Champs magnétiques, interpolation et symétrie

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Outline

- **Without magnetic fields: symmetry and symmetry breaking in interpolation inequalities**
  - Gagliardo-Nirenberg-Sobolev inequalities on the sphere
  - Keller-Lieb-Thirring inequalities on the sphere
  - Caffarelli-Kohn-Nirenberg inequalities

- **With magnetic fields in dimensions 2 and 3**
  - Interpolation inequalities and spectral estimates
  - Estimates, numerics; an open question on constant magnetic fields

- **Magnetic rings: the case of $S^1$**
  - A one-dimensional magnetic interpolation inequality
  - Consequences: Keller-Lieb-Thirring estimates, Aharonov-Bohm magnetic fields and a new Hardy inequality in $\mathbb{R}^2$

- **Aharonov-Bohm magnetic fields in $\mathbb{R}^2$**
  - Aharonov-Bohm effect
  - Interpolation and Keller-Lieb-Thirring inequalities in $\mathbb{R}^2$
  - Aharonov-Symmetry and symmetry breaking
A joint research program (mostly) with...

M.J. Esteban, Ceremade, Université Paris-Dauphine
▷ symmetry, interpolation, Keller-Lieb-Thirring, magnetic fields

M. Loss, Georgia Institute of Technology (Atlanta)
▷ symmetry, interpolation, Keller-Lieb-Thirring, magnetic fields

A. Laptev, Imperial College London
▷ Keller-Lieb-Thirring, magnetic fields

D. Bonheure, Université Libre de Bruxelles
▷ Aharonov-Bohm magnetic fields
Symmetry and symmetry breaking in interpolation inequalities without magnetic field

- Gagliardo-Nirenberg-Sobolev inequalities on the sphere
- Keller-Lieb-Thirring inequalities on the sphere
- Caffarelli-Kohn-Nirenberg inequalities on $\mathbb{R}^2$
A result of uniqueness on a classical example

On the sphere $\mathbb{S}^d$, let us consider the positive solutions of

$$-\Delta u + \lambda u = u^{p-1}$$

$p \in [1, 2) \cup (2, 2^*)$ if $d \geq 3$, $2^* = \frac{2d}{d-2}$

$p \in [1, 2) \cup (2, +\infty)$ if $d = 1, 2$

**Theorem**

*If $\lambda \leq d$, $u \equiv \lambda^{1/(p-2)}$ is the unique solution*

Bifurcation point of view and symmetry breaking

Figure: $(p - 2)\lambda \mapsto (p - 2)\mu(\lambda)$ with $d = 3$

\[ \| \nabla u \|_{L^2(S^d)}^2 + \lambda \| u \|_{L^2(S^d)}^2 \geq \mu(\lambda) \| u \|_{L^p(S^d)}^2 \]

Taylor expansion of $u = 1 + \varepsilon \varphi_1$ as $\varepsilon \to 0$ with $-\Delta \varphi_1 = d \varphi_1$

$\mu(\lambda) < \lambda$ if and only if $\lambda > \frac{d}{p - 2}$

$\triangleright$ The inequality holds with $\mu(\lambda) = \lambda = \frac{d}{p - 2}$ [Bakry & Emery, 1985]
The Bakry-Emery method on the sphere

**Entropy functional**

\[
\mathcal{E}_p[\rho] := \frac{1}{p-2} \left[ \int_{S^d} \rho^{\frac{2}{p}} \, d\mu - \left( \int_{S^d} \rho \, d\mu \right)^{\frac{2}{p}} \right] \quad \text{if} \quad p \neq 2
\]

\[
\mathcal{E}_2[\rho] := \int_{S^d} \rho \log \left( \frac{\rho}{\|\rho\|_{L^1(S^d)}} \right) \, d\mu
\]

**Fisher information functional**

\[
\mathcal{I}_p[\rho] := \int_{S^d} |\nabla \rho|^{\frac{2}{p}} \, d\mu
\]

[Bakry & Emery, 1985] *carré du champ* method: use the heat flow

\[
\frac{\partial \rho}{\partial t} = \Delta \rho
\]

and observe that \( \frac{d}{dt} \mathcal{E}_p[\rho] = -\mathcal{I}_p[\rho] \)

\[
\frac{d}{dt} \left( \mathcal{I}_p[\rho] - d \mathcal{E}_p[\rho] \right) \leq 0 \quad \implies \quad \mathcal{I}_p[\rho] \geq d \mathcal{E}_p[\rho]
\]

with \( \rho = |u|^p \), if \( p \leq 2^\# := \frac{2d^2 + 1}{(d-1)^2} \)
The evolution under the fast diffusion flow

To overcome the limitation \( p \leq 2\# \), one can consider a nonlinear diffusion of fast diffusion / porous medium type

\[
\frac{\partial \rho}{\partial t} = \Delta \rho^m
\]

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any \( p \in [1, 2^*] \)

\[
\mathcal{K}_p[\rho] := \frac{d}{dt} \left( I_p[\rho] - d \mathcal{E}_p[\rho] \right) \leq 0
\]

\((p, m)\) admissible region, \( d = 5 \)

Keller-Lieb-Thirring inequalities on the sphere

The Keller-Lieb-Thirring inequality is equivalent to an interpolation inequality of Gagliardo-Nirenberg-Sobolev type.

We measure a quantitative deviation with respect to the semi-classical regime due to finite size effects.

Joint work with M.J. Esteban and A. Laptev
An introduction to (Keller)-Lieb-Thirring inequalities in $\mathbb{R}^d$

$(\lambda_k)_{k \geq 1}$: eigenvalues of the Schrödinger operator $H = -\Delta - V$ on $\mathbb{R}^d$

- Euclidean case [Keller, 1961]
  \[ |\lambda_1|^\gamma \leq L_{\gamma,d}^1 \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}} \]

  [Lieb-Thirring, 1976]
  \[ \sum_{k \geq 1} |\lambda_k|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}} \]

  $\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$ and $\gamma \geq 0$ if $d \geq 3$ [Weidl], [Cwikel], [Rosenbljum], [Aizenman], [Laptev-Weidl], [Helffer], [Robert], [JD-Felmer-Loss-Paturel], [JD-Laptev-Loss]...[Frank, Hundertmark, Jex, Nam]

- Compact manifolds: log Sobolev case: [Federbusch], [Rothaus]; case $\gamma = 0$ (Rozenbljum-Lieb-Cwikel inequality): [Levin-Solomyak]; [Lieb], [Levin], [Ouabaz-Poupaud]... [Ilyin]

How does one take into account the finite size effects on $S^d$?
Hölder duality and link with interpolation inequalities

Let $p = \frac{q}{q-2}$. Consider the Schrödinger energy

$$\int_{\mathbb{S}^d} |\nabla u|^2 - \int_{\mathbb{S}^d} V |u|^2 \geq \int_{\mathbb{S}^d} |\nabla u|^2 - \mu \|u\|^2_{L^q(\mathbb{S}^d)}$$

$$\geq -\lambda(\mu) \|u\|^2_{L^2(\mathbb{S}^d)} \quad \text{if} \quad \mu = \|V_+\|_{L^p(\mathbb{S}^d)}$$

We deduce from $\|\nabla u\|^2_{L^2(\mathbb{S}^d)} + \lambda \|u\|^2_{L^2(\mathbb{S}^d)} \geq \mu(\lambda) \|u\|^2_{L^q(\mathbb{S}^d)}$ that

$$\|\nabla u\|^2_{L^2(\mathbb{S}^d)} - \mu(\lambda) \|u\|^2_{L^q(\mathbb{S}^d)} \geq -\lambda \|u\|^2_{L^2(\mathbb{S}^d)}$$

$\mu(\lambda)$

$\lambda(\mu)$

$q = d = 3$
A Keller-Lieb-Thirring inequality on the sphere

Let \( d \geq 1 \), \( p \in [\max\{1, d/2\}, +\infty) \) and \( \mu_* := \frac{d}{2} (p - 1) \)

**Theorem (JD-Esteban-Laptev)**

There exists a convex increasing function \( \lambda \) s.t. \( \lambda(\mu) = \mu \) if \( \mu \in [0, \mu_*] \) and \( \lambda(\mu) > \mu \) if \( \mu \in (\mu_*, +\infty) \) and, for any \( p < d/2 \),

\[
|\lambda_1(-\Delta - V)| \leq \lambda(\|V\|_{L^p(S^d)}) \quad \forall V \in L^p(S^d)
\]

This estimate is optimal

For large values of \( \mu \), we have

\[
\lambda(\mu)^{p-\frac{d}{2}} = L_{p-\frac{d}{2},d}^1 (\kappa_{q,d} \mu)^p (1 + o(1))
\]

If \( p = d/2 \) and \( d \geq 3 \), the inequality holds with \( \lambda(\mu) = \mu \) iff \( \mu \in [0, \mu_*] \)
A Keller-Lieb-Thirring inequality: second formulation

