2}{d} \int_{-1}^1 |f'|^2 \nu \, d\nu_d + \int_{-1}^1 |f|^2 \, d\nu_d \geq \left(\int_{-1}^1 |f|^p \, d\nu_d\right)^{\frac{2}{p}}
\]

where \(\nu_d(z) \, dz = d\nu_d(z) := Z_d^{-1} \nu_{\frac{d}{2}-1} \, dz\), \(\nu(z) := 1 - z^2\)
The ultraspherical operator

With \( d \nu_d = Z_d^{-1} \nu_d^{d-1} \, dz \), \( \nu(z) := 1 - z^2 \), consider the space \( L^2((-1, 1), d \nu_d) \) with scalar product

\[
\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1 f_2 \, d \nu_d , \quad ||f||_p = \left( \int_{-1}^{1} f^p \, d \nu_d \right)^{1/p}
\]

The self-adjoint \textit{ultraspherical} operator is

\[
\mathcal{L} f := (1 - z^2) f'' - d \, z \, f' = \nu f'' + \frac{d}{2} \nu' \, f'
\]

which satisfies \( \langle f_1, \mathcal{L} f_2 \rangle = -\int_{-1}^{1} f_1' \, f_2' \, \nu \, d \nu_d \)

Proposition

\( \text{Let } p \in [1, 2) \cup (2, 2^*], \, d \geq 1 \)

\[
-\langle f, \mathcal{L} f \rangle = \int_{-1}^{1} |f'|^2 \, \nu \, d \nu_d \geq d \frac{||f||_p^2 - ||f||_2^2}{p - 2} \quad \forall f \in H^1([-1, 1], d \nu_d)
\]
Flows on the sphere

- Heat flow and the Bakry-Emery method
- Fast diffusion (porous media) flow and the choice of the exponents

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss
Heat flow and the Bakry-Emery method

With $g = f^p$, i.e. $f = g^{\alpha}$ with $\alpha = 1/p$

\[
\langle f, \mathcal{L} f \rangle = -\langle g^{\alpha}, \mathcal{L} g^{\alpha} \rangle =: \mathcal{I}[g] \geq d \frac{\|g\|_1^{2\alpha} - \|g^{2\alpha}\|_1}{p-2} =: \mathcal{F}[g]
\]

Heat flow

\[
\frac{\partial g}{\partial t} = \mathcal{L} g
\]

\[
\frac{d}{dt} \|g\|_1 = 0, \quad \frac{d}{dt} \|g^{2\alpha}\|_1 = -2(p-2) \langle f, \mathcal{L} f \rangle = 2(p-2) \int_{-1}^{1} |f'|^2 \nu \, d\nu_d
\]

which finally gives

\[
\frac{d}{dt} \mathcal{F}[g(t, \cdot)] = -\frac{d}{p-2} \frac{d}{dt} \|g^{2\alpha}\|_1 = -2d \mathcal{I}[g(t, \cdot)]
\]

Ineq. $\iff$ $\frac{d}{dt} \mathcal{F}[g(t, \cdot)] \leq -2d \mathcal{F}[g(t, \cdot)] \iff \frac{d}{dt} \mathcal{I}[g(t, \cdot)] \leq -2d \mathcal{I}[g(t, \cdot)]$
The equation for $g = f^p$ can be rewritten in terms of $f$ as

$$\frac{\partial f}{\partial t} = \mathcal{L} f + (p - 1) \frac{|f'|^2}{f} \nu$$

$$- \frac{1}{2} \frac{d}{dt} \int_{-1}^{1} |f'|^2 \nu \ d\nu_d = \frac{1}{2} \frac{d}{dt} \langle f, \mathcal{L} f \rangle = \langle \mathcal{L} f, \mathcal{L} f \rangle + (p - 1) \langle \frac{|f'|^2}{f} \nu, \mathcal{L} f \rangle$$

$$\frac{d}{dt} \mathcal{I}[g(t, \cdot)] + 2 d \mathcal{I}[g(t, \cdot)] = \frac{d}{dt} \int_{-1}^{1} |f'|^2 \nu \ d\nu_d + 2 d \int_{-1}^{1} |f'|^2 \nu \ d\nu_d$$

$$= -2 \int_{-1}^{1} \left( |f''|^2 + (p - 1) \frac{d}{d + 2} \frac{|f'|^4}{f^2} - 2 (p - 1) \frac{d - 1}{d + 2} \frac{|f'|^2 f''}{f} \right) \nu^2 \ d\nu_d$$

is nonpositive if

$$|f''|^2 + (p - 1) \frac{d}{d + 2} \frac{|f'|^4}{f^2} - 2 (p - 1) \frac{d - 1}{d + 2} \frac{|f'|^2 f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[ (p - 1) \frac{d - 1}{d + 2} \right]^2 \leq (p - 1) \frac{d}{d + 2} \iff p \leq \frac{2 d^2 + 1}{(d - 1)^2} = 2^# < \frac{2 d}{d - 2} = 2^*$$
... up to the critical exponent: a proof in two slides

\[
\left[ \frac{d}{dz}, \mathcal{L} \right] u = (\mathcal{L} u)' - \mathcal{L} u' = -2 z u'' - d u'
\]

\[
\int_{-1}^{1} (\mathcal{L} u)^2 \, d\nu_d = \int_{-1}^{1} |u''|^2 \, \nu^2 \, d\nu_d + d \int_{-1}^{1} |u'|^2 \, \nu \, d\nu_d
\]

\[
\int_{-1}^{1} (\mathcal{L} u) \frac{|u'|^2}{u} \, \nu \, d\nu_d = \frac{d}{d+2} \int_{-1}^{1} \frac{|u'|^4}{u^2} \, \nu^2 \, d\nu_d - 2 \frac{d-1}{d+2} \int_{-1}^{1} \frac{|u'|^2 u''}{u} \, \nu^2 \, d\nu_d
\]

On \((-1,1)\), let us consider the porous medium (fast diffusion) flow

\[
u_t = u^{2-2\beta} \left( \mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right)
\]

If \(\kappa = \beta (p - 2) + 1\), the \(L^p\) norm is conserved

\[
\frac{d}{dt} \int_{-1}^{1} u^{\beta p} \, d\nu_d = \beta p (\kappa - \beta (p - 2) - 1) \int_{-1}^{1} u^{\beta(p-2)} |u'|^2 \nu \, d\nu_d = 0
\]
\( f = u^\beta, \| f' \|_{L^2(S^d)}^2 + \frac{d}{p-2} \left( \| f \|_{L^2(S^d)}^2 - \| f \|_{L^p(S^d)}^2 \right) \geq 0 \) ?

\[
\mathcal{A} := \int_{-1}^{1} |u''|^2 \nu^2 \, d\nu_d - 2 \frac{d - 1}{d + 2} (\kappa + \beta - 1) \int_{-1}^{1} u'' \frac{|u'|^2}{u} \nu^2 \, d\nu_d \\
+ \left[ \kappa (\beta - 1) + \frac{d}{d + 2} (\kappa + \beta - 1) \right] \int_{-1}^{1} \frac{|u'|^4}{u^2} \nu^2 \, d\nu_d
\]

