Méthodes d'entropie pour des EDP diffusives

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Plan de l'exposé

- Introduction: méthodes d'entropie pour les équations de diffusion rapide
- Inégalités de Hardy-Poincaré et applications
- Inégalités de Poincaré généralisées (Beckner): deux approches
  - entropie - production d'entropie
  - approche spectrale
- Autres fonctionnelles de Lyapunov pour des équations d'ordre deux et quatre
- Inégalités de Poincaré L^q pour des mesures quelconques
- Perspectives

Introduction aux méthodes d’entropie pour les équations de diffusion non-linéaires

Une brève revue

- Les méthodes d’entropie (généralisée) pour les équations de diffusion rapide (et des milieux poreux)
- Entropie - production d’entropie et inégalités fonctionnelles
- Quelques directions de recherche voisines

Porous media / fast diffusion equations

Generalized entropies and nonlinear diffusions (EDP, uncomplete): [Carrillo, Toscani], [Del Pino, J.D.], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vazquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub]...

Various approaches:
1) “entropy – entropy-production method”
2) mass transportation techniques
3) hypercontractivity for appropriate semi-groups

4) [J.D., del Pino] relate entropy and entropy-production by Gagliardo-Nirenberg inequalities

Intermediate asymptotics

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^d$$

$$u_{|t=0} = u_0 \geq 0$$

$$u_0(1 + |x|^2) \in L^1 \; , \quad u_0^m \in L^1$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0 \, dx = 1$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$
As $t \to +\infty$, [Friedmann, Kamin, 1980]

$$\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$$

Time-dependent rescaling

Take $u(t,x) = R^{-d}(t)\, v\,(\tau(t), x/R(t))$ where

$$\dot{R} = R^{d(1-m)-1}\;, \quad R(0) = 1\;, \quad \tau = \log R$$

$$v_\tau = \Delta v^m + \nabla \cdot (x\, v)\;, \quad v_{|\tau=0} = u_0$$

[Ralston, Newman, 1984] Lyapunov functional: Entropy

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right)\, dx - \Sigma_0$$

$$\frac{d}{d\tau}\Sigma[v] = -I[v]\;, \quad I[v] = \int v \left| \frac{\nabla v^m}{v} + x \right|^2\, dx$$

Entropy and entropy production

Stationary solution: C s.t. $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left(C + \frac{1-m}{2m}\,|x|^2\right)_+^{-1/(1-m)}$$

Fix Σ_0 so that $\Sigma[v_\infty] = 0$.

$$\Sigma[v] = \int \psi\left(\frac{v^m}{v_\infty^m}\right)\, v_\infty^{m-1}\, dx \quad with\ \psi(t) = \frac{m\, t^{1/m} - 1}{1-m} + 1$$

Theorem 1 $d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2\,\Sigma[v]$$

An equivalent formulation

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0 \leq \frac{1}{2} \int v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} I[v]$$

$p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int |w|^{1+p} dx + K \geq 0$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$ and K depends on $\int v\, dx = \int v^{2p}\, dx$

$m = \frac{d-1}{d}$: Sobolev, $m \to 1$: logarithmic Sobolev

[Del Pino, J.D.], [Carrillo, Toscani], [Otto]

Gagliardo-Nirenberg inequalities

[Del Pino, J.D.]

$$1 < p \leq \frac{d}{d-2} \text{ for } d \geq 3$$

$$\|w\|_{2p} \leq A \, \|\nabla w\|_2^\theta \, \|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}$$

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \; y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$

Uses [Serrin-Pucci], [Serrin-Tang]

$1 < p = \frac{1}{2m-1} \leq \frac{d}{d-2} \Longleftrightarrow$ Fast diffusion case: $\frac{d-1}{d} \leq m < 1$

$0 < p < 1 \Longleftrightarrow$ Porous medium case: $m > 1$

Intermediate asymptotics

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau}$ + Csiszár-Kullback inequalities

[Del Pino, J.D.]

(i) $\frac{d-1}{d} < m < 1$ if $d \geq 3$

$$\limsup_{t\to+\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}} \|u^m - u_\infty^m\|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t\to+\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}} \|\,[u - u_\infty]\; u_\infty^{m-1}\,\|_{L^1} < +\infty$$

$$u_\infty(t,x) = R^{-d}(t)\, v_\infty\,(x/R(t))$$

Optimal L^p-Euclidean logarithmic Sobolev inequality

[Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

Theorem 2 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx \right]$$

$$\mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n \, \frac{p-1}{p}+1)} \right]^{\frac{p}{n}}$$

Equality: $u(x) = \left(\pi^{\frac{n}{2}} \left(\frac{\sigma}{p} \right)^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)} \right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}$

$p = 2$: Gross' logaritmic Sobolev inequality [Gross, 75], [Weissler, 78]
$p = 1$: [Ledoux 96], [Beckner, 99]

Application to $u_t = \Delta_p u^{\frac{1}{p-1}}$

Extensions and related results

- Mass transportation methods: inequalities [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]
- General nonlinearities [Biler, J.D., Esteban], [Carrillo-DiFrancesco], [Carrillo-Juengel-Markowich-Toscani-Unterreiter] and gradient flows [Otto-Kinderlehrer-Jordan], [Ambrosio-Savaré-Gigli], [Otto-Westdickenberg], etc + [J.D.-Nazaret-Savaré, in progress]
- Non-homogeneous nonlinear diffusion equations [Biler, J.D., Esteban], [Carrillo, DiFrancesco]
- Extension to systems and connection with Lieb-Thirring inequalities [J.D.-Felmer-Loss-Paturel, 2006], [J.D.-Felmer-Mayorga]
- Drift-diffusion problems with mean-field terms. An example: the Keller-Segel model [J.D-Perthame, 2004], [Blanchet-J.D-Perthame, 2006], [Biler-Karch-Laurençot-Nadzieja, 2006], [Blanchet-Carrillo-Masmoudi, 2007], etc
- ... connection with linearized problems [Markowich-Lederman], [Carrillo-Vazquez], [Denzler-McCann]

Méthodes d'entropie et linéarisation: asymptotiques intermédiaires, évanescence

A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez

Objectifs

- Utiliser les entropies généralisées ET les propriétés du flot pour obtenir des informations sur le comportement asymptotique, c'est-à-dire...
- Ecrire les solutions en variables relatives aux profils asymptotiques et supposer un bon comportement à l'infini de manière à obtenir une convergence forte
- Comparer la fonctionnelle d'entropie généralisée à la fonctionnelle de production d'entropie généralisée en se ramenant à l'inégalité correspondante pour le problème linéarisé
- En déduire le comportement asymptotique des solutions

$\Longrightarrow$ Etendre le domaine des paramètres au prix d'une restriction de l'espace dans lequel vivent les solutions

Setting of the problem

We consider the solutions $u(\tau, y)$ of

$$\begin{cases} \partial_\tau u = \Delta u^m \\ u(0,\cdot) = u_0 \end{cases}$$

where $m \in (0,1)$ (fast diffusion) and $(\tau, y) \in Q_T = (0,T) \times \mathbb{R}^d$
Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where

$$m_c := \frac{d-2}{d}$$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$
- $0 < m < m_c$, $T < +\infty$: vanishing in finite time

$$\lim_{\tau \nearrow T} u(\tau, y) = 0$$

Barenblatt solutions

$$U_{D,T}(\tau,y) := \frac{1}{R(\tau)^d}\left(D + \frac{1-m}{2\,m}\left|\frac{y}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}}$$

with

- $R(\tau) := \left[d\,(m-m_c)\,(\tau+T)\right]^{\frac{1}{d\,(m-m_c)}}$ if $m_c < m < 1$
- (vanishing in finite time) if $0 < m < m_c$

$$R(\tau) := \left[d\,(m_c-m)\,(T-\tau)\right]^{-\frac{1}{d\,(m_c-m)}}$$

Time-dependent rescaling: $t := \log\left(\frac{R(\tau)}{R(0)}\right)$ and $x := \frac{y}{R(\tau)}$. The function $v(t,x) := R(\tau)^d\,u(\tau,y)$ solves a nonlinear *Fokker-Planck type equation*

$$\begin{cases} \partial_t v(t,x) = \Delta v^m(t,x) + \nabla\cdot(x\,v(t,x)) & (t,x)\in(0,+\infty)\times\mathbb{R}^d \\ v(0,x) = v_0(x) = R(0)^d\,u_0(R(0)\,x) & x\in\mathbb{R}^d \end{cases}$$

Assumptions

(H1) u_0 is a non-negative function in $\mathrm{L}^1_{loc}(\mathbb{R}^d)$ and that there exist positive constants T and $D_0 > D_1$ such that

$$U_{D_0,T}(0,y) \leq u_0(y) \leq U_{D_1,T}(0,y) \quad \forall\, y \in \mathbb{R}^d$$

(H2) If $m \in (0, m_*]$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$u_0(y) = U_{D_*,T}(0,y) + f(y) \quad \forall\, y \in \mathbb{R}^d$$