Let $d \geq 1$, $\gamma = p - d/2$

**Corollary (JD-Esteban-Laptev)**

$$|\lambda_1(-\Delta - V)|^\gamma \lesssim L_{\gamma,d}^1 \int_{\mathbb{S}^d} V^{\gamma + \frac{d}{2}} \quad \text{as} \quad \mu = \|V\|_{L^{\gamma + \frac{d}{2}}(\mathbb{S}^d)} \to \infty$$

if either $\gamma > \max\{0, 1 - d/2\}$ or $\gamma = 1/2$ and $d = 1$

However, if $\mu = \|V\|_{L^{\gamma + \frac{d}{2}}(\mathbb{S}^d)} \leq \mu^*$, then we have

$$|\lambda_1(-\Delta - V)|^{\gamma + \frac{d}{2}} \leq \int_{\mathbb{S}^d} V^{\gamma + \frac{d}{2}}$$

for any $\gamma \geq \max\{0, 1 - d/2\}$ and this estimate is optimal

$L_{\gamma,d}^1$ is the optimal constant in the Euclidean one bound state ineq.

$$|\lambda_1(-\Delta - \phi)|^\gamma \leq L_{\gamma,d}^1 \int_{\mathbb{R}^d} \phi^{\gamma + \frac{d}{2}} \, dx$$
References


Caffarelli-Kohn-Nirenberg, symmetry and symmetry breaking results, and weighted nonlinear flows

Joint work with M.J. Esteban and M. Loss
Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ \nu \in L^p \left( \mathbb{R}^d, |x|^{-b} \, dx \right) : |x|^{-a} |\nabla \nu| \in L^2 \left( \mathbb{R}^d, dx \right) \right\}$

$$
\left( \int_{\mathbb{R}^d} \frac{v^p}{|x|^b} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} \, dx \quad \forall \, v \in \mathcal{D}_{a,b}
$$

holds under conditions on $a$ and $b$

$$
p = \frac{2 \, d}{d - 2 + 2(b - a)} \quad \text{(critical case)}
$$

An optimal function among radial functions:

$$
\nu_*(x) = \left( 1 + |x|^{(p-2)(a_c-a)} \right)^{-\frac{2}{p-2}} \quad \text{and} \quad C_{a,b}^* = \frac{\| |x|^{-b} \nu_* \|^2_p}{\| |x|^{-a} \nabla \nu_* \|^2_2}
$$

Question: $C_{a,b} = C_{a,b}^*$ (symmetry) or $C_{a,b} > C_{a,b}^*$ (symmetry breaking)?
Critical CKN: range of the parameters

Figure: $d = 3$

$$\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^b} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|
abla v|^2}{|x|^{2a}} \, dx$$

\[ p = \frac{2d}{d - 2 + 2(b - a)} \]

\[ a < a_c := \frac{(d - 2)}{2} \]

\[ a \leq b \leq a + 1 \text{ if } d \geq 3, \]

\[ a + 1/2 < b \leq a + 1 \text{ if } d = 1 \]

\[ a < b \leq a + 1 < 1, \]

\[ n = \frac{2}{(b - a)} \text{ if } d = 2 \]

[Il’in (1961)]

[Glaser, Martin, Grosse, Thirring (1976)]

[Caffarelli, Kohn, Nirenberg (1984)]

[F. Catrina, Z.-Q. Wang (2001)]
Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

\[ b_{FS}(a) := \frac{d (a_c - a)}{2 \sqrt{(a_c - a)^2 + d - 1}} + a - a_c \]

The functional

\[ C_{a,b}^* \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} \, dx - \left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b_p}} \, dx \right)^{2/p} \]

is linearly instable at \( v = v^* \).
Symmetry versus symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (2016)]

**Theorem**

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{FS}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric.
The symmetry proof in one slide

- A change of variables: $v(|x|^{\alpha-1} x) = w(x)$, $D_\alpha v = (\alpha \frac{\partial v}{\partial s}, \frac{1}{s} \nabla_\omega v)$

$$\|v\|_{L^{2p},d-n(\mathbb{R}^d)} \leq K_{\alpha,n,p} \|D_\alpha v\|_{L^{2,d-n}(\mathbb{R}^d)}^{\frac{\gamma}{2}} \|v\|_{L^{p+1,d-n}(\mathbb{R}^d)}^{1-\frac{\gamma}{2}} \quad \forall v \in H^{p}_{d-n,d-n}(\mathbb{R}^d)$$

- Concavity of the Rényi entropy power: with $L_\alpha = -D_\alpha^* D_\alpha = \alpha^2 (u'' + \frac{n-1}{s} u') + \frac{1}{s^2} \Delta_\omega u$ and $\frac{\partial u}{\partial t} = L_\alpha u^m$

$$- \frac{d}{dt} G[u(t, \cdot)] \left( \int_{\mathbb{R}^d} u^m d\mu \right)^{1-\sigma} \geq (1 - m) (\sigma - 1) \int_{\mathbb{R}^d} u^m \left| L_\alpha P - \frac{\int_{\mathbb{R}^d} u |D_\alpha P|^2 d\mu}{\int_{\mathbb{R}^d} u^m d\mu} \right|^2 d\mu$$

$$+ 2 \int_{\mathbb{R}^d} \left( \alpha^4 \left( 1 - \frac{1}{n} \right) \left| p'' - \frac{p'}{s} - \frac{\Delta_\omega P}{\alpha^2 (n-1) s^2} \right|^2 + \frac{2 \alpha^2}{s^2} \left| \nabla_\omega p' - \frac{\nabla_\omega P}{s} \right|^2 \right) u^m d\mu$$

$$+ 2 \int_{\mathbb{R}^d} \left( (n-2) \left( \alpha^2 F_S - \alpha^2 \right) \left| \nabla_\omega P \right|^2 + c(n, m, d) \frac{\left| \nabla_\omega P \right|^4}{P^2} \right) u^m d\mu$$

- Elliptic regularity and the Emden-Fowler transformation: justifying the integrations by parts
The variational problem on the cylinder

\[ v(r, \omega) = r^{a - a_c} \varphi(s, \omega) \quad \text{with} \quad r = |x|, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r} \]

the variational problem becomes

\[
\Lambda \mapsto \mu(\Lambda) := \min_{\varphi \in H^1(C)} \frac{\|\partial_s \varphi\|_{L^2(C)}^2 + \|\nabla \omega \varphi\|_{L^2(C)}^2 + \Lambda \|\varphi\|_{L^2(C)}^2}{\|\varphi\|_{L^p(C)}^2} \]

is a concave increasing function

Restricted to symmetric functions, the variational problem becomes

\[
\mu_*(\Lambda) := \min_{\varphi \in H^1(\mathbb{R})} \frac{\|\partial_s \varphi\|_{L^2(\mathbb{R}^d)}^2 + \Lambda \|\varphi\|_{L^2(\mathbb{R}^d)}^2}{\|\varphi\|_{L^p(\mathbb{R}^d)}^2} = \mu_*(1) \Lambda^\alpha
\]

Symmetry means \( \mu(\Lambda) = \mu_*(\Lambda) \)

Symmetry breaking means \( \mu(\Lambda) < \mu_*(\Lambda) \)
Numerical results

Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point $\Lambda_1$ computed by V. Felli and M. Schneider. The branch behaves for large values of $\Lambda$ as shown by F. Catrina and Z.-Q. Wang.
Three references

- Lecture notes on *Symmetry and nonlinear diffusion flows... a course on entropy methods* (see webpage)


Magnetic interpolation inequalities in the Euclidean space

- Three interpolation inequalities and their dual forms
- Estimates in dimension $d = 2$ for constant magnetic fields
  - Lower estimates
  - Upper estimates and numerical results
  - A linear stability result (numerical) and an open question

- Warning: assumptions are not repeated
- Estimates are given only in the case $p > 2$ but similar estimates hold in the other cases

Joint work with M.J. Esteban, A. Laptev and M. Loss
Magnetic Laplacian and spectral gap

In dimensions $d = 2$ and $d = 3$: the magnetic Laplacian is

$$-\Delta_A \psi = -\Delta \psi - 2 i A \cdot \nabla \psi + |A|^2 \psi - i (\text{div} A) \psi$$

where the magnetic potential (resp. field) is $A$ (resp. $B = \text{curl} A$) and

$$H^1_A(\mathbb{R}^d) := \{ \psi \in L^2(\mathbb{R}^d) : \nabla_A \psi \in L^2(\mathbb{R}^d) \} \quad \nabla_A := \nabla + i A$$

**Spectral gap inequality**

$$\| \nabla_A \psi \|_{L^2(\mathbb{R}^d)}^2 \geq \Lambda[B] \| \psi \|_{L^2(\mathbb{R}^d)}^2 \quad \forall \psi \in H^1_A(\mathbb{R}^d)$$