\( \mathcal{A} \) is nonnegative for some \( \beta \) if

\[
\frac{8 d^2}{(d + 2)^2} (p - 1) (2^* - p) \geq 0
\]

\( \mathcal{A} \) is a sum of squares if \( p \in (2, 2^*) \) for an arbitrary choice of \( \beta \) in a certain interval (depending on \( p \) and \( d \))

\[
\mathcal{A} = \int_{-1}^{1} \left| u'' - \frac{p + 2}{6 - p} \frac{|u'|^2}{u} \right|^2 \nu^2 \, d\nu_d \geq 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6 - p}
\]
The rigidity point of view

Which computation have we done? \( u_t = u^{2-2\beta} \left( \mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right) \)

\[-\mathcal{L} u - (\beta - 1) \frac{|u'|^2}{u} \nu + \frac{\lambda}{p-2} u = \frac{\lambda}{p-2} u^\kappa\]

Multiply by \( \mathcal{L} u \) and integrate

\[\ldots \int_{-1}^{1} \mathcal{L} u \ u^\kappa \ d\nu_d = -\kappa \int_{-1}^{1} u^\kappa \frac{|u'|^2}{u} \ d\nu_d\]

Multiply by \( \kappa \frac{|u'|^2}{u} \) and integrate

\[\ldots = +\kappa \int_{-1}^{1} u^\kappa \frac{|u'|^2}{u} \ d\nu_d\]

The two terms cancel and we are left only with the two-homogenous terms
Improvements of the inequalities (subcritical range)

as long as the exponent is either in the range $(1, 2)$ or in the range $(2, 2^*)$, one can establish improved inequalities.

An improvement automatically gives an explicit stability result of the optimal functions in the (non-improved) inequality.

By duality, this provides a stability result for Keller-Lieb-Tirring inequalities.
What does “improvement” mean?

An improved inequality is

\[ d \Phi(e) \leq i \quad \forall \ u \in H^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|^2_{L^2(\mathbb{S}^d)} = 1 \]

for some function \( \Phi \) such that \( \Phi(0) = 0, \Phi'(0) = 1, \Phi' > 0 \) and \( \Phi(s) > s \) for any \( s \). With \( \Psi(s) := s - \Phi^{-1}(s) \)

\[ i - d e \geq d (\Psi \circ \Phi)(e) \quad \forall \ u \in H^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|^2_{L^2(\mathbb{S}^d)} = 1 \]

Lemma (Generalized Csiszár-Kullback inequalities)

\[ \|\nabla u\|^2_{L^2(\mathbb{S}^d)} - \frac{d}{p - 2} \left[ \|u\|^2_{L^p(\mathbb{S}^d)} - \|u\|^2_{L^2(\mathbb{S}^d)} \right] \geq d \|u\|^2_{L^2(\mathbb{S}^d)} (\Psi \circ \Phi) \left( C \frac{\|u\|^2_{L^p(\mathbb{S}^d)}}{\|u\|^2_{L^2(\mathbb{S}^d)}} \|u^r - \bar{u}^r\|^2_{L^q(\mathbb{S}^d)} \right) \quad \forall \ u \in H^1(\mathbb{S}^d) \]

\( s(p) := \max\{2, p\} \) and \( p \in (1, 2) \): \( q(p) := 2/p, r(p) := p; p \in (2, 4) \): \( q = p/2, r = 2; p \geq 4 \): \( q = p/(p - 2), r = p - 2 \)
Fast diffusion equations: new points of view
Introduction to symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities
Spectral estimates: Keller-Lieb-Thirring estimates on manifolds

The sphere
The line
Compact Riemannian manifolds
The Moser-Trudinger-Onofri inequality

**Linear flow: improved Bakry-Emery method**

Cf. [Arnold, JD]

\[
\frac{\partial w}{\partial t} = \mathcal{L} w + \kappa \frac{|w'|^2}{w} \nu
\]

With \(2^\# := \frac{2d^2+1}{(d-1)^2}\)

\[
\gamma_1 := \left(\frac{d-1}{d+2}\right)^2 (p-1)(2^\# - p) \quad \text{if} \quad d > 1, \quad \gamma_1 := \frac{p-1}{3} \quad \text{if} \quad d = 1
\]

If \(p \in [1, 2) \cup (2, 2^\#]\) and \(w\) is a solution, then

\[
\frac{d}{dt} (i - de) \leq -\gamma_1 \int_{-1}^{1} \frac{|w'|^4}{w^2} \, d\nu_d \leq -\gamma_1 \frac{|e'|^2}{1 - (p - 2)e}
\]

Recalling that \(e' = -i\), we get a differential inequality

\[
e'' + de' \geq \gamma_1 \frac{|e'|^2}{1 - (p - 2)e}
\]

After integration: \(d \Phi(e(0)) \leq i(0)\)
Nonlinear flow: the Hölder estimate of J. Demange

\[ w_t = w^{2-2\beta} \left( \mathcal{L} w + \kappa \frac{|w'|^2}{w} \right) \]

For all \( p \in [1, 2^*] \), \( \kappa = \beta (p - 2) + 1 \), \( \frac{d}{dt} \int_{-1}^{1} w^p \, d\nu_d = 0 \)

\[-\frac{1}{2\beta^2} \frac{d}{dt} \int_{-1}^{1} \left( |(w^\beta)'|^2 \nu + \frac{d}{p-2} (w^{2\beta} - \overline{w^{2\beta}}) \right) \, d\nu_d \geq \gamma \int_{-1}^{1} \frac{|w'|^4}{w^4} \, \nu^2 \, d\nu_d \]

**Lemma (Demange)**

*For all \( w \in H^1((-1, 1), d\nu_d) \), such that \( \int_{-1}^{1} w^p \, d\nu_d = 1 \)*

\[ \int_{-1}^{1} \frac{|w'|^4}{w^2} \, \nu^2 \, d\nu_d \geq \frac{1}{\beta^2} \frac{\int_{-1}^{1} |(w^\beta)'|^2 \, \nu \, d\nu_d \int_{-1}^{1} |w'|^2 \, \nu \, d\nu_d}{\left( \int_{-1}^{1} w^{2\beta} \, d\nu_d \right)^{\delta}} \]

.... but there are conditions on \( \beta \)
Admissible \((p, \beta)\) for \(d = 5\)

\[\begin{array}{cccccc}
0.5 & 1.0 & 1.5 & 2.0 & 2.5 & 3.0 \\
2.0 & 2.5 & 3.0 & 3.5 & 4.0 & 4.5
\end{array}\]

J. Dolbeault  Entropy methods and sharp functional inequalities
The line

A first example of a non-compact manifold

Joint work with M.J. Esteban, A. Laptev and M. Loss
One-dimensional Gagliardo-Nirenberg-Sobolev inequalities

\[ \| f \|_{L^p(\mathbb{R})} \leq C_{GN}(p) \| f' \|_{L^2(\mathbb{R})}^{\theta} \| f \|_{L^2(\mathbb{R})}^{1-\theta} \quad \text{if} \quad p \in (2, \infty) \]

\[ \| f \|_{L^2(\mathbb{R})} \leq C_{GN}(p) \| f' \|_{L^2(\mathbb{R})}^{\eta} \| f \|_{L^p(\mathbb{R})}^{1-\eta} \quad \text{if} \quad p \in (1, 2) \]

with \( \theta = \frac{p-2}{2p} \) and \( \eta = \frac{2-p}{2+p} \)