(H1') v_0 is a non-negative function in $\mathrm{L}^1_{loc}(\mathbb{R}^d)$ and there exist positive constants $D_0 > D_1$ such that

$$V_{D_0}(x) \leq v_0(x) \leq V_{D_1}(x) \quad \forall\, x \in \mathbb{R}^d$$

(H2') If $m \in (0, m_*]$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$v_0(x) = V_{D_*}(x) + f(x) \quad \forall\, x \in \mathbb{R}^d$$

Convergence to the asymptotic profile (without rate)

$$m_* := \frac{d-4}{d-2} < m_c := \frac{d-2}{2}\,, \quad p(m) := \frac{d\,(1-m)}{2\,(2-m)}$$

Theorem 1 *Let $d \geq 3$, $m \in (0,1)$. Consider a solution v with initial data satisfying (H1')-(H2')*

(i) For any $m > m_*$, there exists a unique D_* such that $\int_{\mathbb{R}^d}(v(t) - V_{D_*})\,dx = 0$ for any $t > 0$. Moreover, for any $p \in (p(m), \infty]$, $\lim_{t\to\infty} \int_{\mathbb{R}^d} |v(t) - V_{D_*}|^p\,dx = 0$

(ii) For $m \leq m_*$, $v(t) - V_{D_*}$ is integrable, $\int_{\mathbb{R}^d}(v(t) - V_{D_*})\,dx = \int_{\mathbb{R}^d} f\,dx$ and $v(t)$ converges to V_{D_*} in $\mathrm{L}^p(\mathbb{R}^d)$ as $t \to \infty$, for any $p \in (1, \infty]$

(iii) (Convergence in Relative Error) For any $p \in (d/2, \infty]$,

$$\lim_{t\to\infty} \|\, v(t)/\,V_{D_*} - 1\,\|_p = \; 0$$

[Daskalopoulos-Sesum, 06], [Blanchet-Bonforte-Grillo-Vázquez, 06-07]

Convergence with rate

$$q_* := \frac{2\,d\,(1-m)}{2\,(2-m)+d\,(1-m)}$$

Theorem 2 *If $m \neq m_*$, there exist $t_0 \geq 0$ and $\lambda_{m,d} > 0$ such that*

(i) *For any $q \in (q_*, \infty]$, there exists a positive constant C_q such that*

$$\|v(t) - V_{D_*}\|_q \leq C_q\, \mathrm{e}^{-\lambda_{m,d}\,t} \quad \forall\, t \geq t_0$$

(ii) *For any $\vartheta \in [0, (2-m)/(1-m))$, there exists a positive constant C_ϑ such that*

$$\big\|\,|x|^\vartheta (v(t) - V_{D_*})\big\|_2 \leq C_\vartheta\, \mathrm{e}^{-\lambda_{m,d}\,t} \quad \forall\, t \geq t_0$$

(iii) *For any $j \in \mathbb{N}$, there exists a positive constant H_j such that*

$$\|v(t) - V_{D_*}\|_{C^j(\mathbb{R}^d)} \leq H_j\, \mathrm{e}^{-\frac{\lambda_{m,d}}{d+2(j+1)}\,t} \quad \forall\, t \geq t_0$$

Intermediate asymptotics

Corollary 3 *Let* $d \geq 3$, $m \in (0,1)$, $m \neq m_*$. *Consider a solution* u *with initial data satisfying* (H1)-(H2). *For* τ *large enough, for any* $q \in (q_*, \infty]$, *there exists a positive constant* C *such that*

$$\|u(\tau) - U_{D_*}(\tau)\|_q \leq C\, R(\tau)^{-\alpha}$$

where $\alpha = \lambda_{m,d} + d(q-1)/q$ *and large means* $T - \tau > 0$, *small, if* $m < m_c$, *and* $\tau \to \infty$ *if* $m \geq m_c$

For any $p \in (d/2, \infty]$, *there exists a positive constant* $\mathcal{C}$ *and* $\gamma > 0$ *such that*

$$\left\| v(t)/ V_{D_*} - 1 \right\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \mathcal{C}\, \mathrm{e}^{-\gamma\, t} \quad \forall\, t \geq 0$$

Preliminaries

L^1-contraction, Maximum Principle, conservation of relative mass...

Passing to the quotient: the function $w(t,x) := \frac{v(t,x)}{V_{D_*}(x)}$ solves

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}} \nabla \cdot \left[w\, V_{D_*} \nabla \left(\dfrac{m}{m-1} (w^{m-1} - 1)\, V_{D_*}^{m-1} \right) \right] & \text{in } (0,+\infty) \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d \end{cases}$$

with

$$0 < \inf_{x \in \mathbb{R}^d} \frac{V_{D_0}}{V_{D_*}} \leq w(t,x) \leq \sup_{x \in \mathbb{R}^d} \frac{V_{D_1}}{V_{D_*}} < \infty$$

... Harnack Principle

$$\| w(t) \|_{C^k(\mathbb{R}^d)} \leq \overline{H}_k < +\infty \quad \forall\, t \geq t_0$$

$\exists\, t_0 \geq 0$ s.t. (H1) holds if $\exists\, R > 0$, $\sup_{|y|>R} u_0(y)\, |y|^{\frac{2}{1-m}} < \infty$, and $m > m_c$

Relative entropy

Relative entropy

$$\mathcal{F}[w] := \frac{1}{1-m} \int_{\mathbb{R}^d} \left[(w-1) - \frac{1}{m}(w^m - 1) \right] V_{D_*}^m \, dx$$

Relative Fisher information

$$\mathcal{J}[w] := \frac{m}{(m-1)^2} \int_{\mathbb{R}^d} \left| \nabla \left[\left(w^{m-1} - 1 \right) V_{D_*}^{m-1} \right] \right|^2 w \, V_{D_*} \, dx$$

Proposition 4 *Under assumptions (H1)-(H2),*

$$\frac{d}{dt} \mathcal{F}[w(t)] = - \mathcal{J}[w(t)]$$

Proposition 5 *Under assumptions (H1)-(H2), there exists a constant* $\lambda > 0$ *such that*

$$\mathcal{F}[w(t)] \leq \lambda^{-1} \mathcal{J}[w(t)]$$

Heuristics: linearization

Take $w(t,x) = 1 + \varepsilon \, \frac{g(t,x)}{V_{D_*}^{m-1}(x)}$ and formally consider the limit $\varepsilon \to 0$ in

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}} \nabla \cdot \left[w \, V_{D_*} \nabla \left(\dfrac{m}{m-1} (w^{m-1} - 1) \, V_{D_*}^{m-1} \right) \right] & \text{in } (0,+\infty) \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d \end{cases}$$

Then g solves

$$g_t = m \, V_{D_*}^{m-2}(x) \, \nabla \cdot \left[V_{D_*}(x) \, \nabla g(t,x) \right]$$

and the entropy and Fisher information functionals

$$\mathsf{F}[g] := \frac{1}{2} \int_{\mathbb{R}^d} |g|^2 \, V_{D_*}^{2-m} \, dx \quad \text{and} \quad \mathsf{I}[g] := m \int_{\mathbb{R}^d} |\nabla g|^2 \, V_{D_*} \, dx$$

consistently verify $\dfrac{d}{dt} \mathsf{F}[g(t)] = - \, \mathsf{I}[g(t)]$

Comparison of the functionals

Lemma 6 *Let $m \in (0,1)$ and assume that u_0 satisfies* (H1)-(H2)
[Relative entropy]

$$C_1 \int_{\mathbb{R}^d} |w-1|^2 \, V_{D_*}^m \, dx \le \mathcal{F}[w] \le C_2 \int_{\mathbb{R}^d} |w-1|^2 \, V_{D_*}^m \, dx$$

[Fisher information]

$$I[g] \le \beta_1 \, \mathcal{J}[w] + \beta_2 \;\; F[g] \quad \textit{with} \quad g := (w-1) \, V_{D_*}^{m-1}$$

Theorem 7 (Hardy-Poincaré) *There exists a positive constant $\lambda_{m,d}$ such that for any $m \neq m_* = (d-4)/(d-2)$, $m \in (0,1)$, for any $g \in \mathcal{D}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |g - \overline{g}|^2 \; V_{D_*}^{2-m} \, dx \le \mathcal{C}_{m,d} \int_{\mathbb{R}^d} |\nabla g|^2 \, V_{D_*} \, dx$$

with $\overline{g} = \int_{\mathbb{R}^d} g \; V_{D_*}^{2-m} \, dx$ if $m > m_*$, $\overline{g} = 0$ otherwise

Hardy-Poincaré inequalities

With $\alpha = \frac{1}{m-1}$, $\alpha_* = \frac{1}{m_*-1} = 1 - \frac{d}{2}$

Theorem 8 *Assume that* $d \geq 3$, $\alpha \in \mathbb{R} \setminus \{\alpha^*\}$, $d\mu_\alpha(x) := h_\alpha(x)\,dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$. *Then*

$$\int_{\mathbb{R}^d} \frac{|v|^2}{1+|x|^2}\,d\mu_\alpha \leq \mathcal{C}_{\alpha,d} \int_{\mathbb{R}^d} |\nabla v|^2\,d\mu_\alpha$$

holds for some positive constant $\mathcal{C}_{\alpha,d}$, *for any* $v \in \mathcal{D}(\mathbb{R}^d)$, *under the additional condition* $\int_{\mathbb{R}^d} v\,d\mu_{\alpha-1} = 0$ *if* $\alpha \in (-\infty, \alpha^*)$