- $\Lambda$ depends only on $B = \text{curl} A$
- Assumption: *equality holds for some $\psi \in H^1_A(\mathbb{R}^d)$*
- If $B$ is a constant magnetic field, $\Lambda[B] = |B|$
- If $d = 2$, $\text{spec}(-\Delta_A) = \{(2j + 1) |B| : j \in \mathbb{N} \}$ is generated by the Landau levels. The *Lowest Landau Level* corresponds to $j = 0$
Magnetic interpolation inequalities

\[ \| \nabla_A \psi \|_{L^2(\mathbb{R}^d)}^2 + \alpha \| \psi \|_{L^2(\mathbb{R}^d)}^2 \geq \mu_B(\alpha) \| \psi \|_{L^p(\mathbb{R}^d)}^2 \quad \forall \psi \in H^1_A(\mathbb{R}^d) \]

for any \( \alpha \in (-\Lambda[B], +\infty) \) and any \( p \in (2, 2^*) \),

\[ \| \nabla_A \psi \|_{L^2(\mathbb{R}^d)}^2 + \beta \| \psi \|_{L^p(\mathbb{R}^d)}^2 \geq \nu_B(\beta) \| \psi \|_{L^2(\mathbb{R}^d)}^2 \quad \forall \psi \in H^1_A(\mathbb{R}^d) \]

for any \( \beta \in (0, +\infty) \) and any \( p \in (1, 2) \)

\[ \| \nabla_A \psi \|_{L^2(\mathbb{R}^d)}^2 \geq \gamma \int_{\mathbb{R}^d} |\psi|^2 \log \left( \frac{|\psi|^2}{\| \psi \|_{L^2(\mathbb{R}^d)}^2} \right) \, dx + \xi_B(\gamma) \| \psi \|_{L^2(\mathbb{R}^d)}^2 \]

(limit case corresponding to \( p = 2 \)) for any \( \gamma \in (0, +\infty) \)

\[ C_p := \begin{cases} \min_{u \in H^1(\mathbb{R}^d) \setminus \{0\}} \frac{\| \nabla u \|_{L^2(\mathbb{R}^d)}^2 + \| u \|_{L^2(\mathbb{R}^d)}^2}{\| u \|_{L^p(\mathbb{R}^d)}^2} & \text{if } p \in (2, 2^*) \\ \min_{u \in H^1(\mathbb{R}^d) \setminus \{0\}} \frac{\| \nabla u \|_{L^2(\mathbb{R}^d)}^2 + \| u \|_{L^2(\mathbb{R}^d)}^2}{\| u \|_{L^p(\mathbb{R}^d)}^2} & \text{if } p \in (1, 2) \end{cases} \]

\( \mu_0(1) = C_p \) if \( p \in (2, 2^*) \), \( \nu_0(1) = C_p \) if \( p \in (1, 2) \)

\( \xi_0(\gamma) = \gamma \log \left( \frac{\pi e^2}{\gamma} \right) \) if \( p = 2 \)
Technical assumptions

\[ A \in L^\alpha_{\text{loc}}(\mathbb{R}^d), \alpha > 2 \text{ if } d = 2 \text{ or } \alpha = 3 \text{ if } d = 3 \text{ and } \]

\[ \lim_{\sigma \to +\infty} \sigma^{d-2} \int_{\mathbb{R}^d} |A(x)|^2 e^{-\sigma |x|} \, dx = 0 \quad \text{if } p \in (2, 2^*) \]

\[ \lim_{\sigma \to +\infty} \frac{\sigma^{d-1}}{\log \sigma} \int_{\mathbb{R}^d} |A(x)|^2 e^{-\sigma |x|^2} \, dx = 0 \quad \text{if } p = 2 \]

\[ \lim_{\sigma \to +\infty} \sigma^{d-2} \int_{|x| < 1/\sigma} |A(x)|^2 \, dx \quad \text{if } p \in (1, 2) \]

These estimates can be found in [Esteban, Lions, 1989]
Theorem

\[ p \in (2, 2^*): \quad \mu_B \text{ is monotone increasing on } (-\Lambda[B], +\infty), \text{ concave and} \]
\[ \lim_{\alpha \to (\Lambda[B])_+} \mu_B(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to +\infty} \mu_B(\alpha) \alpha^{\frac{d-2}{2} - \frac{d}{p}} = C_p \]

\[ p \in (1, 2): \quad \nu_B \text{ is monotone increasing on } (0, +\infty), \text{ concave and} \]
\[ \lim_{\beta \to 0_+} \nu_B(\beta) = \Lambda[B] \quad \text{and} \quad \lim_{\beta \to +\infty} \nu_B(\beta) \beta^{-\frac{2p}{2p+d(2-p)}} = C_p \]

\[ \xi_B \text{ is continuous on } (0, +\infty), \text{ concave, } \xi_B(0) = \Lambda[B] \quad \text{and} \]
\[ \xi_B(\gamma) = \frac{d}{2} \gamma \log \left( \frac{\pi e^2}{\gamma} \right)(1 + o(1)) \quad \text{as} \quad \gamma \to +\infty \]

Constant magnetic fields: equality is achieved
Nonconstant magnetic fields: only partial answers are known
Figure: Case $d = 2$, $p = 3$, $B = 1$: plot of $\alpha \mapsto (2\pi)^{\frac{2}{p}-1} \mu_B(\alpha)$
Figure: Case $d = 2$, $p = 1.4$, $B = 1$: plot of $\beta \mapsto \nu_B(\beta)$.

The horizontal axis is measured in units of $\left(2\pi\right)^{1 - \frac{2}{p}} \beta$. 
Magnetic Keller-Lieb-Thirring inequalities

\( \lambda_{A,V} \) is the principal eigenvalue of \(-\Delta_A + V\)

\( \alpha_B : (0, +\infty) \to (-\Lambda, +\infty) \) is the inverse function of \( \alpha \mapsto \mu_B(\alpha) \)

**Corollary**

(i) For any \( q = p/(p - 2) \in (d/2, +\infty) \) and any potential \( V \in L^q_+(\mathbb{R}^d) \)

\[
\lambda_{A,V} \geq -\alpha_B(\|V\|_{L^q(\mathbb{R}^d)})
\]

\[
\lim_{\mu \to 0^+} \alpha_B(\mu) = \Lambda \text{ and } \lim_{\mu \to +\infty} \alpha_B(\mu) \mu^{\frac{2(q+1)}{d-2-2q}} = -C_p \frac{2(q+1)}{d-2-2q}
\]

(ii) For any \( q = p/(2 - p) \in (1, +\infty) \) and any \( 0 < W^{-1} \in L^q(\mathbb{R}^d) \)

\[
\lambda_{A,W} \geq \nu_B \left( \|W^{-1}\|_{L^q(\mathbb{R}^d)}^{-1} \right)
\]

(iii) For any \( \gamma > 0 \) and any \( W \geq 0 \) s.t. \( e^{-W/\gamma} \in L^1(\mathbb{R}^d) \)

\[
\lambda_{A,W} \geq \xi_B(\gamma) - \gamma \log \left( \int_{\mathbb{R}^d} e^{-W/\gamma} \, dx \right)
\]
Proofs
Interpolation without magnetic field...

Assume that $p > 2$ and let $C_p$ denote the best constant in

$$
\| \nabla u \|_{L^2(\mathbb{R}^d)}^2 + \| u \|_{L^2(\mathbb{R}^d)}^2 \geq C_p \| u \|_{L^p(\mathbb{R}^d)}^2 \quad \forall u \in H^1(\mathbb{R}^d)
$$

By scaling, if we test the inequality by $u(\cdot / \lambda)$, we find that

$$
\| \nabla u \|_{L^2(\mathbb{R}^d)}^2 + \lambda^2 \| u \|_{L^2(\mathbb{R}^d)}^2 \geq C_p \lambda^{2-d(1-\frac{2}{p})} \| u \|_{L^p(\mathbb{R}^d)}^2 \quad \forall u \in H^1(\mathbb{R}^d) \quad \forall \lambda > 0
$$

An optimization on $\lambda > 0$ shows that the best constant in the scale-invariant inequality

$$
\| \nabla u \|_{L^2(\mathbb{R}^d)}^{d(1-\frac{2}{p})} \| u \|_{L^2(\mathbb{R}^d)}^{2-d(1-\frac{2}{p})} \geq S_p \| u \|_{L^p(\mathbb{R}^d)}^2 \quad \forall u \in H^1(\mathbb{R}^d)
$$

is given by

$$
S_p = \frac{1}{2p} \left( 2p - d(p-2) \right)^{1-d} \frac{p-2}{2p} \left( d(p-2) \right)^{\frac{d(p-2)}{2p}} C_p
$$
Proposition

Let \( d = 2 \) or \( 3 \). For any \( p \in (2, +\infty) \), any \( \alpha > -\Lambda = -\Lambda[\mathcal{B}] < 0 \)

\[
\mu_B(\alpha) \geq \mu_{\text{interp}}(\alpha) := \begin{cases} 
S_p (\alpha + \Lambda) \Lambda^{-d} \frac{p-2}{2p} & \text{if } \alpha \in \left[-\Lambda, \frac{\Lambda (2p-d(p-2))}{d(p-2)}\right] \\
C_p \alpha^{1-d} \frac{p-2}{2p} & \text{if } \alpha \geq \frac{\Lambda (2p-d(p-2))}{d(p-2)}
\end{cases}
\]