The threshold case corresponding to the limit as \( p \to 2 \) is the logarithmic Sobolev inequality

\[ \int_{\mathbb{R}} u^2 \log \left( \frac{u^2}{\| u \|_{L^2(\mathbb{R})}^2} \right) \, dx \leq \frac{1}{2} \| u \|_{L^2(\mathbb{R})}^2 \log \left( \frac{2}{\pi e} \frac{\| u' \|_{L^2(\mathbb{R})}^2}{\| u \|_{L^2(\mathbb{R})}^2} \right) \]

If \( p > 2 \), \( u_*(x) = (\cosh x)^{-\frac{2}{p-2}} \) solves

\[ -(p-2)^2 u'' + 4u - 2p |u|^{p-2} u = 0 \]

If \( p \in (1, 2) \) consider \( u_*(x) = (\cos x)^{\frac{2}{p}}, \, x \in (-\pi/2, \pi/2) \)
Flow

Let us define on $H^1(\mathbb{R})$ the functional

$$\mathcal{F}[v] := \|v'\|^2_{L^2(\mathbb{R})} + \frac{4}{(p-2)^2} \|v\|^2_{L^2(\mathbb{R})} - C \|v\|^2_{L^p(\mathbb{R})} \quad \text{s.t. } \mathcal{F}[u_*] = 0$$

With $z(x) := \tanh x$, consider the flow

$$v_t = \frac{v^{1-\frac{p}{2}}}{\sqrt{1 - z^2}} \left[ v'' + \frac{2p}{p-2} z v' + \frac{p}{2} \frac{|v'|^2}{v} + \frac{2}{p-2} \frac{v}{v} \right]$$

Theorem (Dolbeault-Esteban-Laptev-Loss)

Let $p \in (2, \infty)$. Then

$$\frac{d}{dt} \mathcal{F}[v(t)] \leq 0 \quad \text{and} \quad \lim_{t \to \infty} \mathcal{F}[v(t)] = 0$$

$$\frac{d}{dt} \mathcal{F}[v(t)] = 0 \quad \iff \quad v_0(x) = u_*(x - x_0)$$

Similar results for $p \in (1, 2)$
The inequality \((p > 2)\) and the ultraspherical operator

The problem on the line is equivalent to the critical problem for the ultraspherical operator

\[
\int_{\mathbb{R}} |v'|^2 \, dx + \frac{4}{(p - 2)^2} \int_{\mathbb{R}} |v|^2 \, dx \geq C \left( \int_{\mathbb{R}} |v|^p \, dx \right)^{\frac{2}{p}}
\]

With

\[z(x) = \tanh x, \quad v_\ast = (1 - z^2)^{\frac{1}{p-2}} \quad \text{and} \quad v(x) = v_\ast(x) f(z(x))\]

equality is achieved for \(f = 1\) and, if we let \(\nu(z) := 1 - z^2\), then

\[
\int_{-1}^{1} |f'|^2 \nu \, d\nu_d + \frac{2p}{(p - 2)^2} \int_{-1}^{1} |f|^2 \, d\nu_d \geq \frac{2p}{(p - 2)^2} \left( \int_{-1}^{1} |f|^p \, d\nu_d \right)^{\frac{2}{p}}
\]

where \(d\nu_p\) denotes the probability measure \(d\nu_p(z) := \frac{1}{\zeta_p} \nu^{\frac{2}{p-2}} \, dz\)

\[d = \frac{2p}{p-2} \quad \iff \quad p = \frac{2d}{d-2}\]
Change of variables = stereographic projection + Emden-Fowler
Compact Riemannian manifolds

no sign is required on the Ricci tensor and an improved integral criterion is established

the flow explores the energy landscape... and shows the non-optimality of the improved criterion
Riemannian manifolds with positive curvature

$(\mathcal{M}, g)$ is a smooth closed compact connected Riemannian manifold dimension $d$, no boundary, $\Delta_g$ is the Laplace-Beltrami operator, $\text{vol}(\mathcal{M}) = 1$, $\mathcal{R}$ is the Ricci tensor, $\lambda_1 = \lambda_1(-\Delta_g)$

$$\rho := \inf_{\mathcal{M}} \inf_{\xi \in S^{d-1}} \mathcal{R}(\xi, \xi)$$

**Theorem (Licois-Véron, Bakry-Ledoux)**

Assume $d \geq 2$ and $\rho > 0$. If

$$\lambda \leq (1 - \theta) \lambda_1 + \theta \frac{d \rho}{d - 1}$$

where

$$\theta = \frac{(d - 1)^2 (p - 1)}{d (d + 2) + p - 1} > 0$$

then for any $p \in (2, 2^*)$, the equation

$$-\Delta_g v + \frac{\lambda}{p - 2} (v - v^{p-1}) = 0$$

has a unique positive solution $v \in C^2(\mathcal{M})$: $v \equiv 1$
Theorem (Dolbeault-Esteban-Loss)

For any $p \in (1, 2) \cup (2, 2^*)$

$$0 < \lambda < \lambda_* = \inf_{u \in H^2(\mathcal{M})} \frac{\int_{\mathcal{M}} \left[(1 - \theta)(\Delta_g u)^2 + \frac{\theta}{d-1} \mathcal{R}(\nabla u, \nabla u)\right] d \nu_g}{\int_{\mathcal{M}} |\nabla u|^2 d \nu_g}$$

there is a unique positive solution in $C^2(\mathcal{M})$: $u \equiv 1$

$$\lim_{p \to 1^+} \theta(p) = 0 \implies \lim_{p \to 1^+} \lambda_*(p) = \lambda_1$$ if $\rho$ is bounded

$$\lambda_* = \lambda_1 = d \rho / (d - 1) = d$$ if $\mathcal{M} = S^d$ since $\rho = d - 1$

$$(1 - \theta) \lambda_1 + \theta \frac{d \rho}{d - 1} \leq \lambda_* \leq \lambda_1$$
Riemannian manifolds: second improvement

\[ H_g u \] denotes Hessian of \( u \) and \( \theta = \frac{(d - 1)^2 (p - 1)}{d (d + 2) + p - 1} \)

\[ Q_g u := H_g u - \frac{g}{d} \Delta_g u - \frac{(d - 1)(p - 1)}{\theta (d + 3 - p)} \left[ \frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right] \]

\[ \Lambda_* := \inf_{u \in H^2(M) \setminus \{0\}} \frac{(1 - \theta) \int_M (\Delta_g u)^2 \, dv_g + \frac{\theta d}{d-1} \int_M \left[ \|Q_g u\|^2 + \mathcal{K}(\nabla u, \nabla u) \right]}{\int_M |\nabla u|^2 \, dv_g} \]

**Theorem (Dolbeault-Esteban-Loss)**

Assume that \( \Lambda_* > 0 \). For any \( p \in (1, 2) \cup (2, 2^*) \), the equation has a unique positive solution in \( C^2(M) \) if \( \lambda \in (0, \Lambda_*): u \equiv 1 \)
Optimal interpolation inequality