Limit cases

Poincaré inequality: take $\alpha = -1/\epsilon^2$ to $v_\epsilon(x) := \epsilon^{-d/2}\, v(x/\epsilon)$ and let $\epsilon \to 0$

$$\int_{\mathbb{R}^d} |v|^2 \, d\nu_\infty \le \frac{1}{2} \int_{\mathbb{R}^d} |\nabla v|^2 \, d\nu_\infty \quad \text{with} \quad d\nu_\infty(x) := e^{-|x|^2} dx$$

... under the additional condition $\int_{\mathbb{R}^d} v \; e^{-|x|^2} dx = 0$

Hardy's inequality: take $v_{1/\epsilon}(x) := \epsilon^{d/2}\, v(\epsilon\, x)$ and let $\epsilon \to 0$

$$\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^2} \, d\nu_{0,\alpha} \le \frac{1}{(\alpha - \alpha_*)^2} \int_{\mathbb{R}^d} |\nabla v|^2 \, d\nu_{0,\alpha} \quad \text{with} \quad d\nu_{0,\alpha}(x) := |x|^{2\alpha} \, dx$$

... under the additional condition $\bar{v}_\alpha := \int_{\mathbb{R}^d} v \, d\nu_{0,\alpha} = 0$ if $\alpha < \alpha^*$

Some estimates of $\mathcal{C}_{\alpha,d}$

α	$-\infty < \alpha \leq -d$	$-d < \alpha < \alpha^*$	$\alpha^* < \alpha \leq 1$
$\mathcal{C}_{\alpha,d}$	$\frac{1}{2\,\lvert\alpha\rvert}$	$\mathcal{C}_{\alpha,d} \geq \frac{4}{(d+2\alpha-2)^2}$	$\frac{4}{(d+2\alpha-2)^2}$
Optimality	?	?	yes

α	$1 \leq \alpha \leq \bar{\alpha}(d)$	$\bar{\alpha}(d) \leq \alpha \leq d$	d	$\alpha > d$
$\mathcal{C}_{\alpha,d}$	$\frac{4}{d(d+2\alpha-2)}$	$\frac{1}{\alpha(d+\alpha-2)}$	$\frac{1}{2d(d-1)}$	$\frac{1}{d(d+\alpha-2)}$
Optimality	?	?	yes	?

Hardy's inequality: the "completing the square method"

Let $v \in \mathcal{D}(\mathbb{R}^d)$ with supp$(v) \subset \mathbb{R}^d \setminus \{0\}$ if $\alpha < \alpha^*$

$$\begin{aligned} 0 \;\; &\leq \;\; \int_{\mathbb{R}^d} \left| \nabla v + \lambda \, \frac{x}{|x|^2} \, v \right|^2 |x|^{2\alpha} \, dx \\ &= \int_{\mathbb{R}^d} |\nabla v|^2 \, |x|^{2\alpha} \, dx + \Big[\lambda^2 - \lambda \, (d + 2\alpha - 2) \Big] \int_{\mathbb{R}^d} \frac{|v|^2}{|x|^2} \, |x|^{2\alpha} \, dx \end{aligned}$$

An optimization of the right hand side with respect to λ gives $\lambda = \alpha - \alpha^*$, that is $(d + 2\alpha - 2)^2/4 = \lambda^2$. Such an inequality is optimal, with optimal constant λ^2, as follows by considering the test functions:

1) if $\alpha > \alpha^*$: $v_\epsilon(x) = \min\{\epsilon^{-\lambda}, (|x|^{-\lambda} - \epsilon^\lambda)_+\}$

2) if $\alpha < \alpha^*$: $v_\epsilon(x) = |x|^{1-\alpha-d/2+\epsilon}$ for $|x| < 1$
$v_\epsilon(x) = (2 - |x|)_+$ for $|x| \geq 1$

and letting $\epsilon \to 0$ in both cases

The optimality case: Davies' method

Proposition 9 *Let $d \geq 3$, $\alpha \in (\alpha^*, \infty)$. Then the Hardy-Poincaré inequality holds for any $v \in \mathcal{D}(\mathbb{R}^d)$ with $\mathcal{C}_{\alpha,d} := 4/(d-2+2\,\alpha)^2$ if $\alpha \in (\alpha^*, 1]$ and $\mathcal{C}_{\alpha,d} := 4/[d\,(d-2+2\,\alpha)]$ if $\alpha \geq 1$. The constant $\mathcal{C}_{\alpha,d}$ is optimal for any $\alpha \in (\alpha^*, 1]$.*

Proof: $\nabla h_\alpha = 2\alpha\, x\, h_{\alpha-1}$, $\Delta h_\alpha = 2\alpha\, h_{\alpha-2}[d + 2(\alpha-\alpha^*)\,|x|^2] > 0$.
By Cauchy-Schwarz

$$\begin{aligned} \left|\int_{\mathbb{R}^d} |v|^2\, \Delta h_\alpha\, dx\right|^2 &\leq 4\left(\int_{\mathbb{R}^d} |v|\,|\nabla v|\,|\nabla h_\alpha|\, dx\right)^2 \\ &\leq 4\int_{\mathbb{R}^d} |v|^2\,|\Delta h_\alpha|\, dx \int_{\mathbb{R}^d} |\nabla v|^2\,|\nabla h_\alpha|^2\,|\Delta h_\alpha|^{-1}\, dx \end{aligned}$$

$|\Delta h_\alpha| \geq 2\,|\alpha|\, \min\{d, (d-2+2\,\alpha)\}\, \frac{h_\alpha(x)}{1+|x|^2}$

$\frac{|\nabla h_\alpha|^2}{|\Delta h_\alpha|} \leq \frac{2\,|\alpha|}{d-2+2\,\alpha}\, h_\alpha(x)$

Inégalités de Poincaré généralisées et application aux équations linéaires (Fokker-Planck)

Coll. A. Arnold, J.-P. Bartier, J.D.

- Méthode d'entropie - production d'entropie
- Une amélioration basée sur le critère de Bakry-Emery
- Le point de vue spectral

Entropy-entropy production method

[Bakry, Emery, 1984]
[Toscani 1996], [Arnold, Markowich, Toscani, Unterreiter, 2001]
Relative entropy of $u = u(x)$ w.r.t. $u_\infty(x)$

$$\Sigma[u|u_\infty] := \int_{\mathbb{R}^d} \psi\left(\frac{u}{u_\infty}\right) u_\infty \, dx \geq 0$$

$$\begin{aligned} \psi(w) &\geq 0 \text{ for } w \geq 0, \quad \text{convex} \\ \psi(1) &= \psi'(1) = 0 \end{aligned}$$

Admissibility condition $\quad (\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV}$

Examples

$\psi_1 = w \ln w - w + 1$, $\Sigma_1(u|u_\infty) = \int u \ln\left(\frac{u}{u_\infty}\right) dx \ldots$ "physical entropy"

$\psi_p = \frac{w^p - p(w-1) - 1}{p-1}$, $1 < p \leq 2$, $\quad \Sigma_2(u|u_\infty) = \int_{\mathbb{R}^d} (u - u_\infty)^2 \, u_\infty^{-1} \, dx$

Exponential decay of entropy production

Fokker-Planck equation

$$u_t = \Delta u + \nabla \cdot (u\,\nabla V)$$

$$-I(u(t)|u_\infty) := \frac{d}{dt}\Sigma[u(t)|u_\infty] = -\int \psi''\left(\frac{u}{u_\infty}\right) \underbrace{\left|\nabla\left(\frac{u}{u_\infty}\right)\right|^2}_{=:v} u_\infty\,dx \leq 0$$

$V(x) = -\log u_\infty \ldots$ unif. convex: $\underbrace{\dfrac{\partial^2 V}{\partial x^2}}_{\text{Hessian}} \geq \lambda_1 \mathrm{Id},\ \lambda_1 > 0$

Entropy production rate

$$\begin{aligned} -I' &= 2\int \psi''\left(\frac{u}{u_\infty}\right) v^T \cdot \frac{\partial^2 V}{\partial x^2} \cdot v\,u_\infty\,dx + 2\underbrace{\int \mathrm{Tr}\,(XY)\,u_\infty\,dx}_{\geq 0} \\ &\geq +2\,\lambda_1\,I \end{aligned}$$

Positivity of Tr (XY) ?