Diamagnetic inequality: \( \|\nabla|\psi|\|_{L^2(\mathbb{R}^d)} \leq \|\nabla A \psi\|_{L^2(\mathbb{R}^d)} \)

Non-magnetic inequality with \( \lambda = \frac{\alpha + \Lambda}{1-t}, t \in [0, 1] \)

\[
\|\nabla A \psi\|_{L^2(\mathbb{R}^d)}^2 + \alpha \|\psi\|_{L^2(\mathbb{R}^d)}^2 \geq t \left( \|\nabla A \psi\|_{L^2(\mathbb{R}^d)}^2 - \Lambda \|\psi\|_{L^2(\mathbb{R}^d)}^2 \right)
\]

\[
+ (1 - t) \left( \|\nabla |\psi|\|_{L^2(\mathbb{R}^d)}^2 + \frac{\alpha + \Lambda}{1-t} \|\psi\|_{L^2(\mathbb{R}^d)}^2 \right)
\]

\[
\geq C_p (1 - t) \frac{d(p-2)}{2p} (\alpha + t \Lambda)^{1-d} \frac{p-2}{2p} \|\psi\|_{L^p(\mathbb{R}^d)}^2
\]

and optimize on \( t \in [\max\{0, -\alpha/\Lambda\}, 1] \)
The special case of constant magnetic field in dimension $d = 2$
Constant magnetic field, $d = 2$

Assume that $B = (0, B)$ is constant, $d = 2$ and choose

$$A_1 = \frac{B}{2}x_2, \quad A_2 = -\frac{B}{2}x_1 \quad \forall x = (x_1, x_2) \in \mathbb{R}^2$$

**Proposition**

[Loss, Thaller, 1997] Consider a constant magnetic field with field strength $B$ in two dimensions. For every $c \in [0, 1]$, we have

$$\int_{\mathbb{R}^2} |\nabla_A \psi|^2 \, dx \geq (1 - c^2) \int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx + c B \int_{\mathbb{R}^2} \psi^2 \, dx$$

and equality holds with $\psi = u e^{iS}$ and $u > 0$ if and only if

$$(- \partial_2 u^2, \partial_1 u^2) = \frac{2 u^2}{c} (A + \nabla S)$$
... a computation ($d = 2$, constant magnetic field)

\[
\int_{\mathbb{R}^2} |\nabla A\psi|^2 \, dx = \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} |A + \nabla S|^2 u^2 \, dx
\]

\[
= (1 - c^2) \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} \left(c^2 |\nabla u|^2 + |A + \nabla S|^2 u^2 \right) \, dx
\]

\[
\geq \int_{\mathbb{R}^2} 2c |\nabla u| |A + \nabla S| u \, dx
\]

with equality only if $c|\nabla u| = |A + \nabla S| u$

\[
2 |\nabla u| |A + \nabla S| u = |\nabla u|^2 |A + \nabla S| \geq (\nabla u^2)^\perp \cdot (A + \nabla S)
\]

where $(\nabla u^2)^\perp := (-\partial_2 u^2, \partial_1 u^2)$

Equality case: $(-\partial_2 u^2, \partial_1 u^2) = \gamma (A + \nabla S)$ for $\gamma = 2 u^2 / c$

Integration by parts yields

\[
\int_{\mathbb{R}^2} \left(c^2 |\nabla u|^2 + |A + \nabla S|^2 u^2 \right) \, dx \geq B c \int_{\mathbb{R}^2} u^2 \, dx
\]
Proposition

Consider a constant magnetic field with field strength $B$ in two dimensions. Given any $p \in (2, +\infty)$, and any $\alpha > -B$, we have

$$
\mu_B(\alpha) \geq C_p \left(1 - c^2\right) \frac{1-\frac{2}{p}}{\frac{2}{p}} \left(\alpha + c B\right)^{\frac{2}{p}} =: \mu_{LT}(\alpha)
$$

with

$$
c = c(p, \eta) = \frac{\sqrt{\eta^2 + p - 1} - \eta}{p - 1} = \frac{1}{\eta + \sqrt{\eta^2 + p - 1}} \in (0, 1)
$$

and $\eta = \frac{\alpha (p - 2)}{2 B}$
Upper estimate (1): \( d = 2 \), constant magnetic field

For every integer \( k \in \mathbb{N} \) we introduce the special symmetry class

\[
\psi(x) = \left( \frac{x_2 + i x_1}{|x|} \right)^k \nu(|x|) \quad \forall x = (x_1, x_2) \in \mathbb{R}^2
\]

\((C_k)\)

[Esteban, Lions, 1989]: if \( \psi \in C_k \), then

\[
\frac{1}{2 \pi} \int_{\mathbb{R}^2} |\nabla_A \psi|^2 \, dx = \int_0^{+\infty} |\nu'|^2 \, r \, dr + \int_0^{+\infty} \left( \frac{k}{r} - \frac{B}{2} r \right)^2 |\nu| \, r \, dr
\]

and optimality is achieved in \( C_k \)

Test function \( \nu_\sigma(r) = e^{-r^2/(2\sigma)} \): an optimization on \( \sigma > 0 \) provides an explicit expression of \( \mu_{\text{Gauss}}(\alpha) \) such that

**Proposition**

*If \( p > 2 \), then*

\[
\mu_B(\alpha) \leq \mu_{\text{Gauss}}(\alpha) \quad \forall \alpha > -\Lambda[B]
\]

This estimate is not optimal because \( \nu_\sigma \) does not solve the Euler-Lagrange equations
Upper estimate (2): $d = 2$, constant magnetic field

A more numerical point of view. The Euler-Lagrange equation in $C_0$ is

$$-v'' - \frac{v'}{r} + \left( \frac{B^2}{4} r^2 + \alpha \right) v = \mu_{EL}(\alpha) \left( \int_0^{+\infty} |v|^p r \ dr \right)^{\frac{2}{p}-1} |v|^{p-2} v$$

We can restrict the problem to positive solutions such that

$$\mu_{EL}(\alpha) = \left( \int_0^{+\infty} |v|^p r \ dr \right)^{1-\frac{2}{p}}$$

and then we have to solve the reduced problem

$$-v'' - \frac{v'}{r} + \left( \frac{B^2}{4} r^2 + \alpha \right) v = |v|^{p-2} v$$
Numerical results and the symmetry issue
Figure: Case $d = 2$, $p = 3$, $B = 1$

Upper estimates: $\alpha \mapsto \mu_{\text{Gauss}}(\alpha), \mu_{\text{EL}}(\alpha)$

Lower estimates: $\alpha \mapsto \mu_{\text{interp}}(\alpha), \mu_{\text{LT}}(\alpha)$

The exact value associated with $\mu_B$ lies in the grey area.

Plots represent the curves $\log_{10}(\mu/\mu_{\text{EL}})$
Asymptotics (1): \textbf{Lowest Landau Level}

**Proposition**

Let $d = 2$ and consider a constant magnetic field with field strength $B$. If $\psi_\alpha$ is a minimizer for $\mu_B(\alpha)$ such that $\|\psi_\alpha\|_{L^p(\mathbb{R}^d)} = 1$, then there exists a non trivial $\varphi_\alpha \in \text{LLL}$ such that

$$\lim_{\alpha \to (-B)_+} \|\psi_\alpha - \varphi_\alpha\|_{H^1_A(\mathbb{R}^2)} = 0$$

Let $\psi_\alpha \in H^1_A(\mathbb{R}^2)$ be an optimal function such that $\|\psi_\alpha\|_{L^p(\mathbb{R}^d)} = 1$ and let us decompose it as $\psi_\alpha = \varphi_\alpha + \chi_\alpha$, where $\varphi_\alpha \in \text{LLL}$ and $\chi_\alpha$ is in the orthogonal of LLL

$$\mu_B(\alpha) \geq (\alpha + B) \|\varphi_\alpha\|_{L^2(\mathbb{R}^d)}^2 + (\alpha + 3B) \|\chi_\alpha\|_{L^2(\mathbb{R}^d)}^2 \geq (\alpha + 3B) \|\chi_\alpha\|_{L^2(\mathbb{R}^d)}^2 \sim 2B \|\chi_\alpha\|_{L^2(\mathbb{R}^d)}^2$$

as $\alpha \to (-B)_+$ because $\|\nabla \chi_\alpha\|_{L^2(\mathbb{R}^d)}^2 \geq 3B \|\chi_\alpha\|_{L^2(\mathbb{R}^d)}^2$

Since $\lim_{\alpha \to (-B)_+} \mu_B(\alpha) = 0$, $\lim_{\alpha \to (-B)_+} \|\chi_\alpha\|_{L^2(\mathbb{R}^d)} = 0$ and

$$\mu_B(\alpha) = (\alpha + B) \|\varphi_\alpha\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla A \chi_\alpha\|_{L^2(\mathbb{R}^d)}^2 + \alpha \|\chi_\alpha\|_{L^2(\mathbb{R}^d)}^2 \geq \frac{2}{3} \|\nabla A \chi_\alpha\|_{L^2(\mathbb{R}^d)}^2$$

concludes the proof.
Asymptotics (2): \textit{semi-classical regime}

Let us consider the small magnetic field regime. We assume that the magnetic potential is given by

\[ A_1 = \frac{B}{2}x_2, \quad A_2 = -\frac{B}{2}x_1 \quad \forall \, \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \]

if \( d = 2 \). In dimension \( d = 3 \), we choose \( A = \frac{B}{2}(-x_2, x_1, 0) \) and observe that the constant magnetic field is \( B = (0, 0, B) \), while the spectral gap is \( \Lambda[B] = B \).