For any \( p \in (1, 2) \cup (2, 2^*) \) or \( p = 2^* \) if \( d \geq 3 \)

\[
\| \nabla v \|_{L^2(\mathcal{M})}^2 \geq \frac{\lambda}{p-2} \left[ \| v \|_{L^p(\mathcal{M})}^2 - \| v \|_{L^2(\mathcal{M})}^2 \right] \quad \forall v \in H^1(\mathcal{M})
\]

**Theorem (Dolbeault-Esteban-Loss)**

Assume \( \Lambda_*>0 \). The above inequality holds for some \( \lambda = \Lambda \in [\Lambda_*, \lambda_1] \)
If \( \Lambda_* < \lambda_1 \), then the optimal constant \( \Lambda \) is such that

\[ \Lambda_* < \Lambda \leq \lambda_1 \]

If \( p = 1 \), then \( \Lambda = \lambda_1 \)

Using \( u = 1 + \varepsilon \varphi \) as a test function where \( \varphi \) we get \( \lambda \leq \lambda_1 \)

A minimum of

\[ v \mapsto \| \nabla v \|_{L^2(\mathcal{M})}^2 - \frac{\lambda}{p-2} \left[ \| v \|_{L^p(\mathcal{M})}^2 - \| v \|_{L^2(\mathcal{M})}^2 \right] \]

under the constraint \( \| v \|_{L^p(\mathcal{M})} = 1 \) is negative if \( \lambda > \lambda_1 \).
The flow

The key tool is the flow

$$u_t = u^{2-2\beta} \left( \Delta_g u + \kappa \frac{\nabla u^2}{u} \right), \quad \kappa = 1 + \beta (p - 2)$$

If $v = u^\beta$, then $\frac{d}{dt} ||v||_{L^p(\mathcal{M})} = 0$ and the functional

$$F[u] := \int_{\mathcal{M}} |\nabla (u^\beta)|^2 \, dv_g + \frac{\lambda}{p - 2} \left[ \int_{\mathcal{M}} u^{2\beta} \, dv_g - \left( \int_{\mathcal{M}} u^{\beta p} \, dv_g \right)^{2/p} \right]$$

is monotone decaying
Let $d \geq 2$, $u \in C^2 (\mathcal{M})$, and consider the trace free Hessian

$$L_g u := H_g u - \frac{g}{d} \Delta_g u$$

**Lemma**

$$\int_{\mathcal{M}} (\Delta_g u)^2 d\nu_g = \frac{d}{d-1} \int_{\mathcal{M}} \|L_g u\|^2 d\nu_g + \frac{d}{d-1} \int_{\mathcal{M}} R(\nabla u, \nabla u) d\nu_g$$

Based on the Bochner-Lichnerovich-Weitzenböck formula

$$\frac{1}{2} \Delta |\nabla u|^2 = \|H_g u\|^2 + \nabla (\Delta_g u) \cdot \nabla u + R(\nabla u, \nabla u)$$
Elementary observations (2/2)

Lemma

\[ \int_M \Delta_g u \left\vert \nabla u \right\vert^2 \frac{d \nu_g}{u} = \frac{d}{d+2} \int_M \left\vert \nabla u \right\vert^4 \frac{d \nu_g}{u^2} - 2 \frac{d}{d+2} \int_M [L_g u] : \left[ \frac{\nabla u \otimes \nabla u}{u} \right] d \nu_g \]

Lemma

\[ \int_M (\Delta_g u)^2 \, d \nu_g \geq \lambda_1 \int_M \left\vert \nabla u \right\vert^2 \, d \nu_g \quad \forall u \in H^2(M) \]

and \( \lambda_1 \) is the optimal constant in the above inequality
The key estimates

\[ G[u] := \int_{\mathcal{M}} \left[ \theta (\Delta_g u)^2 + (\kappa + \beta - 1) \Delta_g u \frac{\nabla u}{u}^2 + \kappa (\beta - 1) \frac{\nabla u}{u}^4 \right] d\nu_g \]

**Lemma**

\[ \frac{1}{2 \beta^2} \frac{d}{dt} \mathcal{F}[u] = -(1 - \theta) \int_{\mathcal{M}} (\Delta_g u)^2 d\nu_g - G[u] + \lambda \int_{\mathcal{M}} |\nabla u|^2 d\nu_g \]

\[ Q^\theta_g u := L_g u - \frac{1}{\theta} \frac{d-1}{d+2} (\kappa + \beta - 1) \left[ \frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right] \]

**Lemma**

\[ G[u] = \frac{\theta}{d-1} \left[ \int_{\mathcal{M}} \| Q^\theta_g u \|^2 d\nu_g + \int_{\mathcal{M}} \mathcal{K}(\nabla u, \nabla u) d\nu_g \right] - \mu \int_{\mathcal{M}} \frac{|\nabla u|^4}{u^2} d\nu_g \]

with \[ \mu := \frac{1}{\theta} \left( \frac{d-1}{d+2} \right)^2 (\kappa + \beta - 1)^2 - \kappa (\beta - 1) - (\kappa + \beta - 1) \frac{d}{d + 2} \]
Assume that $d \geq 2$. If $\theta = 1$, then $\mu$ is nonpositive if

$$\beta_-(p) \leq \beta \leq \beta_+(p) \quad \forall \, p \in (1, 2^*)$$

where $\beta_\pm := \frac{b \pm \sqrt{b^2 - a}}{2a}$ with $a = 2 - p + \left[\frac{(d-1)(p-1)}{d+2}\right]^2$ and $b = \frac{d+3-p}{d+2}$.

Notice that $\beta_-(p) < \beta_+(p)$ if $p \in (1, 2^*)$ and $\beta_-(2^*) = \beta_+(2^*)$

$$\theta = \frac{(d-1)^2(p-1)}{d(d+2)+p-1} \quad \text{and} \quad \beta = \frac{d+2}{d+3-p}$$

**Proposition**

Let $d \geq 2$, $p \in (1, 2) \cup (2, 2^*)$ ($p \neq 5$ or $d \neq 2$)

$$\frac{1}{2} \beta^2 \frac{d}{dt} \mathcal{F}[u] \leq \left(\lambda - \Lambda_\star\right) \int_M |\nabla u|^2 \, dv_g$$
The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban

Extension to compact Riemannian manifolds of dimension 2...
We shall also denote by $\mathcal{R}$ the Ricci tensor, by $H_g u$ the Hessian of $u$ and by

$$L_g u := H_g u - \frac{g}{d} \Delta_g u$$

the trace free Hessian. Let us denote by $M_g u$ the trace free tensor

$$M_g u := \nabla u \otimes \nabla u - \frac{g}{d} |\nabla u|^2$$

We define

$$\lambda_* := \inf_{u \in H^2(\mathcal{M}) \setminus \{0\}} \frac{\int_{\mathcal{M}} \left[ \| L_g u - \frac{1}{2} M_g u \|^2 + \mathcal{R}(\nabla u, \nabla u) \right] e^{-u/2} d\nu_g}{\int_{\mathcal{M}} |\nabla u|^2 e^{-u/2} d\nu_g}$$
Theorem