Admissibility condition $\Longleftrightarrow (\psi''')^2 \leq \frac{1}{2}\,\psi''\psi^{IV}$

$$X = \begin{pmatrix} \psi''\left(\frac{u}{u_\infty}\right) & \psi'''\left(\frac{u}{u_\infty}\right) \\ \psi'''\left(\frac{u}{u_\infty}\right) & \frac{1}{2}\,\psi^{IV}\left(\frac{u}{u_\infty}\right) \end{pmatrix} \geq 0$$

$$Y = \begin{pmatrix} \sum_{ij}\left(\frac{\partial v_i}{\partial x_j}\right)^2 & v^T \cdot \frac{\partial v}{\partial x} \cdot v \\ v^T \cdot \frac{\partial v}{\partial x} \cdot v & |v|^4 \end{pmatrix} \geq 0$$

$$\Rightarrow \quad I(t) \leq e^{-2\lambda_1 t}\, I(t=0) \qquad \forall\, t > 0$$

$\forall\, u_0$ with $I(t=0) = I(u_0|u_\infty) < \infty$

Exponential decay of relative entropy

Known: $-I' \geq \ 2\,\lambda_1 \underbrace{I}_{=\Sigma'}$

$\int_t^\infty ...dt \qquad \Rightarrow \Sigma' = I \leq -2\lambda_1 \Sigma$

Theorem 1 *[Bakry, Emery] [Arnold, Markowich, Toscani, Unterreiter]*
Under the "Bakry–Emery condition"

$$\frac{\partial^2 V}{\partial x^2} \geq \lambda_1 \,\mathrm{Id}$$

if $\Sigma[u_0|u_\infty] < \infty$*, then*

$$\Sigma[u(t)|u_\infty] \leq \Sigma[u_0|u_\infty]\, e^{-2\lambda_1 t} \qquad \forall\, t > 0$$

Convex Sobolev inequalities

$$\Sigma[u|u_\infty] \leq \frac{1}{2\lambda_1} |I(u|u_\infty)|$$

Example 1 logarithmic entropy $\psi_1(w) = w \ln w - w + 1$

$$\int u \ln\left(\frac{u}{u_\infty}\right) dx \leq \frac{1}{2\lambda_1} \int u \left|\nabla \ln\left(\frac{u}{u_\infty}\right)\right|^2 dx$$

$$\forall u,\ u_\infty \in L^1_+(\mathbb{R}^d),\ \int u\, dx = \int u_\infty\, dx = 1$$

Example 2 power law entropies

$\psi_p(w) = w^p - p(w-1) - 1, \qquad 1 < p \leq 2$

$$\frac{p}{p-1}\left[\int f^2\, u_\infty\, dx - \left(\int |f|^{\frac{2}{p}}\, u_\infty\, dx\right)^p\right] \leq \frac{2}{\lambda_1} \int |\nabla f|^2\, u_\infty\, dx$$

Use $\frac{u}{u_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}}\, u_\infty\, dx}$, $f \in L^{\frac{2}{p}}(\mathbb{R}^d, u_\infty dx)$

Refined convex Sobolev inequalities

Estimate of entropy production rate / entropy production

$$I' = 2\int \psi''\left(\frac{u}{u_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u u_\infty dx + \underbrace{2\int \mathrm{Tr}\,(XY) u_\infty dx}_{\geq 0}$$

$$\geq -2\lambda_1 I$$

[Arnold, J.D.] Observe that $\psi_p(w) = \frac{w^p - p(w-1) - 1}{p-1}$, $1 < p < 2$

$$X = \begin{pmatrix} \psi''\left(\frac{u}{u_\infty}\right) & \psi'''\left(\frac{u}{u_\infty}\right) \\ \psi'''\left(\frac{u}{u_\infty}\right) & \frac{1}{2}\,\psi^{IV}\left(\frac{u}{u_\infty}\right) \end{pmatrix} > 0$$

- Assume $\frac{\partial V^2}{\partial x^2} \geq \lambda_1$ Id $\Rightarrow \Sigma'' \geq -2\lambda_1 \Sigma' + \kappa \frac{|\Sigma'|^2}{1+\Sigma}$ $\qquad \kappa = \frac{2-p}{p} < 1$

$$\Rightarrow \boxed{k(\Sigma[u|u_\infty]) \leq \frac{1}{2\lambda_1} |\Sigma'|} = \frac{1}{2\lambda_1} \int \psi'' \left(\frac{u}{u_\infty}\right) |\nabla \frac{u}{u_\infty}|^2 \, u_\infty \, dx$$

Refined convex Sobolev inequality with $x \leq k(x) = \frac{1+x-(1+x)^\kappa}{1-\kappa}$

- Set $\frac{u}{u_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}} \, u_\infty \, dx} \Rightarrow$ Refined Beckner inequality [Arnold, J.D.]

$$\frac{1}{2}\left(\frac{p}{p-1}\right)^2 \left[\int f^2 \, u_\infty \, dx - \left(\int |f|^{\frac{2}{p}} \, u_\infty \, dx\right)^{2(p-1)} \left(\int f^2 \, u_\infty \, dx\right)^{\frac{2-p}{p}}\right]$$

$$\leq \frac{2}{\lambda_1} \int |\nabla f|^2 \, u_\infty \, dx \quad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^d, \, u_\infty \, dx)$$

...another formulation: $p \mapsto 2/p$ (inversion on $(0,1)$)

A refined interpolation inequality

Theorem 2 *[Arnold, J.D.] For all* $p \in [1,2)$

$$\frac{1}{(2-p)^2}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2\left(\frac{2}{p}-1\right)}\left(\int_{\mathbb{R}^d} f^2\,d\nu\right)^{p-1}\right] \leq \frac{1}{\kappa}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu$$

for any $f \in H^1(d\nu)$, *where* κ *is the uniform convexity bound of* $-\log\nu(x)$

The generalized Poincaré inequality is a consequence of this result:

use Hölder's inequality, $\left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p} \leq \int_{\mathbb{R}^d} f^2\,d\nu$

use the inequality $(1-t^{2-p})/(2-p) \geq 1-t$ for any $t \in [0,1]$, $p \in (1,2)$

Gaussian measures

[W. Beckner, 1989]: a family of generalized Poincaré inequalities (GPI)

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\mu - \left(\int_{\mathbb{R}^d} |f|^p\,d\mu\right)^{2/p}\right] \leq \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu \quad \forall f \in H^1(d\mu) \qquad (1)$$

where $\mu(x) := \frac{e^{-\frac{1}{2}|x|^2}}{(2\pi)^{d/2}}$ denotes the normal centered Gaussian distribution on $\mathbb{R}^d$. For $p = 1$: the Poincaré inequality

$$\int_{\mathbb{R}^d} f^2\,d\mu - \left(\int_{\mathbb{R}^d} f\,d\mu\right)^2 \leq \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu \quad \forall f \in H^1(d\mu)$$

In the limit $p \to 2$: the logarithmic Sobolev inequality (LSI) [L. Gross 1975]

$$\int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right)\,d\mu \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu \quad \forall\; f \in H^1(d\mu)$$

Sufficient conditions for generalized Poincaré inequalities ?

[AMTU]: for strictly log-concave distribution functions $\nu(x)$

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \le \frac{1}{\kappa}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \quad \forall f\in H^1(d\nu)$$

where κ is the uniform convexity bound of $-\log\nu(x)$...

...the Bakry-Emery criterion

[Latała and Oleszkiewicz]: under the weaker assumption that $\nu(x)$ satisfies a LSI with constant $0<\mathcal{C}<\infty$

$$\int_{\mathbb{R}^d} f^2\log\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\nu}\right)d\nu \le 2\,\mathcal{C}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \quad \forall\, f\in H^1(d\nu) \qquad (2)$$

for $1\le p<2$, L-O proved that

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \le \mathcal{C}\min\left\{\frac{2}{p},\frac{1}{2-p}\right\}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu$$

Proof of the result of Latała and Oleszkiewicz (1/2)

The function $q \mapsto \alpha(q) := q \log \left(\int_{\mathbb{R}^d} |f|^{2/q} \, d\nu \right)$ is convex since

$$\alpha''(q) = \frac{4}{q^3} \, \frac{\left(\int_{\mathbb{R}^d} |f|^{2/q} \left(\log |f|\right)^2 \, d\nu \right) \left(\int_{\mathbb{R}^d} |f|^{2/q} \, d\nu \right) - \left(\int_{\mathbb{R}^d} |f|^{2/q} \log |f| \, d\nu \right)^2}{\left(\int_{\mathbb{R}^d} |f|^{2/q} \, d\nu \right)^2}$$

is nonnegative: $q \mapsto e^{\alpha(q)}$ is also convex, $\varphi(q) := \frac{e^{\alpha(1)} - e^{\alpha(q)}}{q-1}$ is $\searrow$

$$\varphi(q) \leq \lim_{q_1 \to 1} \varphi(q_1) = \int_{\mathbb{R}^d} f^2 \log \left(\frac{f^2}{\|f\|^2_{L^2(d\mu)}} \right) d\nu$$

$$\frac{1}{q-1} \left[\int_{\mathbb{R}^d} f^2 \, d\nu - \left(\int_{\mathbb{R}^d} |f|^{2/q} \, d\nu \right)^q \right] \leq 2\mathcal{C} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\nu$$

$$\tfrac{1}{q-1} \left[\int_{\mathbb{R}^d} f^2 \, d\nu - \left(\int_{\mathbb{R}^d} |f|^{2/q} \, d\nu \right)^q \right] = \tfrac{p}{2-p} \left[\int_{\mathbb{R}^d} f^2 \, d\nu - \left(\int_{\mathbb{R}^d} |f|^p \, d\nu \right)^{2/p} \right]$$

if $p = 2/q$: $C_p \leq 2\,\mathcal{C}/p$

Proof of the result of Latała and Oleszkiewicz (2/2)