**Proposition**

Let \( d = 2 \) or \( 3 \) and consider a constant magnetic field \( B \) of intensity \( B \) with magnetic potential \( A \)

For any \( p \in (2, 2^*) \) and any fixed \( \alpha \) and \( \mu > 0 \), we have

\[
\lim_{\varepsilon \to 0^+} \mu \varepsilon B(\alpha) = C_p \, \alpha^\frac{d}{p} - \frac{d^2}{2}
\]

Consider any function \( \psi \in H^1_A(\mathbb{R}^d) \) and let \( \psi(x) = \chi(\sqrt{\varepsilon} \, x), \sqrt{\varepsilon} \, A(x/\sqrt{\varepsilon}) = A(x) \) with our conventions on \( A \).
Numerical stability of radial optimal functions

Let us denote by $\psi_0$ an optimal function in $(C_0)$ such that

$$-\psi_0'' - \frac{\psi_0'}{r} + \left(\frac{B^2}{4} r^2 + \alpha\right) \psi_0 = |\psi_0|^{p-2} \psi_0$$

and consider the test function

$$\psi_\varepsilon = \psi_0 + \varepsilon e^{i\theta} \nu$$

where $\nu = \nu(r)$ and $e^{i\theta} = (x_1 + i x_2)/r$

As $\varepsilon \to 0^+$, the leading order term is

$$2\pi \left[ \int_{\mathbb{R}^2} |\nu'|^2 \, dx + \int_{\mathbb{R}^2} \left( \left(\frac{1}{r} - \frac{Br}{2} \right)^2 + \alpha \right) |\nu|^2 \, dx - \frac{p}{2} \int_0^{+\infty} |\psi_0|^{p-2} \nu^2 \, r \, dr \right] \varepsilon^2$$

and we have to solve the eigenvalue problem

$$-\nu'' - \frac{\nu'}{r} + \left(\frac{1}{r} - \frac{Br}{2} \right)^2 + \alpha \right) \nu - \frac{p}{2} |\psi_0|^{p-2} \nu = \mu \nu$$
Figure: Case $p = 3$ and $B = 1$: plot of the eigenvalue $\mu$ as a function of $\alpha$. A careful investigation shows that $\mu$ is always positive, including in the limiting case as $\alpha \to (-B)_+$, thus proving the numerical stability of the optimal function in $C_0$ with respect to perturbations in $C_1$. 
An open question of symmetry

- [Bonheure, Nys, Van Schaftingen, 2016] for a fixed $\alpha > 0$ and for $B$ small enough, the optimal functions are radially symmetric functions, i.e., belong to $C_0$
- This regime is equivalent to the regime as $\alpha \to +\infty$ for a given $B$, at least if the magnetic field is constant
- Numerically our upper and lower bounds are (in dimension $d = 2$, for a constant magnetic field) numerically extremely close
- The optimal function in $C_0$ is linearly stable with respect to perturbations in $C_1$

$\triangleright$ Prove that the optimality case is achieved among radial function if $d = 2$ and $B$ is a constant magnetic field
Reference

Magnetic rings

▷ A magnetic interpolation inequality on $\mathbb{S}^1$: with $p > 2$

\[
\|\psi' + ia\psi\|^2_{L^2(S^1)} + \alpha \|\psi\|^2_{L^2(S^1)} \geq \mu_{a,p}(\alpha) \|\psi\|^2_{L^p(S^1)}
\]

▷ Consequences

- A Keller-Lieb-Thirring inequality
- A new Hardy inequality for Aharonov-Bohm magnetic fields in $\mathbb{R}^2$

Joint work with M.J. Esteban, A. Laptev and M. Loss
Magnetic flux, a reduction

Assume that $a : \mathbb{R} \to \mathbb{R}$ is a $2\pi$-periodic function such that its restriction to $(-\pi, \pi] \approx \mathbb{S}^1$ is in $L^{1}(\mathbb{S}^1)$ and define the space

$$X_a := \{ \psi \in C_{\text{per}}(\mathbb{R}) : \psi' + i a \psi \in L^{2}(\mathbb{S}^1) \}$$

A standard change of gauge (see e.g. [Ilyin, Laptev, Loss, Zelik, 2016])

$$\psi(s) \mapsto e^{i \int_{-\pi}^{s} (a(s) - \bar{a}) \, d\sigma} \psi(s)$$

where $\bar{a} := \int_{-\pi}^{\pi} a(s) \, d\sigma$ is the magnetic flux, reduces the problem to $a$ is a constant function

For any $k \in \mathbb{Z}$, $\psi$ by $s \mapsto e^{iks} \psi(s)$ shows that $\mu_{a,p}(\alpha) = \mu_{k+a,p}(\alpha)$

$a \in [0, 1)$

$\mu_{a,p}(\alpha) = \mu_{1-a,p}(\alpha)$ because

$$|\psi' + i a \psi|^2 = |\chi' + i (1-a) \chi|^2 = |\overline{\psi}' - i a \overline{\psi}|^2$$

if $\chi(s) = e^{-is} \overline{\psi(s)}$

$a \in [0, 1/2]$
Optimal interpolation

We want to characterize the *optimal constant* in the inequality

$$\|\psi' + i a \psi\|_{L^2(S^1)}^2 + \alpha \|\psi\|_{L^2(S^1)}^2 \geq \mu_{a,p}(\alpha) \|\psi\|_{L^p(S^1)}^2$$

written for any $p > 2$, $a \in (0, 1/2]$, $\alpha \in (-a^2, +\infty)$, $\psi \in X_a$

$$\mu_{a,p}(\alpha) := \inf_{\psi \in X_a \setminus \{0\}} \frac{\int_{-\pi}^{\pi} (|\psi' + i a \psi|^2 + \alpha |\psi|^2) \, d\sigma}{\|\psi\|_{L^p(S^1)}^2}$$

$p = -2 = 2d/(d-2)$ with $d = 1$ [Exner, Harrell, Loss, 1998]
$p = +\infty$ [Galunov, Oliën, 1995] [Ilyin, Laptev, Loss, Zelik, 2016]
$\lim_{\alpha \to -a^2} \mu_{a,p}(\alpha) = 0$ [J. Dolbeault, Esteban, Laptev, Loss, 2016]

Using a Fourier series $\psi(s) = \sum_{k \in \mathbb{Z}} \psi_k e^{iks}$, we obtain that

$$\|\psi' + i a \psi\|_{L^2(S^1)}^2 = \sum_{k \in \mathbb{Z}} (a + k)^2 \left|\psi_k\right|^2 \geq a^2 \|\psi\|_{L^2(S^1)}^2$$

$\psi \mapsto \|\psi' + i a \psi\|_{L^2(S^1)}^2 + \alpha \|\psi\|_{L^2(S^1)}^2$ is coercive for any $\alpha > -a^2$
An interpolation result for the magnetic ring

**Theorem**

For any \( p > 2 \), \( a \in \mathbb{R} \), and \( \alpha > -a^2 \), \( \mu_{a,p}(\alpha) \) is achieved and
(i) if \( a \in [0, 1/2] \) and \( a^2 (p + 2) + \alpha (p - 2) \leq 1 \), then \( \mu_{a,p}(\alpha) = a^2 + \alpha \) and equality is achieved only by the constant functions
(ii) if \( a \in [0, 1/2] \) and \( a^2 (p + 2) + \alpha (p - 2) > 1 \), then \( \mu_{a,p}(\alpha) < a^2 + \alpha \) and equality is not achieved by the constant functions

If \( \alpha > -a^2 \), \( a \mapsto \mu_{a,p}(\alpha) \) is monotone increasing on \((0, 1/2)\)

**Figure:** \( \alpha \mapsto \mu_{a,p}(\alpha) \) with \( p = 4 \) and (left) \( a = 0.45 \) or (right) \( a = 0.2 \)
The proof: how to eliminate the phase
Reformulations of the interpolation problem (1/3)

Any minimizer $\psi \in X_a$ of $\mu_{a,p}(\alpha)$ satisfies the Euler-Lagrange equation

$$(H_a + \alpha) \psi = |\psi|^{p-2} \psi, \quad H_a \psi = -\left( \frac{d}{ds} + ia \right)^2 \psi$$