Assume that $d = 2$ and $\lambda_* > 0$. If $u$ is a smooth solution to

$$- \frac{1}{2} \Delta_g u + \lambda = e^u$$

then $u$ is a constant function if $\lambda \in (0, \lambda_*)$

The Moser-Trudinger-Onofri inequality on $\mathcal{M}$

$$\frac{1}{4} \| \nabla u \|_{L^2(\mathcal{M})}^2 + \lambda \int_{\mathcal{M}} u \, d\nu_g \geq \lambda \log \left( \int_{\mathcal{M}} e^u \, d\nu_g \right) \quad \forall u \in H^1(\mathcal{M})$$

for some constant $\lambda > 0$. Let us denote by $\lambda_1$ the first positive eigenvalue of $-\Delta_g$

Corollary

If $d = 2$, then the MTO inequality holds with $\lambda = \Lambda := \min\{4 \pi, \lambda_*\}$. Moreover, if $\Lambda$ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than $\Lambda$
The flow

\[
\frac{\partial f}{\partial t} = \Delta_g (e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2}
\]

\[
G_\lambda[f] := \int_M \| L_g f - \frac{1}{2} M_g f \|^2 e^{-f/2} \, d\nu_g + \int_M \mathcal{R}(\nabla f, \nabla f) e^{-f/2} \, d\nu_g
\]

\[
- \lambda \int_M |\nabla f|^2 e^{-f/2} \, d\nu_g
\]

Then for any \( \lambda \leq \lambda^* \) we have

\[
\frac{d}{dt} F_\lambda[f(t, \cdot)] = \int_M \left( -\frac{1}{2} \Delta_g f + \lambda \right) \left( \Delta_g (e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2} \right) \, d\nu_g
\]

\[
= -G_\lambda[f(t, \cdot)]
\]

Since \( F_\lambda \) is nonnegative and \( \lim_{t \to \infty} F_\lambda[f(t, \cdot)] = 0 \), we obtain that

\[
F_\lambda[u] \geq \int_0^\infty G_\lambda[f(t, \cdot)] \, dt
\]
Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

On the Euclidean space $\mathbb{R}^2$, given a general probability measure $\mu$ does the inequality

$$\frac{1}{16 \pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq \lambda \left[ \log \left( \int_{\mathbb{R}^d} e^u \, d\mu \right) - \int_{\mathbb{R}^d} u \, d\mu \right]$$

hold for some $\lambda > 0$? Let

$$\Lambda_* := \inf_{x \in \mathbb{R}^2} \frac{-\Delta \log \mu}{8 \pi \mu}$$

**Theorem**

Assume that $\mu$ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda < \Lambda_*$ and the inequality holds with $\lambda = \Lambda_*$ if equality is achieved among radial functions.
Nonlinear flows (fast diffusion equation) can be used as a tool for the investigation of sharp functional inequalities
Caffarelli-Kohn-Nirenberg inequalities and the symmetry breaking issue

Let \( D_{a,b} := \left\{ v \in L^p(\mathbb{R}^d, |x|^{-b} \, dx) : |x|^{-a} |\nabla v| \in L^2(\mathbb{R}^d, dx) \right\} \)

\[
\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b \, p}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} \, dx \quad \forall v \in D_{a,b}
\]

hold under the conditions that \( a \leq b \leq a + 1 \) if \( d \geq 3 \), \( a < b \leq a + 1 \) if \( d = 2 \), \( a + 1/2 < b \leq a + 1 \) if \( d = 1 \), and \( a < a_c := (d - 2)/2 \)

\[
p = \frac{2 \, d}{d - 2 + 2 \, (b - a)}
\]

\( \triangleright \) With

\[
v_\star(x) = \left( 1 + |x|^{(p-2) \, (a_c-a)} \right)^{- \frac{2}{p-2}} \quad \text{and} \quad C_{a,b}^\star = \frac{\| |x|^{-b} \, v_\star \|^2_p}{\| |x|^{-a} \, \nabla v_\star \|^2_2}
\]

do we have \( C_{a,b} = C_{a,b}^\star \) (symmetry) \( \text{or} \ C_{a,b} > C_{a,b}^\star \) (symmetry breaking) ?
Fast diffusion equations: new points of view
Fast diffusion equations on manifolds and sharp functional inequalities
Introduction to symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities
Spectral estimates: Keller-Lieb-Thirring estimates on manifolds

CKN: range of the parameters

Figure: $d = 3$

$$
\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|
abla v|^2}{|x|^{2a}} \, dx
$$

$a \leq b \leq a + 1$ if $d \geq 3$

$a < b \leq a + 1$ if $d = 2$, $a + 1/2 < b \leq a + 1$ if $d = 1$

and $a < a_c := (d - 2)/2$

$$p = \frac{2d}{d - 2 + 2(b - a)}$$
The Emden-Fowler transformation and the cylinder

With an Emden-Fowler transformation, Caffarelli-Kohn-Nirenberg inequalities on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder

\[ v(r, \omega) = r^{a-c} \varphi(s, \omega) \quad \text{with} \quad r = |x|, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r} \]

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as

\[ \| \partial_s \varphi \|_{L^2(C)}^2 + \| \nabla \omega \varphi \|_{L^2(C)}^2 + \Lambda \| \varphi \|_{L^2(C)}^2 \geq \mu(\Lambda) \| \varphi \|_{L^p(C)}^2 \quad \forall \varphi \in H^1(C) \]

where \( \Lambda := (a_c - a)^2 \), \( C = \mathbb{R} \times S^{d-1} \) and the optimal constant \( \mu(\Lambda) \) is

\[ \mu(\Lambda) = \frac{1}{C_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda} \]
Symmetry vs. symmetry breaking: the sharp result

A result based on entropies and nonlinear flows

Spectral estimates

- Spectral estimates on the sphere
- Spectral estimates on compact Riemannian manifolds
- Spectral estimates on the cylinder
Spectral estimates on the sphere

- The Keller-Lieb-Tirring inequality is equivalent to an interpolation inequality of Gagliardo-Nirenberg-Sobolev type

- We measure a quantitative deviation with respect to the semi-classical regime due to finite size effects

Joint work with M.J. Esteban and A. Laptev
An introduction to Lieb-Thirring inequalities

Consider the Schrödinger operator \( H = -\Delta - V \) on \( \mathbb{R}^d \) and denote by \((\lambda_k)_{k\geq 1}\) its eigenvalues.