Linearization $f = 1 + \varepsilon\, g$ with $\int_{\mathbb{R}^d} g\, d\nu = 0$, limit $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} f\, d\nu \right)^2 \leq \mathfrak{C} \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu$$

Hölder's inequality, $\left(\int_{\mathbb{R}^d} f\, d\nu \right)^2 \leq \left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu \right)^q$

$$\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu \right)^q \leq \int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} f\, d\nu \right)^2 \leq \mathfrak{C} \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu$$

Generalized Poincaré inequalities for the Gaussian measure

The spectrum of the Ornstein-Uhlenbeck operator $\mathsf{N} := -\Delta + x \cdot \nabla$ is made of all nonnegative integers $k \in \mathbb{N}$, the corresponding eigenfunctions are the Hermite polynomials. Observe that

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu = \int_{\mathbb{R}^d} f \cdot \mathsf{N} f \, d\mu \quad \forall \, f \in H^1(d\mu)$$

Strategy of Beckner (improved): consider the $L^2(d\mu)$-orthogonal decomposition of f on the eigenspaces of N, i.e.

$$f = \sum_{k \in \mathbb{N}} f_k \,,$$

where $\mathsf{N} \, f_k = k \, f_k$. If we denote by π_k the orthogonal projection on the eigenspace of N associated to the eigenvalue $k \in \mathbb{N}$, then $f_k = \pi_k[f]$.

$$a_k := \|f_k\|^2_{L^2(d\mu)} \,, \quad \|f\|^2_{L^2(d\mu)} = \sum_{k\in\mathbb{N}} a_k \quad \text{and} \quad \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu = \sum_{k\in\mathbb{N}} k \, a_k$$

The solution of the evolution equation associated to N

$$u_t = -\mathsf{N}\, u = \Delta u - x\cdot\nabla u \,, \quad u(t=0) = f$$

is given by $u(x,t) = \left(e^{-t\,\mathsf{N}}\, f\right)(x) = \sum_{k\in\mathbb{N}} e^{-k\,t}\, f_k(x)$

$$\left\|e^{-t\,\mathsf{N}}\, f\right\|^2_{L^2(d\mu)} = \sum_{k\in\mathbb{N}} e^{-2\,k\,t} a_k$$

Lemma 3 *Let $f \in H^1(d\mu)$. If $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0 \geq 1$, then*

$$\int_{\mathbb{R}^d} |f|^2 \, d\mu - \int_{\mathbb{R}^d} \left|e^{-t\,\mathsf{N}} f\right|^2 \, d\mu \leq \frac{1-e^{-2k_0\,t}}{k_0} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu$$

Proof

We use the decomposition on the eigenspaces of N

$$\int_{\mathbb{R}^d} |f_k|^2 \, d\mu - \int_{\mathbb{R}^d} \left| e^{-t\,\mathsf{N}} f_k \right|^2 \, d\mu = \left(1 - e^{-2\,k\,t}\right) a_k$$

For any fixed $t > 0$, the function $k \mapsto \frac{1-e^{-2\,k\,t}}{k}$ is monotone decreasing: if $k \geq k_0$, then

$$1 - e^{-2\,k\,t} \leq \frac{1 - e^{-2\,k_0\,t}}{k_0}\, k$$

Thus we get

$$\int_{\mathbb{R}^d} |f_k|^2 \, d\mu - \int_{\mathbb{R}^d} \left| e^{-t\,\mathsf{N}} f_k \right|^2 \, d\mu \leq \frac{1 - e^{-2\,k_0\,t}}{k_0} \int_{\mathbb{R}^d} |\nabla f_k|^2 \, d\mu$$

which proves the result by summation $\square$

Nelson's hypercontractive estimates

Lemma 4 *For any $f \in L^p(d\mu)$, $p \in (1,2)$, it holds*

$$\left\| e^{-t\mathsf{N}} f \right\|_{L^2(d\mu)} \leq \| f \|_{L^p(d\mu)} \quad \forall\, t \geq -\frac{1}{2} \log(p-1)$$

Proof. We set $F(t) := \left(\int_{\mathbb{R}^d} |u(t)|^{q(t)}\, d\mu \right)^{1/q(t)}$ with $q(t)$ to be chosen later and $u(x,t) := \left(e^{-t\mathsf{N}} f \right)(x)$. A direct computation gives

$$\frac{F'(t)}{F(t)} = \frac{q'(t)}{q^2(t)} \int_{\mathbb{R}^d} \frac{|u|^q}{F^q} \log\left(\frac{|u|^q}{F^q} \right) d\mu - \frac{4}{F^q}\, \frac{q-1}{q^2} \int_{\mathbb{R}^d} \left| \nabla \left(|u|^{q/2} \right) \right|^2 d\mu$$

We set $v := |u|^{q/2}$, use the logarithmic Sobolev inequality with $\nu = \mu$ and $\mathcal{C} = 1$, and choose q such that $4\,(q-1) = 2q'$, $q(0) = p$ and $q(t) = 2$. This implies $F'(t) \leq 0$ and ends the proof with $2 = q(t) = 1 + (p-1)\, e^{2t}$ $\square$

A generalization of Beckner's estimates

[Arnold, Bartier, J.D.] First result, for the Gaussian distribution $\mu(x)$

Theorem 5 *Let* $f \in H^1(d\mu)$*. If* $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ *for some* $k_0 \geq 1$*, then*

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} |f|^2 \, d\mu - \left(\int_{\mathbb{R}^d} |f|^p \, d\mu\right)^{2/p}\right] \leq \frac{1-(p-1)^{k_0}}{k_0\,(2-p)} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu$$

holds for $1 \leq p < 2$

- In the special case $k_0 = 1$ this is exactly the generalized Poincaré inequality due to Beckner, and for $k_0 > 1$ it is a strict improvement for any $p \in [1, 2)$
- Easy to generalize to other measures

$$\nu(x) := e^{-V(x)}$$

using the spectrum of $\mathsf{N} := -\Delta + \nabla V \cdot \nabla$

Equations de Poincaré généralisées

Coll. J. Carrillo, J.D. , I. Gentil, A. Jüngel

Higher order diffusion equations

The one dimensional porous medium/fast diffusion equation

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \,, \quad x \in S^1 \,, \quad t > 0$$

The thin film equation

$$u_t = -(u^m \, u_{xxx})_x \,, \quad x \in S^1 \,, \quad t > 0$$

The Derrida-Lebowitz-Speer-Spohn (DLSS) equation

$$u_t = -(u \, (\log u)_{xx})_{xx} \,, \quad x \in S^1 \,, \quad t > 0$$

... with initial condition $u(\cdot, 0) = u_0 \geq 0$ in $S^1 \equiv [0, 1)$

Entropies and energies

Averages:

$$\mu_p[v] := \left(\int_{S^1} v^{1/p} \, dx \right)^p \quad \text{and} \quad \bar{v} := \int_{S^1} v \, dx$$

Entropies: $p \in (0, +\infty)$, $q \in \mathbb{R}$, $v \in H^1_+(S^1)$, $v \not\equiv 0 \, a.e.$

$$\Sigma_{p,q}[v] := \frac{1}{p\,q\,(p\,q-1)} \left[\int_{S^1} v^q \, dx - (\mu_p[v])^q \right] \qquad \text{if } p\,q \neq 1 \text{ and } q \neq 0 \,,$$

$$\Sigma_{1/q,q}[v] := \int_{S^1} v^q \, \log\left(\frac{v^q}{\int_{S^1} v^q \, dx} \right) dx \qquad \text{if } p\,q = 1 \text{ and } q \neq 0 \,,$$

$$\Sigma_{p,0}[v] := -\frac{1}{p} \int_{S^1} \log\left(\frac{v}{\mu_p[v]} \right) dx \qquad \text{if } q = 0$$

Convexity

$\Sigma_{p,q}[v]$ is non-negative by convexity of

$$u \mapsto \frac{u^{p\,q} - 1 - p\,q\,(u-1)}{p\,q\,(p\,q-1)} =: \sigma_{p,q}(u)$$

By Jensen's inequality,

$$\begin{aligned}
\Sigma_{p,q}[v] &= \mu_p[v]^q \int_{S^1} \sigma_{p,q}\left(\frac{v^{1/p}}{(\mu_p[v])^{1/p}}\right) dx \\
&\geq \mu_p[v]^q \sigma_{p,q}\left(\int_{S^1} \frac{v^{1/p}}{(\mu_p[v])^{1/p}}\, dx\right) = \mu_p[v]^q \sigma_{p,q}(1) = 0
\end{aligned}$$