(*)

up to a multiplication by a constant and $v(s) = \psi(s) e^{ias}$ satisfies the condition

$$v(s + 2\pi) = e^{2i\pi a} v(s) \quad \forall s \in \mathbb{R}$$

Hence

$$\mu_{a,p}(\alpha) = \min_{v \in Y_a \setminus \{0\}} Q_{p,\alpha}[v]$$

where $Y_a := \{ v \in C(\mathbb{R}) : v' \in L^2(\mathbb{S}^1), \ (*) \ holds \}$ and

$$Q_{p,\alpha}[v] := \frac{\|v'\|^2_{L^2(\mathbb{S}^1)} + \alpha \|v\|^2_{L^2(\mathbb{S}^1)}}{\|v\|^2_{L^p(\mathbb{S}^1)}}$$
With \( v = u e^{i\phi} \) the boundary condition becomes

\[
\begin{align*}
u(\pi) &= u(-\pi), \\
\phi(\pi) &= 2\pi (a + k) + \phi(-\pi)
\end{align*}
\]

for some \( k \in \mathbb{Z} \), and \( \| v' \|_{L^2(S^1)}^2 = \| u' \|_{L^2(S^1)}^2 + \| u \phi' \|_{L^2(S^1)}^2 \)

Hence

\[
\mu_{a,p}(\alpha) = \min_{(u,\phi) \in Z_a \setminus \{0\}} \frac{\| u' \|_{L^2(S^1)}^2 + \| u \phi' \|_{L^2(S^1)}^2 + \alpha \| u \|_{L^p(S^1)}^2}{\| u \|_{L^p(S^1)}^2}
\]

where \( Z_a := \{(u, \phi) \in C(\mathbb{R})^2 : u', u \phi' \in L^2(S^1), (** \text{ holds})\} \)
Reformulations of the interpolation problem (3/3)

We use the Euler-Lagrange equations

\[-u'' + |\phi'|^2 u + \alpha u = |u|^{p-2} u \quad \text{and} \quad (\phi' u^2)' = 0\]

Integrating the second equation, and assuming that $u$ never vanishes, we find a constant $L$ such that $\phi' = L/u^2$. Taking (*) into account, we deduce from

\[L \int_{-\pi}^{\pi} \frac{ds}{u^2} = \int_{-\pi}^{\pi} \phi' \, ds = 2\pi (a + k)\]

that

\[\|u \phi'\|_{L^2(S^1)}^2 = L^2 \int_{-\pi}^{\pi} \frac{d\sigma}{u^2} = \frac{(a + k)^2}{\|u^{-1}\|_{L^2(S^1)}^2}\]

Hence

\[\phi(s) - \phi(0) = \frac{a + k}{\|u^{-1}\|_{L^2(S^1)}^2} \int_{-\pi}^{s} \frac{ds}{u^2}\]
Let us define

\[ Q_{a,p,\alpha}[u] := \frac{\|u'\|^2_{L^2(S^1)} + a^2 \|u^{-1}\|_{L^2(S^1)}^{-2} + \alpha \|u\|_{L^2(S^1)}^2}{\|u\|_{L^p(S^1)}^2} \]

**Lemma**

For any \( a \in (0, 1/2), \ p > 2, \ \alpha > -a^2 \),

\[ \mu_{a,p}(\alpha) = \min_{u \in H^1(S^1) \setminus \{0\}} Q_{a,p,\alpha}[u] \]

is achieved by a function \( u > 0 \)
Proofs

The existence proof is done on the original formulation of the problem using the diamagnetic inequality

\[ \psi(s) e^{ias} = v_1(s) + i v_2(s), \] solves

\[ -v_j'' + \alpha v_j = \left( v_1^2 + v_2^2 \right)^{\frac{p}{2} - 1} v_j, \quad j = 1, 2 \]

and the Wronskian \( w = ( v_1 v_2' - v_1' v_2 ) \) is constant so that \( \psi(s) = 0 \) is incompatible with the twisted boundary condition

if \( a^2 (p + 2) + \alpha (p - 2) \leq 1 \), then \( \mu_{a,p}(\alpha) = a^2 + \alpha \) because

\[
\| u' \|^2_{L^2(S^1)} + a^2 \| u^{-1} \|_{L^2(S^1)}^{-2} + \alpha \| u \|^2_{L^2(S^1)} = (1 - 4 a^2) \| u' \|^2_{L^2(S^1)} + \alpha \| u \|^2_{L^2(S^1)} \\
+ 4 a^2 \left( \| u' \|^2_{L^2(S^1)} + \frac{1}{4} \| u^{-1} \|^2_{L^2(S^1)} \right)
\]

if \( a^2 (p + 2) + \alpha (p - 2) > 1 \), the test function \( u_\epsilon := 1 + \epsilon w_1 \)

\[ Q_{a,p,\alpha}[u_\epsilon] = a^2 + \alpha + (1 - a^2 (p + 2) - \alpha (p - 2)) \epsilon^2 + o(\epsilon^2) \]

proves the linear instability of the constants and \( \mu_{a,p}(\alpha) < a^2 + \alpha \).
Symmetry in non-magnetic interpolation inequalities
Magnetic interpolation in the Euclidean space
Magnetic rings: the one-dimensional periodic case
Symmetry in Aharonov-Bohm magnetic fields

Magnetic interpolation on the circle
Proof: how to eliminate the phase
Consequences: Keller-Lieb-Thirring and Hardy inequalities

\[ Q_{a,p,\alpha}[u] := \frac{\|u'\|^2_{L^2(S^1)} + a^2 \|u^{-1}\|^2_{L^2(S^1)} + \alpha \|u\|^2_{L^2(S^1)}}{\|u\|^2_{L^p(S^1)}} \]

\[ \mu_{a,p}(\alpha) = \min_{u \in H^1(S^1) \setminus \{0\}} Q_{a,p,\alpha}[u] \]

\[ Q_{p,\alpha}[u] = Q_{a=0,p,\alpha}[u], \quad \nu_p(\alpha) := \inf_{v \in H^1_0(S^1) \setminus \{0\}} Q_{p,\alpha}[v] \]

**Proposition**

\( \forall \ p > 2, \ \alpha > -a^2, \) we have \( \mu_{a,p}(\alpha) < \mu_{1/2,p}(\alpha) \leq \nu_p(\alpha) = \mu_{1/2,p}(\alpha) \)

\[ \begin{align*}
\text{Figure:} & \quad p = 4, \ \alpha = 0, \ a = 0.40, 0.41, \ldots 0.49; \ u'' + u^{p-1} = 0 
\end{align*} \]
Consequences: Keller-Lieb-Thirring inequalities and Hardy inequalities for Aharonov-Bohm magnetic fields
A Keller-Lieb-Thirring inequality

Magnetic Schrödinger operator \( H_a - \varphi = -\left( \frac{d}{ds} + i a \right)^2 \psi - \varphi \)

- The function \( \alpha \mapsto \mu_{a,p}(\alpha) \) is monotone increasing, concave, and therefore has an inverse, denoted by \( \alpha_{a,p} : \mathbb{R}^+ \to (-a^2, +\infty) \), which is monotone increasing, and convex

**Corollary**

Let \( p > 2 \), \( a \in [0, 1/2] \), \( q = p/(p-2) \) and assume that \( \varphi \) is a non-negative function in \( L^q(S^1) \). Then

\[
\lambda_1(H_a - \varphi) \geq -\alpha_{a,p}(\|\varphi\|_{L^q(S^1)})
\]

and \( \alpha_{a,p}(\mu) = \mu - a^2 \) iff \( 4a^2 + \mu(p-2) \leq 1 \) (optimal \( \varphi \) is constant)

Equality is achieved
Aharonov-Bohm magnetic fields

On the two-dimensional Euclidean space $\mathbb{R}^2$, let us introduce the polar coordinates $(r, \vartheta) \in [0, +\infty) \times S^1$ of $x \in \mathbb{R}^2$ and consider a magnetic potential $a$ in a transversal (Poincaré) gauge, or Poincaré gauge

$$(a, e_r) = 0 \quad \text{and} \quad (a, e_\vartheta) = a_\vartheta(r, \vartheta)$$

Magnetic Schrödinger energy

$$\int_{\mathbb{R}^2} |(i \nabla + a) \psi|^2 \, dx = \int_0^{+\infty} \int_{-\pi}^{\pi} \left( |\partial_r \psi|^2 + \frac{1}{r^2} |\partial_\vartheta \psi + i r a_\vartheta \psi|^2 \right) r \, d\vartheta \, dr$$