- **Euclidean case** [Keller, 1961]
  \[
  |\lambda_1|^{\gamma} \leq L^{1}_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma+\frac{d}{2}}
  \]

  [Lieb-Thirring, 1976]
  \[
  \sum_{k \geq 1} |\lambda_k|^{\gamma} \leq L_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma+\frac{d}{2}}
  \]

  \(\gamma \geq 1/2\) if \(d = 1\), \(\gamma > 0\) if \(d = 2\) and \(\gamma \geq 0\) if \(d \geq 3\) [Weidl], [Cwikel], [Rosenbljum], [Aizenman], [Laptev-Weidl], [Helffer], [Robert], [Dolbeault-Felmer-Loss-Paturel]... [Dolbeault-Laptev-Loss 2008]

- **Compact manifolds**: log Sobolev case: [Federbusch], [Rothaus]; case \(\gamma = 0\) (Rozenbljum-Lieb-Cwikel inequality): [Levin-Solomyak]; [Lieb], [Levin], [Ouabaz-Poupaud]... [Ilyin]

> How does one take into account the finite size effects in the case of...
A Keller-Lieb-Thirring inequality on the sphere

Let $d \geq 1$, $p \in \left[ \max\{1, d/2\}, +\infty \right)$ and

$$\mu_* := \frac{d}{2} (p - 1)$$

**Theorem (Dolbeault-Esteban-Laptev)**

There exists a convex increasing function $\alpha$ s.t. $\alpha(\mu) = \mu$ if $\mu \in [0, \mu_*]$ and $\alpha(\mu) > \mu$ if $\mu \in (\mu_*, +\infty)$ and, for any $p < d/2$,

$$|\lambda_1(-\Delta - V)| \leq \alpha\left(\|V\|_{L^p(\mathbb{R}^d)}\right) \quad \forall V \in L^p(S^d)$$

This estimate is optimal

For large values of $\mu$, we have

$$\alpha(\mu)^{p - \frac{d}{2}} = L_{p - \frac{d}{2}, d}^1 \left(k_{q, d} \mu\right)^p (1 + o(1))$$

If $p = d/2$ and $d \geq 3$, the inequality holds with $\alpha(\mu) = \mu$ iff $\mu \in [0, \mu_*]$
Let $d \geq 1$, $\gamma = p - d/2$

**Corollary (Dolbeault-Esteban-Laptev)**

$$|\lambda_1(-\Delta - V)|^\gamma \lesssim L^1_{\gamma,d} \int_{S^d} V^{\gamma + \frac{d}{2}} \quad \text{as} \quad \mu = \|V\|_{L^{\gamma + \frac{d}{2}}(\mathbb{R}^d)} \to \infty$$

if either $\gamma > \max\{0, 1 - d/2\}$ or $\gamma = 1/2$ and $d = 1$

However, if $\mu = \|V\|_{L^{\gamma + \frac{d}{2}}(\mathbb{R}^d)} \leq \mu_*$, then we have

$$|\lambda_1(-\Delta - V)|^{\gamma + \frac{d}{2}} \leq \int_{S^d} V^{\gamma + \frac{d}{2}}$$

for any $\gamma \geq \max\{0, 1 - d/2\}$ and this estimate is optimal

$L^1_{\gamma,d}$ is the optimal constant in the Euclidean one bound state ineq.

$$|\lambda_1(-\Delta - \phi)|^\gamma \leq L^1_{\gamma,d} \int_{\mathbb{R}^d} \phi^{\gamma + \frac{d}{2}} \, dx$$
Consider the Schrödinger operator $-\Delta - V$ and the energy

$$E[u] := \int_{S^d} |\nabla u|^2 - \int_{S^d} V |u|^2 \geq \int_{S^d} |\nabla u|^2 - \mu \|u\|^2_{L^q(\mathbb{R}^d)} \geq -\alpha(\mu) \|u\|^2_{L^2(\mathbb{R}^d)} \text{ if } \mu = \|V_+\|_{L^p(\mathbb{R}^d)}$$

\[\Rightarrow \text{Is it true that}\]

$$\|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \alpha \|u\|^2_{L^2(\mathbb{R}^d)} \geq \mu(\alpha) \|u\|^2_{L^q(\mathbb{R}^d)} \quad ?$$

In other words, what are the properties of the minimum of

$$Q_\alpha[u] := \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \alpha \|u\|^2_{L^2(\mathbb{R}^d)}}{\|u\|^2_{L^q(\mathbb{R}^d)}}$$

An important convention (for the numerical value of the constants): we consider the uniform probability measure on the unit sphere $S^d$. 
\( \mu_{\text{asymp}}(\alpha) := \frac{K_{q,d}}{\kappa_{q,d}} \alpha^{1-q}, \vartheta := d \frac{q-2}{2q} \) corresponds to the *semi-classical regime* and \( K_{q,d} \) is the optimal constant in the *Euclidean Gagliardo-Nirenberg-Sobolev inequality*

\[
K_{q,d} \| \nabla v \|_{L^q(\mathbb{R}^d)}^2 \leq \| \nabla v \|_{L^2(\mathbb{R}^d)}^2 + \| v \|_{L^2(\mathbb{R}^d)}^2 \quad \forall v \in H^1(\mathbb{R}^d)
\]

Let \( \varphi \) be a non-trivial eigenfunction of the Laplace-Beltrami operator corresponding the first nonzero eigenvalue

\[
-\Delta \varphi = d \varphi
\]

Consider \( u = 1 + \varepsilon \varphi \) as \( \varepsilon \to 0 \) Taylor expand \( Q_\alpha \) around \( u = 1 \)

\[
\mu(\alpha) \leq Q_\alpha [1 + \varepsilon \varphi] = \alpha + [d + \alpha (2 - q)] \varepsilon^2 \int_{S^d} |\varphi|^2 \, d\nu_g + o(\varepsilon^2)
\]

By taking \( \varepsilon \) small enough, we get \( \mu(\alpha) < \alpha \) for all \( \alpha > d/(q - 2) \)

Optimizing on the value of \( \varepsilon > 0 \) (not necessarily small) provides an interesting test function...
Another inequality

Let \( d \geq 1 \) and \( \gamma > d/2 \) and assume that \( L_{-\gamma,d}^1 \) is the optimal constant in

\[
\lambda_1(-\Delta + \phi)^{-\gamma} \leq L_{-\gamma,d}^1 \int_{\mathbb{R}^d} \phi^\frac{d}{2} - \gamma \, dx
\]

\[
q = 2 \frac{2 \gamma - d}{2 \gamma - d + 2} \quad \text{and} \quad p = \frac{q}{2-q} = \gamma - \frac{d}{2}
\]

**Theorem (Dolbeault-Esteban-Laptev)**

\[
\left( \lambda_1(-\Delta + W) \right)^{-\gamma} \lesssim L_{-\gamma,d}^1 \int_{S^d} W^\frac{d}{2} - \gamma \quad \text{as} \quad \beta = \| W^{-1} \|_{L_\gamma^{-\frac{d}{2}}(\mathbb{R}^d)}^{-1} \rightarrow \infty
\]

However, if \( \gamma \geq \frac{d}{2} + 1 \) and \( \beta = \| W^{-1} \|_{L_\gamma^{-\frac{d}{2}}(\mathbb{R}^d)}^{-1} \leq \frac{1}{4} d (2 \gamma - d + 2) \)

\[
\left( \lambda_1(-\Delta + W) \right)^\frac{d}{2} - \gamma \leq \int_{S^d} W^\frac{d}{2} - \gamma
\]

and this estimate is optimal
$K_{q,d}^*$ is the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality

$$K_{q,d}^* \|v\|_{L^2(\mathbb{R}^d)}^2 \leq \|\nabla v\|_{L^2(\mathbb{R}^d)}^2 + \|v\|_{L^q(\mathbb{R}^d)}^2 \quad \forall \, v \in H^1(\mathbb{R}^d)$$

and $\mathcal{L}_{-\gamma,d}^{1} := \left( K_{q,d}^* \right)^{-\gamma}$ with $q = 2 \frac{2\gamma-d}{2\gamma-d+2}$, $\delta := \frac{2q}{2d-q(d-2)}$.