Limit cases

$p\,q = 1$:

$$\lim_{p \to 1/q} \Sigma_{p,q}[v] = \Sigma_{1/q,q}[v] \quad \text{for } q > 0$$

$q = 0$:

$$\lim_{q \to 0} \Sigma_{p,q}[v] = \Sigma_{p,0}[v] \quad \text{for } p > 0$$

$p = q = 0$:

$$\Sigma_{0,0}[v] = -\int_{S^1} \log\left(\frac{v}{\|v\|_\infty}\right) dx$$

Some references (>2005):

[M. J. Cáceres, J. A. Carrillo, and G. Toscani]

[M. Gualdani, A. Jüngel, and G. Toscani]

[A. Jüngel and D. Matthes]

[R. Laugesen]

Global functional inequalities

Theorem 1 *For all $p \in (0,+\infty)$ and $q \in (0,2)$, there exists a positive constant $\kappa_{p,q}$ such that, for any $v \in H^1_+(S^1)$,*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{\kappa_{p,q}} J_1[v] := \frac{1}{\kappa_{p,q}} \int_{S^1} |v'|^2 \, dx$$

Corollary 1 *Let $p \in (0,+\infty)$ and $q \in (0,2)$. Then, for any $v \in H^1_+(S^1)$,*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{4\pi^2\, \kappa_{p,q}} J_2[v] := \frac{1}{4\pi^2\, \kappa_{p,q}} \int_{S^1} |v''|^2 \, dx$$

A minimizing sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(S^1)$

$$v_n \rightharpoonup v \quad \text{in } H^1(S^1) \quad \text{and} \quad \Sigma_{p,q}[v_n] \to \Sigma_{p,q}[v] \quad \text{as} \quad n \to \infty$$

If $\Sigma_{p,q}[v] = 0$, $\lim_{n\to\infty} J_1[v_n] = 0$. Let $\varepsilon_n := J_1[v_n]$, $w_n := \frac{v_n - 1}{\sqrt{\varepsilon_n}}$ and make a Taylor expansion

$$\left| \, (1+\sqrt{\varepsilon}\, x)^{1/p} - 1 - \frac{\sqrt{\varepsilon}}{p}\, x \, \right| \le \frac{1}{p}\, r(\varepsilon_0, p)\, \varepsilon \quad \forall\, (x,\varepsilon) \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \times (0, \varepsilon_0)$$

$$\varepsilon_n := J_1[v_n]\ , \quad \Sigma_{p,q}[v_n] \le c(\varepsilon_0, p, q)\, \varepsilon_n$$

Hence, since $q < 2$,

$$\frac{J_1[v_n]}{\Sigma_{p,q}[v_n]^{2/q}} = \frac{\varepsilon_n\, J_1[w_n]}{\Sigma_{p,q}[v_n]^{2/q}} \ge [c(\varepsilon_0, p, q)]^{-2/q}\, \varepsilon_n^{1-2/q} \to \infty$$

gives a contradiction

Asymptotic functional inequalities

The regime of small entropies:

$$\mathcal{X}_\varepsilon^{p,q} := \left\{ v \in H^1_+(S^1) \;:\; \Sigma_{p,q}[v] \le \varepsilon \text{ and } \mu_p[v] = 1 \right\}$$

Theorem 2 *For any $p > 0$, $q \in \mathbb{R}$ and $\varepsilon_0 > 0$, there exists a positive constant C such that, for any $\varepsilon \in (0, \varepsilon_0]$,*

$$\Sigma_{p,q}[v] \le \frac{1 + C\sqrt{\varepsilon}}{8\, p^2\, \pi^2}\, J_1[v] \quad \forall\, v \in \mathcal{X}_\varepsilon^{p,q}$$

Without the condition $\mu_p[v] = 1$:

$$\Sigma_{p,q}[v] \le \frac{1 + C\sqrt{\varepsilon}}{8\, p^2\, \pi^2}\, (\mu_p[v])^{q-2}\, J_1[v]$$

If $J_1[v] \leq 8\,p^2\,\pi^2\,\varepsilon$, define $w := (v-1)/(\kappa_p^\infty\,\sqrt{\varepsilon})$: $J_1[w] \leq 1$.

$$\begin{aligned}
\Sigma_{p,q}[v] &= \frac{1}{pq(pq-1)}\left[\int_{S^1}(1+\kappa_p^\infty\sqrt{\varepsilon}w)^q dx - \left(\int_{S^1}(1+\kappa_p^\infty\sqrt{\varepsilon}w)^{1/p} dx\right)^{pq}\right] \\
&= \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\left[\int_{S^1} w^2\;dx - \left(\int_{S^1} w\;dx\right)^2\right] + O(\varepsilon^{3/2}) \\
&= \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\int_{S^1}(w-\bar{w})^2\;dx + O(\varepsilon^{3/2}) \\
&\leq \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\,\frac{J_1[w]}{(2\pi)^2} + O(\varepsilon^{3/2}) = \frac{J_1[v]}{8\,p^2\,\pi^2} + O(\varepsilon^{3/2})
\end{aligned}$$

using Poincaré's inequality

1^{st} application: Porous media

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \quad x \in S^1,\ t > 0$$

A one parameter family of entropies :

$$\Sigma_k[u] := \begin{cases} \dfrac{1}{k\,(k+1)} \displaystyle\int_{S^1} \left(u^{k+1} - \bar{u}^{k+1}\right)\ dx & \text{if} \quad k \in \mathbb{R} \setminus \{-1, 0\} \\ \displaystyle\int_{S^1} u \log\left(\frac{u}{\bar{u}}\right)\ dx & \text{if} \quad k = 0 \\ -\displaystyle\int_{S^1} \log\left(\frac{u}{\bar{u}}\right)\ dx & \text{if} \quad k = -1 \end{cases}$$

With $v := u^p$, $p := \frac{m+k}{2}$, $q := \frac{k+1}{p} = 2\,\frac{k+1}{m+k}$, $\Sigma_k[u] = \Sigma_{p,q}[v]$

Lemma 1 *Let $k \in \mathbb{R}$. If u is a smooth positive solution*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \lambda \int_{S^1} \left|(u^{(k+m)/2})_x\right|^2 dx = 0$$

with $\lambda := 4\,m/(m+k)^2$ whenever $k+m \neq 0$, and

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \lambda \int_{S^1} |(\log u)_x|^2 \, dx = 0$$

with $\lambda := m$ for $k+m=0$.

Decay rates

Proposition 1 *Let* $m \in (0, +\infty)$, $k \in \mathbb{R} \setminus \{-m\}$, $q = 2\,(k+1)/(m+k)$, $p = (m+k)/2$ *and* u *be a smooth positive solution*

i) *Short-time Algebraic Decay: If* $m > 1$ *and* $k > -1$, *then*

$$\Sigma_k[u(\cdot,t)] \le \left[\Sigma_k[u_0]^{-(2-q)/q} + \frac{2-q}{q}\,\lambda\,\kappa_{p,q}\,t\right]^{-q/(2-q)}$$

ii) *Asymptotically Exponential Decay: If* $m > 0$ *and* $m + k > 0$, *there exists* $C > 0$ *and* $t_1 > 0$ *such that for* $t \ge t_1$,

$$\Sigma_k[u(\cdot,t)] \le \Sigma_k[u(\cdot,t_1)]\,\exp\left(-\frac{8\,p^2\,\pi^2\,\lambda\,\bar{u}^{p(2-q)}\,(t-t_1)}{1 + C\sqrt{\Sigma_k[u(\cdot,t_1)]}}\right)$$

2^{nd} Application: fourth order equations

$$u_t = -\Big(u^m \;\big(u_{xxx} + a\,u^{-1}\,u_x\,u_{xx} + b\,u^{-2}\,u_x^3\big)\Big)_x\,, \quad x \in S^1,\; t > 0$$

Example 1. The thin film equation: $a = b = 0$

$$u_t = -(u^m\,u_{xxx})_x\,,$$

Example 2. The DLSS equation: $m = 0$, $a = -2$, and $b = 1$

$$u_t = -\Big(u\,(\log u)_{xx}\Big)_{xx}\,,$$

$L_\pm := \frac{1}{4}(3\,a + 5) \pm \frac{3}{4}\sqrt{(a-1)^2 - 8\,b}$

$A := (k + m + 1)^2 - 9\,(k + m - 1)^2 + 12\,a\,(k + m - 2) - 36\,b$

Theorem 3 *Assume* $(a-1)^2 \geq 8\,b$

i) Entropy production: If $L_- \leq k+m \leq L_+$

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] \leq 0 \quad \forall\, t>0$$

ii) Entropy production: If $k+m+1 \neq 0$ *and* $L_- < k+m < L_+$*,*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \mu \int_{S^1} \left|(u^{(k+m+1)/2})_{xx}\right|^2 dx \leq 0 \quad \forall\, t>0$$

If $k+m+1=0$ *and* $a+b+2-\mu \leq 0$ *for some* $0<\mu<1$*, then*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \mu \int_{S^1} |(\log u)_{xx}|^2\, dx \leq 0 \quad \forall\, t>0$$