Aharonov-Bohm magnetic fields: $a_\vartheta(r, \vartheta) = a/r$ for some constant $a \in \mathbb{R}$ ($a$ is the magnetic flux), with magnetic field $b = \text{curl} \, a$

$$\int_{\mathbb{R}^2} |(i \nabla + a) \psi|^2 \, dx \geq \int_{\mathbb{R}^2} \frac{\varphi(|x|/|x|)}{|x|^2} |\psi|^2 \, dx \quad \forall \varphi \in L^q(S^1), \quad q \in (1, +\infty)$$

$$\Longrightarrow \tau = \tau (a, \|\varphi\|_{L^q(S^1)})$$
Hardy inequalities

[Hoffmann-Ostenhof, Laptev, 2015] proved Hardy’s inequality

\[
\int_{\mathbb{R}^d} |\nabla \psi|^2 \, dx \geq \tau \int_{\mathbb{R}^d} \frac{\varphi(x/|x|)}{|x|^2} |\psi|^2 \, dx
\]

where the constant \(\tau\) depends on the value of \(\|\varphi\|_{L^q(S^{d-1})}\) and \(d \geq 3\)

*Aharonov-Bohm vector potential* in dimension \(d = 2\)

\[
a(x) = a \left( \frac{x_2}{|x|^2}, \frac{-x_1}{|x|^2} \right), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad a \in \mathbb{R}
\]

and recall the inequality [Laptev, Weidl, 1999]

\[
\int_{\mathbb{R}^2} |(i \nabla + a) \psi|^2 \, dx \geq \min_{k \in \mathbb{Z}} (a - k)^2 \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2} \, dx
\]
A new Hardy inequality

\[ \int_{\mathbb{R}^2} |(i \nabla + a) \psi|^2 \, dx \geq \tau \int_{\mathbb{R}^2} \frac{\varphi(x/|x|)}{|x|^2} |\psi|^2 \, dx \quad \forall \varphi \in L^q(S^1), \quad q \in (1, +\infty) \]

Corollary

Let \( p > 2 \), \( a \in [0, 1/2] \), \( q = p/(p - 2) \) and assume that \( \varphi \) is a non-negative function in \( L^q(S^1) \). Then the inequality holds with \( \tau > 0 \) given by

\[ \alpha_{a,p} \left( \tau \| \varphi \|_{L^q(S^1)} \right) = 0 \]

Moreover, \( \tau = a^2/\| \varphi \|_{L^q(S^1)} \) if \( 4a^2 + \| \varphi \|_{L^q(S^1)} (p - 2) \leq 1 \)

For any \( a \in (0, 1/2) \), by taking \( \varphi \) constant, small enough in order that \( 4a^2 + \| \varphi \|_{L^q(S^1)} (p - 2) \leq 1 \), we recover the inequality

\[ \int_{\mathbb{R}^2} |(i \nabla + a) \psi|^2 \, dx \geq a^2 \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2} \, dx \]
Proofs (Keller-Lieb-Thirring inequality)

Hölder’s inequality

\[ \| \psi' + ia \psi \|^2_{L^2(S^d)} - \int_{-\pi}^{\pi} \varphi |\psi|^2 \, d\sigma \geq \| \psi' + ia \psi \|^2_{L^2(S^d)} - \mu \| \psi \|^2_{L^p(S^d)} \]

where \( \mu = \| \varphi \|_{L^q(S^d)} \) and \( \frac{1}{q} + \frac{2}{p} = 1 \): choose \( \mu_{a,p}(\alpha) = \mu \)

\[ \| \psi' + ia \psi \|^2_{L^2(S^d)} - \mu \| \psi \|^2_{L^p(S^d)} \geq -\alpha \| \psi \|^2_{L^2(S^d)} \]
Proofs (Hardy inequality)

Let $\tau \geq 0$, $x = (r, \vartheta) \in \mathbb{R}^2$ be polar coordinates in $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} \left( |(i \nabla + a) \psi|^2 - \tau \frac{\varphi}{|x|^2} |\psi|^2 \right) \, dx$$

$$= \int_0^\infty \int_{S^1} \left( r \left| \partial_r \psi \right|^2 + \frac{1}{r} \left| \partial_\vartheta \psi + i a \psi \right|^2 - \tau \frac{\varphi}{r} |\psi|^2 \right) \, d\vartheta \, dr \geq 0$$

$$\geq \lambda_1 (H_a - \tau \varphi) \int_0^\infty \int_{S^1} \frac{1}{r} |\psi|^2 \, d\vartheta \, dr$$

$$\geq - \alpha_{a,p}(\tau \|\varphi\|_{L^q(S^d)}) \int_0^\infty \int_{S^1} \frac{1}{r} |\psi|^2 \, d\vartheta$$

- If $\tau = 0$, then $\alpha_{a,p}(\tau \|\varphi\|_{L^q(S^d)}) = \alpha_{a,p}(0) = -a^2$
- $\alpha_{a,p}(\tau \|\varphi\|_{L^q(S^d)}) > 0$ for $\tau$ large

$\implies \exists! \, \tau > 0$ such that $\alpha_{a,p}(\tau \|\varphi\|_{L^q(S^d)}) = 0$
The region $a^2 (p + 2) + \alpha (p - 2) < 1$ is exactly the set where the constant functions are linearly stable critical points.

The proof of the rigidity result is based
- neither on the carré du champ method, at least directly
- nor on a Fourier representation of the operator as it was the case in earlier proofs ($p = +\infty$, or $p > 2$ and $\alpha = 0$)

Magnetic rings: see [Bonnaillie-Noël, Hérau, Raymond, 2017]

Deducing Hardy’s inequality applied with Aharonov-Bohm magnetic fields from a Keller-Lieb-Thirring inequality is an extension of [Hoffmann-Ostenhof, Laptev, 2015] to the magnetic case

Our results are not limited to the semi-classical regime
Symmetry in Aharonov-Bohm magnetic fields

- Aharonov-Bohm effect
- Interpolation and Keller-Lieb-Thirring inequalities in $\mathbb{R}^2$
  - Statements
  - Constants and numerics
- Symmetry and symmetry breaking

Joint work with D. Bonheure, M.J. Esteban, A. Laptev, & M. Loss
Aharonov-Bohm effect

A major difference between classical mechanics and quantum mechanics is that particles are described by a non-local object, the wave function. In 1959 Y. Aharonov and D. Bohm proposed a series of experiments intended to put in evidence such phenomena which are nowadays called *Aharonov-Bohm effects*.

One of the proposed experiments relies on a long, thin solenoid which produces a magnetic field such that the region in which the magnetic field is non-zero can be approximated by a line in dimension $d = 3$ and by a point in dimension $d = 2$.

▷ [Physics today, 2009] *“The notion, introduced 50 years ago, that electrons could be affected by electromagnetic potentials without coming in contact with actual force fields was received with a skepticism that has spawned a flourishing of experimental tests and expansions of the original idea.”* Problem solved by considering appropriate weak solutions!

▷ Is the wave function a physical object or is the modulus the only relevant quantity? Decisive experiments have been done only 20
The interpolation inequality

Let us consider an Aharonov-Bohm vector potential

\[ \mathbf{A}(x) = \frac{a}{|x|^2} (x_2, -x_1), \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}, \quad a \in \mathbb{R} \]

Magnetic Hardy inequality [Laptev, Weidl, 1999]

\[ \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 \, dx \geq \min_{k \in \mathbb{Z}} (a - k)^2 \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2} \, dx \]

where \( \nabla_{\mathbf{A}} \psi := \nabla \psi + i \mathbf{A} \psi \), so that, with \( \psi = |\psi| e^{iS} \)

\[ \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 \, dx = \int_{\mathbb{R}^2} \left[ (\partial_r |\psi|)^2 + (\partial_r S)^2 |\psi|^2 + \frac{1}{r^2} (\partial_\theta S + A)^2 |\psi|^2 \right] \, dx \]

Magnetic interpolation inequality

\[ \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 \, dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2} \, dx \geq \mu(\lambda) \left( \int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2} \, dx \right)^{2/p} \]

\( \triangleright \) Symmetrization: [Erdös, 1996], [Boulenger, Lenzmann], [Lenzmann, Sok]
A magnetic Hardy-Sobolev inequality

**Theorem**

Let $a \in [0, 1/2]$ and $p > 2$. For any $\lambda > -a^2$, there is an optimal, monotone increasing, concave function $\lambda \mapsto \mu(\lambda)$ which is such that

$$\int_{\mathbb{R}^2} |\nabla_A \psi|^2 \, dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2} \, dx \geq \mu(\lambda) \left( \int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2} \, dx \right)^{2/p}$$

If $\lambda \leq \lambda_* = 4 \frac{1 - 4a^2}{p^2 - 4} - a^2$ equality is achieved by

$$\psi(x) = (|x|^{\alpha} + |x|^{-\alpha})^{-\frac{2}{p-2}} \quad \forall \, x \in \mathbb{R}^2, \quad \text{with} \quad \alpha = \frac{p - 2}{2} \sqrt{\lambda + a^2}$$

If $\lambda > \lambda_\bullet$ with

$$\lambda_\bullet := \frac{8 \left( \sqrt{p^4 - a^2} (p-2)^2 (p+2) (3p-2)+2 \right) - 4p(p+4)}{(p-2)^3 (p+2)} - a^2$$

there is symmetry breaking: optimal functions are not radially symmetric.
A magnetic Keller-Lieb-Thirring estimate

Let $q \in (1, +\infty)$ and denote by $L^q_*(\mathbb{R}^2)$ the space defined using the weighted norm $|||\phi|||_q := \left(\int_{\mathbb{R}^2} |\phi|^q |x|^{2(q-1)} \, dx\right)^{1/q}$.