**Lemma (Dolbeault-Esteban-Laptev)**

Let $q \in (0, 2)$ and $d \geq 1$. There exists a concave increasing function $\nu$

$$\nu(\beta) \leq \beta \quad \forall \, \beta > 0 \quad \text{and} \quad \nu(\beta) < \beta \quad \forall \, \beta \in \left( \frac{d}{2-q}, +\infty \right)$$

$$\nu(\beta) = \beta \quad \forall \, \beta \in \left[ 0, \frac{d}{2-q} \right] \quad \text{if} \quad q \in [1, 2)$$

$$\nu(\beta) = K_{q,d}^* \left( \kappa_{q,d} \beta \right)^\delta \left( 1 + o(1) \right) \quad \text{as} \quad \beta \to +\infty$$

such that

$$\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \beta \|u\|_{L^q(\mathbb{R}^d)}^2 \geq \nu(\beta) \|u\|_{L^2(\mathbb{R}^d)}^2 \quad \forall \, u \in H^1(\mathbb{S}^d)$$
The threshold case: $q = 2$

Lemma (Dolbeault-Esteban-Laptev)

Let $p > \max\{1, d/2\}$. There exists a concave nondecreasing function $\xi$

$$\xi(\alpha) = \alpha \quad \forall \alpha \in (0, \alpha_0) \quad \text{and} \quad \xi(\alpha) < \alpha \quad \forall \alpha > \alpha_0$$

for some $\alpha_0 \in \left[\frac{d}{2} (p - 1), \frac{d}{2} p\right]$, and $\xi(\alpha) \sim \alpha^{1 - \frac{d}{2p}}$ as $\alpha \to +\infty$

such that, for any $u \in H^1(S^d)$ with $\|u\|_{L^2(\mathbb{R}^d)} = 1$

$$\int_{S^d} |u|^2 \log |u|^2 \, dv_g + p \log \left(\frac{\xi(\alpha)}{\alpha}\right) \leq p \log \left(1 + \frac{1}{\alpha} \|\nabla u\|_{L^2(\mathbb{R}^d)}^2\right)$$

Corollary (Dolbeault-Esteban-Laptev)

$$e^{-\frac{\lambda_1(-\Delta - W)}{\alpha}} \leq \frac{\alpha}{\xi(\alpha)} \left(\int_{S^d} e^{-p \frac{W}{\alpha}} \, dv_g\right)^{1/p}$$
Spectral estimates on compact Riemannian manifolds

Joint work with M.J. Esteban, A. Laptev, and M. Loss

The same kind of results as for the sphere. However, estimates are not, in general, sharp.
Let us define
\[ \kappa := \text{vol}_g(\mathcal{M})^{1-2/q} \]

**Proposition**

Assume that \( q \in (2, 2^*) \) if \( d \geq 3 \), or \( q \in (2, \infty) \) if \( d = 1 \) or 2. There exists a concave increasing function \( \mu : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \mu(\alpha) = \kappa \alpha \) for any \( \alpha \leq \frac{\Lambda}{q-2} \), \( \mu(\alpha) < \kappa \alpha \) for \( \alpha > \frac{\Lambda}{q-2} \) and

\[ \| \nabla u \|_{L^2(\mathcal{M})}^2 + \alpha \| u \|_{L^2(\mathcal{M})}^2 \geq \mu(\alpha) \| u \|_{L^q(\mathcal{M})}^2 \quad \forall u \in H^1(\mathcal{M}) \]

The asymptotic behaviour of \( \mu \) is given by \( \mu(\alpha) \sim K_{q,d} \alpha^{1-\vartheta} \) as \( \alpha \to +\infty \), with \( \vartheta = d \frac{q-2}{2q} \) and \( K_{q,d} \) defined by

\[ K_{q,d} := \inf_{v \in H^1(\mathbb{R}^d) \setminus \{0\}} \frac{\| \nabla v \|_{L^2(\mathbb{R}^d)}^2 + \| v \|_{L^2(\mathbb{R}^d)}^2}{\| v \|_{L^q(\mathbb{R}^d)}^2} \]
Manifolds: the first Keller-Lieb-Thirring estimate

We consider $\|V\|_{L^p(M)} = \mu \mapsto \alpha(\mu)$

\[
\int_M |\nabla u|^2 \, d\nu_g - \int_M V |u|^2 \, d\nu_g + \alpha(\mu) \int_M |u|^2 \, d\nu_g
\geq \|\nabla u\|_{L^2(M)}^2 - \mu \|u\|_{L^q(M)}^2 + \alpha(\mu) \|u\|_{L^2(M)}^2
\]

$p$ and $\frac{q}{2}$ are Hölder conjugate exponents.

**Theorem**

Let $d \geq 1$, $p \in (1, +\infty)$ if $d = 1$ and $p \in \left(\frac{d}{2}, +\infty\right)$ if $d \geq 2$ and assume that $\Lambda_* > 0$. With the above notations and definitions, for any nonnegative $V \in L^p(M)$, we have

\[
|\lambda_1(-\Delta_g - V)| \leq \alpha(\|V\|_{L^p(M)})
\]

Moreover, we have $\alpha(\mu)^{p - \frac{d}{2}} = L_{1,\gamma,d}^1 \mu^p \left(1 + o(1)\right)$ as $\mu \to +\infty$ with

$L_{1,\gamma,d}^1 := (K_{q,d})^{-p}$, $\gamma = p - \frac{d}{2}$
Theorem

Let $d \geq 1$, $p \in (0, +\infty)$. There exists an increasing concave function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\nu(\beta) = \beta/\kappa$, for any $\beta \in (0, \frac{p+1}{2} \kappa \Lambda)$ if $p > 1$, such that for any positive potential $W$ we have

$$\lambda_1(-\Delta + W) \geq \nu(\beta) \quad \text{with} \quad \beta = \left( \int_{\mathbb{M}} W^{-p} \, d\nu_g \right)^{1/p}$$

Moreover, for large values of $\beta$, we have

$$\nu(\beta)^{-(p+d)} = L^1_{-(p+d),d} \beta^{-p} (1 + o(1)) \quad \text{as} \quad \beta \to +\infty$$
Spectral estimates on the cylinder

Joint work with M.J. Esteban and M. Loss
Spectral estimates and the symmetry breaking problem on the cylinder

Let \((\mathcal{M}, g)\) be a smooth compact connected Riemannian manifold of dimension \(d - 1\) (no boundary) with \(\text{vol}_g(\mathcal{M}) = 1\), and let

\[
\mathcal{C} := \mathbb{R} \times \mathcal{M} \ni x = (s, z)
\]

be the cylinder. \(\lambda_1^\mathcal{M}\) is the lowest positive eigenvalue of the Laplace-Beltrami operator, \(\kappa := \inf_{\mathcal{M}} \inf_{\xi \in \mathbb{S}^{d-2}} \text{Ric}(\xi, \xi)\)

\[\triangleright \text{Is} \]
\[
\Lambda(\mu) := \sup \{ \lambda_1^\mathcal{C}[V] : V \in L^q(\mathcal{C}), \|V\|_{L^q(\mathcal{C})} = \mu \}
\]

equal to

\[
\Lambda_*(\mu) := \sup \{ \lambda_1^\mathbb{R}[V] : V \in L^q(\mathbb{R}), \|V\|_{L^q(\mathbb{R})} = \mu \}
\]