Decay rates

Theorem 4 *Let* $k, m \in \mathbb{R}$ *be such that* $L_- \leq k + m \leq L_+$

i) Short-time Algebraic Decay: If $k > -1$ *and* $m > 0$*, then*

$$\Sigma_k[u(\cdot,t)] \leq \left[\Sigma_k[u_0]^{-(2-q)/q} + 4\pi^2\,\mu\,\kappa_{p,q}\left(\frac{2}{q}-1\right)\,t\right]^{-q/(2-q)}$$

ii) Asymptotically Exponential Decay: If $m + k + 1 > 0$*, then there exists* $C > 0$ *and* $t_1 > 0$ *such that*

$$\Sigma_k[u(\cdot,t)] \leq \Sigma_k[u(\cdot,t_1)]\,\exp\left(-\frac{32\,p^2\,\pi^4\,\mu\,\bar{u}^{p(2-q)}\,(t-t_1)}{1+C\sqrt{\Sigma_k[u(\cdot,t_1)]}}\right)$$

Inégalités de Poincaré L^q pour des mesures quelconques et application à l'équation des milieux poreux

J.D., Ivan Gentil, Arnaud Guillin and Feng-Yu Wang

Goal

L^q-Poincaré inequalities, $q \in (1/2, 1]$

$$\left[\mathbf{Var}_\mu(f^q)\right]^{1/q} := \left[\int f^{2q}\,d\mu - \left(\int f^q\,d\mu\right)^2\right]^{1/q} \leqslant \mathrm{C_P} \int |\nabla f|^2\,d\nu$$

Application to the weighted porous media equation, $m \geq 1$

$$\frac{\partial u}{\partial t} = \Delta u^m - \nabla\psi \cdot \nabla u^m\,, \quad t \geqslant 0\,, \quad x \in \mathbb{R}^d$$

(Ornstein-Uhlenbeck form). With $d\mu = d\nu = d\mu_\psi = e^{-\psi}\,dx / \int e^{-\psi}\,dx$

$$\frac{d}{dt}\mathbf{Var}_{\mu_\psi}(u) = -\,\frac{8}{(m+1)^2}\int |\,\nabla u^{\frac{m+1}{2}}\,|^2\,d\mu_\psi$$

Outline

Equivalence between the following properties:

- L^q-Poincaré inequality
- Capacity-measure criterion
- Weak Poincaré inequality
- BCR (Barthe-Cattiaux-Roberto) criterion

In dimension $d = 1$, there are necessary and sufficient conditions to satisfy the BCR criterion

Motivation: large time asymptotics in connection with functional inequalities

L^q-Poincaré inequality

M Riemanian manifold
Let μ a probability measure, ν a positive measure on M

We shall say that (μ, ν) satisfies a L^q-Poincaré inequality with constant $\mathrm{C_P}$ if for all non-negative functions $f \in \mathcal{C}^1(M)$ one has

$$\Big[\mathbf{Var}_\mu(f^q)\,\Big]^{1/q} \leqslant \mathrm{C_P} \int |\nabla f|^2\, d\nu$$

$q \in (0,1]$ (false for $q > 1$ unless μ is a Dirac measure)
$\mathbf{Var}_\mu(g^2) = \int g^2\, d\mu - \left(\int g\ d\mu\right)^2 = \mu(g^2) - \mu(g)^2$
$q \mapsto \Big[\mathbf{Var}_\mu(f^q)\,\Big]^{1/q}$ increasing wrt $q \in (0,1]$: L^q-Poincaré inequalities form a hierarchy

Capacity-measure criterion

Capacity $\mathrm{Cap}_\nu(A,\Omega)$ of two measurable sets A and Ω such that $A\subset\Omega\subset M$

$$\mathrm{Cap}_\nu(A,\Omega) := \inf\left\{\int |\nabla f|^2\,d\nu \;:\; f\in\mathcal{C}^1(M)\,,\ \mathbb{I}_A \leqslant f \leqslant \mathbb{I}_\Omega\right\}$$

$$\beta_{\mathrm{P}} := \sup\left\{\sum_{k\in\mathbb{Z}} \frac{\left[\mu(\Omega_k)\right]^{1/(1-q)}}{\left[\mathrm{Cap}_\nu(\Omega_k,\Omega_{k+1})\right]^{q/(1-q)}}\right\}^{(1-q)/q}$$

over all $\Omega\subset M$ with $\mu(\Omega)\leq 1/2$ and all sequences $(\Omega_k)_{k\in\mathbb{Z}}$ such that for all $k\in\mathbb{Z}$, $\Omega_k\subset\Omega_{k+1}\subset\Omega$

Theorem 1 (i) *If* $q\in[1/2,1)$, *then* $\beta_{\mathrm{P}}\leqslant 2^{1/q}\,\mathrm{C}_{\mathrm{P}}$

(ii) *If* $q\in(0,1)$ *and* $\beta_{\mathrm{P}}<+\infty$, *then* $\mathrm{C}_{\mathrm{P}}\leqslant\kappa_{\mathrm{P}}\,\beta_{\mathrm{P}}$

Weak Poincaré inequalities

Definition 2 [Röckner and Wang] (μ, ν) *satisfies a weak Poincaré inequality if there exists a non-negative non increasing function* $\beta_{\mathrm{WP}}(s)$ *on* $(0, 1/4)$ *such that, for any bounded function* $f \in \mathcal{C}^1(M)$*,*

$$\forall\, s > 0\,, \quad \mathbf{Var}_\mu(f) \leqslant \beta_{\mathrm{WP}}(s) \int |\nabla f|^2 \, d\nu + s \left[\mathbf{Osc}_\mu(f)\right]^2$$

$\mathbf{Var}_\mu(f) \leqslant \mu((f-a)^2)\ \forall\, a \in \mathbb{R}$

For $a = (\mathrm{supess}_\mu f + \mathrm{infess}_\mu f)/2$, $\mathbf{Var}_\mu(f) \leqslant \left[\mathbf{Osc}_\mu(f)\right]^2/4$: $s \leqslant 1/4$.

Proposition 3 *Let* $q \in [1/2, 1)$. *If* (μ, ν) *satisfies the* L^q*-Poincaré inequality, then it also satisfies a weak Poincaré inequality with* $\beta_{\mathrm{WP}}(s) = (11 + 5\sqrt{5})\, \beta_{\mathrm{P}}\, s^{1-1/q}/2$, $K := (11 + 5\sqrt{5})/2$.

L^q-Poincaré $\Longrightarrow$ BCR criterion $\Longrightarrow$ weak Poincaré

Theorem 4 [Maz'ja] *Let $q \in [1/2, 1)$. For all bounded open set $\Omega \subset M$, if $(\Omega_k)_{k\in\mathbb{Z}}$ is a sequence of open sets such that $\Omega_k \subset \Omega_{k+1} \subset \Omega$, then*

$$\sum_{k\in\mathbb{Z}} \frac{\mu(\Omega_k)^{1/(1-q)}}{[\mathrm{Cap}_\nu(\Omega_k, \Omega_{k+1})]^{q/(1-q)}} \leqslant \frac{1}{1-q} \int_0^{\mu(\Omega)} \left(\frac{t}{\Phi(t)}\right)^{q/(1-q)} dt$$

where $\Phi(t) := \inf\{\mathrm{Cap}_\nu(A, \Omega) \,:\, A \subset \Omega\,,\ \mu(A) \geqslant t\}$

As a consequence: $\beta_{\mathrm{P}} \leqslant (1-q)^{-(1-q)/q} \, \|t/\Phi(t)\|_{L^{q/(1-q)}(0,\mu(\Omega))}$

Corollary 5 *Let $q \in [1/2, 1)$. If (μ, ν) satisfies a weak Poincaré inequality with function β_{WP}, then it satisfies a L^q-Poincaré inequality with*

$$\beta_{\mathrm{P}} \leqslant \frac{11 + 5\sqrt{5}}{2} \left(\frac{4}{1-q}\right)^{\frac{1-q}{q}} \|\beta_{\mathrm{WP}}(\cdot/4)\|_{L^{\frac{q}{1-q}}(0,1/2)}$$

L^q-Poincaré $\Longrightarrow$ Weak Poincaré with $\beta_{\mathrm{WP}}(s) = C\, s^{\frac{q-1}{q}}$ $\Longrightarrow$ $L^{q'}$-Poincaré $\forall\, q' \in (0, q)$

BCR criterion (1/2)

A variant of two results of [Barthe, Cattiaux, Roberto, 2005] (no absolute continuity of the measure μ with respect to the volume measure)

Theorem 6 [BCR] *Let μ be a probability measure and ν a positive measure on M such that (μ, ν) satisfies a weak Poincaré inequality with function $\beta_{\mathrm{WP}}(s)$. Then for every measurable subsets A, B of M such that $A \subset B$ and $\mu(B) \leqslant 1/2$,*

$$\mathrm{Cap}_\nu(A, B) \geq \frac{\mu(A)}{\gamma(\mu(A))} \quad \textit{with} \quad \gamma(s) := 4\,\beta_{\mathrm{WP}}(s/4)$$

Proof $\triangleleft$ Take f such that $\mathbb{I}_A \leqslant f \leqslant \mathbb{I}_B$: $\mathbf{Osc}_\mu(f) \leqslant 1$