**Theorem**

*Let* $a \in (0, 1/2)$, $q \in (1, \infty)$ and $\phi \in L^q_*(\mathbb{R}^2)$: $\mu \mapsto \lambda(\mu)$ *is a convex monotone increasing function such that* $\lim_{\mu \to 0^+} \lambda(\mu) = -\min_{k \in \mathbb{Z}} (a - k)^2$ *and* $\lambda_1(-\Delta_A - \phi) \geq -\lambda \left(|||\phi|||_q\right)$.

There is an explicit $\mu_0 > 0$ such that the equality case is achieved for any $\mu \leq \mu_0$ by

$\phi(x) = \left(|x|^\alpha + |x|^{-\alpha}\right)^{-2} \quad \forall \, x \in \mathbb{R}^2,$

*with* $\alpha = \frac{p - 2}{2} \sqrt{\lambda(\mu) + a^2}$.

There is an explicit $\mu_\bullet > \mu_0$ such that the equality case is achieved only by non-radial functions if $\mu > \mu_\bullet$. 

J. Dolbeault  
Magnetic fields, interpolation & symmetry
Constants are explicit...

- For \( a = 1/2 \), we shall see that \( \mu_\bullet = \mu_* = -1/4 \)

- The function \( \lambda \mapsto \mu(\lambda) \) is the inverse of \( \mu \mapsto \lambda(\mu) \) and

\[
\mu_* = h(\lambda_*) \quad \text{and} \quad \mu_\bullet = h(\lambda_\bullet)
\]

with

\[
h(\lambda) := \frac{p}{2} (2 \pi)^{1-{2 \over p}} (\lambda + a^2)^{1+{2 \over p}} \left( \frac{2 \sqrt{\pi} \Gamma(\frac{p}{p-2})}{(p-2) \Gamma(\frac{p}{p-2} + \frac{1}{2})} \right)^{1-{2 \over p}}
\]
Figure: Case $p = 4$
Symmetry breaking region: $\lambda > \lambda_*(a)$
Symmetry breaking region: $\lambda < \lambda_*$

Figure: The curve $a \mapsto \lambda_*(a) - \lambda_*(a)$
Lemma

Let $a \in [0, 1/2]$ and $\psi = u e^{iS} \in C^1 \cap H^1_A$ such that $|\psi| > 0$

\[
\int_{\mathbb{R}^2} |\nabla_A \psi|^2 \, dx \geq \int_{\mathbb{R}^2} \left( |\partial_r u|^2 + \frac{1}{r^2} |\partial_\theta u|^2 + \frac{1}{r^2} \frac{a^2}{\int_{S^2} u^{-2} \, d\sigma} \right) \, dx
\]

Equality holds if and only if $\partial_r S \equiv 0$ and

\[
\partial_\theta S = a - \frac{a}{u^2} \frac{1}{\int_{S^2} u^{-2} \, d\sigma}
\]

When $u$ does not depend on $\theta$, equality is achieved iff $S$ is constant.

Lemma

For all $a \in [0, 1/2]$ and $\psi \in H^1(S^1)$ with $u = |\psi|$, we have

\[
\int_{S^2} |\partial_\theta \psi - i a \psi|^2 \, d\sigma \geq (1 - 4a^2) \int_{S^2} |\partial_\theta u|^2 \, d\sigma + a^2 \int_{S^2} u^2 \, d\sigma
\]
Proof (1/3): the inequality with a non-optimal constant

Diamagnetic inequality: $\| \nabla_A \psi \|_{L^2(\mathbb{R}^2)} \geq \| \nabla u \|_{L^2(\mathbb{R}^2)}$, $u = |\psi|$

$$\int_{\mathbb{R}^2} |\nabla_A \psi|^2 \, dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2} \, dx$$

$$\geq t \left( \| \nabla_A \psi \|_{L^2(\mathbb{R}^2)}^2 - a^2 \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \, dx \right)$$

$$+ (1 - t) \left( \| \nabla u \|_{L^2(\mathbb{R}^2)}^2 + \frac{\lambda + a^2 t}{1 - t} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \, dx \right)$$

With $a^2 = \frac{\lambda + a^2 t}{1 - t}$, $t \in (0, 1)$ such that $\lambda + a^2 t > 0$: existence of a positive constant $\mu(\lambda)$
Proof (2/3): optimal estimate in the symmetry range

With $a \in [0, 1/2]$, $\psi \in H^1(\mathbb{R}^2)$ and $u = |\psi|$

$$\int_{\mathbb{R}^2} |\nabla_A \psi|^2 \, dx \geq \int_{\mathbb{R}^2} |\partial_r u|^2 \, dx + (1 - 4a^2) \int_{\mathbb{R}^2} \frac{1}{r^2} |\partial_\theta u|^2 \, dx + a^2 \int_{\mathbb{R}^2} u^2 \, dx$$

The relaxed inequality

$$\int_{\mathbb{R}^2} \left( |\partial_r u|^2 + \frac{1-4a^2}{r^2} |\partial_\theta u|^2 \right) \, dx + (\lambda + a^2) \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} \, dx \geq \mu_{\text{rel}}(\lambda) \left( \int_{\mathbb{R}^2} \frac{|u|^p}{|x|^2} \, dx \right)^{\frac{2}{p}}$$

is rewritten on the cylinder $C := \mathbb{R} \times S^1$ using the Emden-Fowler transformation as

$$\int_{C} \left( |\partial_s w|^2 + (1 - 4a^2) |\partial_\theta w|^2 \right) \, dy + (\lambda + a^2) \int_{C} |w|^2 \, dy$$

$$\geq (2\pi)^{\frac{2}{p} - 1} \mu_{\text{rel}}(\lambda) \left( \int_{C} |w|^p \, dy \right)^{\frac{2}{p}}$$

If $(\lambda + a^2) \left( p^2 - 4 \right) \leq 4 \left( 1 - 4a^2 \right) \iff \lambda \leq \lambda_*$, the minimizer is symmetric
Proof (3/3): symmetry breaking range

\[ \mathcal{E}_{a,\lambda}[\psi] := \int_{\mathbb{R}^2} |\nabla A \psi|^2 \, dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2} \, dx - \mu \left( \int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2} \, dx \right)^{2/p} \]

\[ \mu = (2\pi \int_{C} |w_{*}|^p \, dy)^{1-2/p}, \quad w_{*}(s) = \zeta_{*} \left( \cosh(\omega s) \right)^{-\frac{2}{p-2}} \]

\[ s = - \log r \text{ and } \psi_\varepsilon(r, \theta) := \left( w_{*}(s) + \varepsilon \varphi(s, \theta) \right) e^{i \varepsilon \chi(s, \theta)} \]

\[ \mathcal{E}_{a,\lambda}[\psi_\varepsilon] = \varepsilon^2 Q[\varphi, \chi] + o(\varepsilon^2) \]

\[ Q[\varphi, \chi] = \int_{C} w_{*}^2 \left( |\partial_s \chi|^2 + |\partial_\theta \chi - a|^2 - a^2 \right) \, dy - 4a \int_{C} w_{*} \varphi \partial_\theta \chi \, dy \]

\[ + \int_{C} \left( |\partial_s \varphi|^2 + |\partial_\theta \varphi|^2 + (\lambda + a^2) \varphi^2 \right) \, dy \]

\[ - (p-1) \int_{C} |w_{*}|^{p-2} |\varphi|^2 \, dy \]

\[ \varphi(s, \theta) = \frac{\cos \theta}{\cosh(\omega s)^{\frac{p}{p-2}}}, \quad \chi(s, \theta) = \frac{\zeta}{\zeta_{*}} \frac{\sin \theta}{\cosh(\omega s)} : \quad Q[\varphi, \chi] < 0 \implies \lambda > \lambda_{\bullet} \]
These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
▷ Lectures

The papers can be found at

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▷ Preprints / papers

For final versions, use Dolbeault as login and Jean as password

Thank you for your attention!
1. Symmetry in non-magnetic interpolation inequalities
   - Interpolation on the sphere
   - Keller-Lieb-Thirring inequalities on the sphere
   - CKN inequalities, symmetry breaking and weighted nonlinear flows

2. Magnetic interpolation in the Euclidean space
   - Three interpolation inequalities and their dual forms
   - Proofs for general magnetic fields
   - Estimates in dimension $d = 2$ for constant magnetic fields
   - Numerical results and the symmetry issue

3. Magnetic rings: the one-dimensional periodic case
   - Magnetic interpolation on the circle
   - Proof: how to eliminate the phase
   - Consequences: Keller-Lieb-Thirring and Hardy inequalities

4. Symmetry in Aharonov-Bohm magnetic fields
   - Aharonov-Bohm effect
   - Interpolation and Keller-Lieb-Thirring inequalities
   - Symmetry and symmetry breaking