\(-\lambda_1^\mathcal{C}[V]\) is the lowest eigenvalue of \(-\partial_s^2 - \Delta_g - V\) and \(-\partial_s^2 - V\) on \(\mathcal{C}\)
The Keller-Lieb-Thirring inequality on the line

Assume that \( q \in (1, +\infty) \), \( \beta = \frac{2q}{2q-1} \), \( \mu_1 := q (q - 1) \left( \frac{\sqrt{\pi} \Gamma(q)}{\Gamma(q+1/2)} \right)^{1/q} \).

\[ \Lambda_\star(\mu) = (q - 1)^2 \left( \frac{\mu}{\mu_1} \right)^\beta \quad \forall \mu > 0, \]

If \( V \) is a nonnegative real valued potential in \( L^q(\mathbb{R}) \), then we have

\[ \lambda_1^\mathbb{R}[V] \leq \Lambda_\star(\|V\|_{L^q(\mathbb{R})}) \quad \text{where} \quad \Lambda_\star(\mu) = (q - 1)^2 \left( \frac{\mu}{\mu_1} \right)^\beta \quad \forall \mu > 0 \]

and equality holds if and only if, up to scalings, translations and multiplications by a positive constant,

\[ V(s) = \frac{q (q - 1)}{(\cosh s)^2} =: V_1(s) \quad \forall s \in \mathbb{R} \]

where \( \|V_1\|_{L^q(\mathbb{R})} = \mu_1 \), \( \lambda_1^\mathbb{R}[V_1] = (q - 1)^2 \) and \( \varphi(s) = (\cosh s)^{1-q} \).
\[ \lambda_\theta := \left(1 + \delta \theta \frac{d-1}{d-2}\right) \kappa + \delta (1 - \theta) \lambda_1^m \] with \[ \delta = \frac{n-d}{(d-1)(n-1)} \]

\[ \lambda_* := \lambda_{\theta_*} \quad \text{where} \quad \theta_* := \frac{(d-2)(n-1) \left(3n+1-d(3n+5)\right)}{(d+1) \left(d(n^2-n-4)-n^2+3n+2\right)} \]

**Theorem**

Let \( d \geq 2 \) and \( q \in \left(\min\{4, d/2\}, +\infty\right) \). The function \( \mu \mapsto \Lambda(\mu) \) is convex, positive and such that

\[ \Lambda(\mu)^{q-d/2} \sim L^1_{q-d/2, d} \mu^q \quad \text{as} \quad \mu \to +\infty \]

Moreover, there exists a positive \( \mu_* \) with

\[ \frac{\lambda_*}{2(q-1)} \mu_1^\beta \leq \mu_*^\beta \leq \frac{\lambda_1^m}{2q-1} \mu_1^\beta \]

such that

\[ \Lambda(\mu) = \Lambda_*(\mu) \quad \forall \mu \in (0, \mu_*) \quad \text{and} \quad \Lambda(\mu) > \Lambda_*(\mu) \quad \forall \mu > \mu_* \]

As a special case, if \( \mathcal{M} = \mathbb{S}^{d-1} \), inequalities are in fact equalities
The upper estimate

**Lemma**

If \( \Lambda_* (\mu) > \frac{4 \lambda_1^m}{p^2 - 4} \), then

\[
\sup \{ \lambda_1^c [V] : V \in L^q(C), \|V\|_{L^q(C)} = \mu \} > \Lambda_* (\mu)
\]

\[
\phi_\varepsilon (s,z) := \varphi_\mu (s) + \varepsilon \left( \varphi_\mu (s) \right)^{p/2} \psi_1 (z) \quad \text{and} \quad V_\varepsilon (s,z) := \mu \frac{|\phi_\varepsilon (s,z)|^{p-2}}{\|\phi_\varepsilon \|_{L^p(C)}^{p-2}}
\]

where \( \psi_1 \) is an eigenfunction of \( \lambda_1^m \) and \( \varphi_\mu \) is optimal for \( \Lambda_* (\mu) \)

\[
- \lambda_1^c [V_\varepsilon] + \Lambda_* (\mu) \leq \frac{4 \varepsilon^2}{p + 2} \left( \lambda_1^m - \frac{1}{4} (p^2 - 4) \Lambda_* (\mu) \right) + o(\varepsilon^2)
\]
The lower estimate

\[ J[V] := \frac{\| V \|^q_{L^q(C)} - \| \partial_s V^{(q-1)/2} \|^2_{L^2(C)} - \| \nabla_g V^{(q-1)/2} \|^2_{L^2(C)}}{\| V^{(q-1)/2} \|^2_{L^2(C)}} \]

Lemma

\[ \Lambda(\mu) = \sup \{ J[V] : \| V \|_{L^q(C)} = \mu \} \]

With \( \alpha = \frac{1}{q-1} \sqrt{\Lambda_*(\mu)} \), let us consider the operator \( \mathcal{L} \) such that

\[ \mathcal{L} u^m := -\frac{m}{m-1} \partial_s \left( u e^{-2\alpha s} \partial_s \left( u^{m-1} e^{\alpha s} \right) \right) + e^{-\alpha s} \Delta_g u^m \]

where \( m = 1 - \frac{1}{n}, \ n = 2q \). To any potential \( V \geq 0 \) we associate the pressure function

\[ p_V(r) := r V(s)^{-\frac{q-1}{4q}} \quad \forall r = e^{-\alpha s} \]
\[
K[p] := \frac{n-1}{n} \alpha^4 \int_{\mathbb{R}^d} \left| p'' - \frac{p'}{r} - \frac{\Delta_g p}{\alpha^2 (n-1) r^2} \right|^2 p^{1-n} \, d\mu \\
+ 2 \alpha^2 \int_{\mathbb{R}^d} \frac{1}{r^2} \left| \nabla_g p' - \frac{\nabla_g p}{r} \right|^2 p^{1-n} \, d\mu \\
+ \left( \lambda_* - \frac{2}{q-1} \Lambda_*(\mu) \right) \int_{\mathbb{R}^d} \frac{\left| \nabla_g p \right|^2}{r^4} p^{1-n} \, d\mu
\]

where \(d\mu\) is the measure on \(\mathbb{R}^+ \times \mathcal{M}\) with density \(r^{n-1}\), and \('\) denotes the derivative with respect to \(r\).

**Lemma**

There exists a positive constant \(c\) such that, if \(V\) is a critical point of \(J\) under the constraint \(\|V\|_{L^q(C)} = \mu\) and \(u_V = V^{(q-1)/2}\), then we have

\[
J[V + \varepsilon u_V^{-1} \ell u_V^m] - J[V] \geq c \varepsilon K[p_V] + o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0
\]
These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
▷ Lectures

Thank you for your attention!