By Cauchy-Schwarz, $\left(\int f\,d\mu\right)^2 \leqslant \mu(B) \int f^2\,d\mu \leqslant \frac{1}{2} \int f^2\,d\mu$

$$\beta_{\mathrm{WP}}(s) \int |\nabla f|^2\,d\nu + s \geqslant \mathbf{Var}_\mu(f) \geq \frac{1}{2} \int f^2\,d\mu \geq \frac{\mu(A)}{2}$$

$\frac{a}{\gamma(a)} = \frac{a}{4\,\beta_{\mathrm{WP}}(a/4)} \leqslant \sup_{s \in (0,1/4)} \frac{a/2 - s}{\beta_{\mathrm{WP}}(s)}$ with $a/2 = \mu(A)/2 \leqslant 1/4$ $\triangleright$

BCR criterion (2/2)

Lemma 7 *Take μ and ν as before, $\theta \in (0,1)$, γ a positive non increasing function on $(0,\theta)$. If $\forall\ A, B \subset M$ such that $A \subset B$ are measurable and $\mu(B) \leqslant \theta$,*

$$\mathrm{Cap}_\nu(A,B) \geq \frac{\mu(A)}{\gamma(\mu(A))}$$

then for every function $f \in \mathcal{C}^1(M)$ such that $\mu(\Omega_+) \leqslant \theta$, $\Omega_+ := \{f > 0\}$

$$\int f_+^2 \leq \frac{11+5\sqrt{5}}{2}\,\gamma(s)\int_{\Omega_+} |\nabla f|^2\, d\nu + s\left[\text{supess}_\mu\, f\right]^2 \quad \forall\ s \in (0,1)$$

Theorem 8 *Same assumptions, $\theta = 1/2$. Then $\forall f \in \mathcal{C}^1(M)$*

$$\mathbf{Var}_\mu(f) \leq \frac{11+5\sqrt{5}}{2}\,\gamma(s)\int |\nabla f|^2\, d\nu + s\left[\mathbf{Osc}_\mu(f)\right] \quad \forall\ s \in (0,1/4)$$

$\theta = 1/2$: use the median $m_\mu(f)$, $\mu(f \geqslant m_\mu(f)) \geqslant 1/2$, $\mu(f \leqslant m_\mu(f)) \geqslant 1/2$

Using the BCR criterion: a "Hardy condition"

[Muckenhoupt, 1972] [Bobkov-Götze, 1999] [Barthe-Roberto, 2003] [Barthe-Cattiaux-Roberto, 2005]

$M = \mathbb{R}$, $d\mu = \rho_\nu \, dx$ with median m_μ, $d\nu = \rho_\nu \, dx$

$$R(x) := \mu([x, +\infty)) \; , \quad L(x) := \mu((-\infty, x])$$

$$r(x) := \int_{m_\mu}^{x} \frac{1}{\rho_\nu} \, dx \quad \text{and} \quad \ell(x) := \int_{x}^{m_\mu} \frac{1}{\rho_\nu} \, dx$$

Proposition 9 *Let $q \in [1/2, 1]$. (μ, ν) satisfies a L^q-Poincaré inequality if*

$$\int_{m_\mu}^{\infty} |\, r\, R \,|^{q/(1-q)} \, d\mu < \infty \quad \textit{and} \quad \int_{-\infty}^{m_\mu} |\, \ell\, L \,|^{q/(1-q)} \, d\mu < \infty$$

Proof

Proof $\lhd$ Method: $\mathbf{Var}_\mu(f) \leqslant \mu(|F_-|^2) + \mu(|F_+|^2))$ with $g = (f - f(m_\mu))_\pm$ and prove that

$$\mu(|g|^2) \leqslant \frac{11+5\sqrt{5}}{2}\,\gamma(s) \int |\nabla g|^2\, d\nu + s\left[\text{supess}_\mu g\right]^2 \quad \forall\, s \in (0,1/2)$$

Let $A \subset B \subset M = (m_\mu, \infty)$ such that $A \subset B$ and $\mu(B) \leqslant 1/2$

$$\mathrm{Cap}_\nu(A,B) \geqslant \mathrm{Cap}_\nu\big(A,(m_\mu,\infty)\big) = \mathrm{Cap}_\nu\big((a,\infty),(m_\mu,\infty)\big) = \frac{1}{r(a)}$$

where $a = \inf A$. Change variables: $t = R(a)$ and choose $\gamma(t) := t\,(r \circ R)^{-1}(t)$ for any $t \in (0,1/2)$ $\rhd$

Porous media equation

With $\psi \in \mathcal{C}^2(\mathbb{R}^d)$, $d\mu_\psi := \frac{e^{-\psi}\,dx}{Z_\psi}$, define $\mathcal{L}$ on $\mathcal{C}^2(\mathbb{R}^d)$ by

$$\forall\ f \in \mathcal{C}^2(\mathbb{R}^d) \quad \mathcal{L}f := \Delta f - \nabla\psi \cdot \nabla f$$

Such a generator $\mathcal{L}$ is symmetric in $L^2_{\mu_\psi}(\mathbb{R}^d)$,

$$\forall\ f,\ g \in \mathcal{C}^1(\mathbb{R}^d) \quad \int f\,\mathcal{L}g\,d\mu_\psi = -\int \nabla f \cdot \nabla g\,d\mu_\psi$$

Consider for $m > 1$ the weighted porous media equation

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}\,u^m & \text{in} \quad \mathrm{Q} \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega \\ n \cdot \nabla u = 0 & \text{on} \quad \Sigma \end{cases}$$

$\Omega \subset \mathbb{R}^d$, $Q = \Omega \times [0, +\infty)$, $\Sigma = \partial\Omega \times [0, +\infty)$

$u \in \mathcal{C}^2$, L^1-contraction, existence and uniqueness

Asymptotic behavior

Theorem 10 *Let* $m \geqslant 1$ *and assume that* (μ_ψ, μ_ψ) *satisfies a* L^q*-Poincaré inequality,* $q = 2/(m+1)$

$$\mathbf{Var}_{\mu_\psi}(u(\cdot,t)) \leqslant \left(\left[\mathbf{Var}_{\mu_\psi}(u_0) \right]^{-(m-1)/2} + \frac{4\,m\,(m-1)}{(m+1)^2}\, \mathrm{C_P}\, t \right)^{-2/(m-1)}$$

Reciprocally, if the above inequality is satisfied for any u_0*, then* (μ_ψ, μ_ψ) *satisfies a* L^q*-Poincaré inequality with constant* $\mathrm{C_P}$

Proof $\triangleleft$

$$\frac{d}{dt}\mathbf{Var}_{\mu_\psi}(u) = 2\int u_t\, u\, d\mu_\psi = 2\int u\, \mathcal{L} u^m\, d\mu_\psi = -\frac{8m}{(m+1)^2}\int |\nabla u^{\frac{m+1}{2}}|^2\, d\mu_\psi$$

Apply the L^q-Poincaré inequality with $u = f^{2/(m+1)}$, $q = 2/(m+1)$

Reciprocally, a derivation at $t = 0$ gives the L^q-Poincaré inequality $\triangleright$

A conclusion on L^q-Poincaré inequalities

- Observe that we have only algebraic rates
- Weak logarithmic Sobolev inequalities [Cattiaux-Gentil-Guillin, 2006], L^q-logarithmic Sobolev inequalities [D.-Gentil-Guillin-Wang, 2006]

$$\left(\int f^{2q} \frac{\log f^{2q}}{\int f^{2q}\, d\mu}\, d\mu\right) =: \mathbf{Ent}_\mu\left(f^{2q}\right)^{1/q} \leq \mathrm{C}_{\mathrm{LS}} \int |\nabla f|^2\, d\mu$$

- Orlicz spaces, duality, connections with mass transport theory [Bobkov-Götze, 1999] [Cattiaux-Gentil-Guillin, 2006] [Wang, 2006] [Roberto-Zegarlinski, 2003] [Barthe-Cattiaux-Roberto, 2005]

Perspectives

- Etudier des équations de diffusions avec des non-linéarités non-homogènes ou des termes de champs moyen, avec application à des problèmes de traffic automobile ou en maths-bio. Il s'agit de cataloguer des comportements qualitatifs pour contribuer à la modélisation.
- Limites de diffusion. On sait passer d'une équation cinétique à une équation de diffusion non-linéaire de type milieux poreux. Au niveau de l'équation de diffusion, on sait donner des taux de convergence. Qu'en est-il au niveau cinétique, en-dehors des quelques cas qui relèvent des méthodes d'hypo-ellipticité / hypo-coercivité ?

Perspectives

- Equations d'ordre quatre: comment contruire des relations entre entropies et production d'entropie qui ne soient pas basées sur la positivité purement algébrique, ponctuelle, des termes calculés en effectuant des dérivées, mais soient plutôt basés sur des inégalités fonctionnelles, si possible globales ?
- Equations de (dérive-)diffusion pour la mécanique quantique (faisant intervenir, par exemple, le terme de Bohm): quels taux de convergence? Quelles asymptotiques intermédiaires ? Quels flots-gradients pour quelles entropies ?