L^2 Hypocoercivity

Jean Dolbeault

http://www.ceremade.dauphine.fr/∼dolbeaul

Ceremade, Université Paris-Dauphine

October 16, 2019

CIRM conference on PDE/Probability Interactions
Particle Systems, Hyperbolic Conservation Laws
October 14-18, 2019

Outline

- **Abstract method and motivation**
  ▷ Abstract statement in a Hilbert space
  ▷ Diffusion limit, toy model
- **The compact case**
  ▷ Strong confinement
  ▷ Mode-by-mode decomposition
  ▷ Application to the torus
  ▷ Further results
- **The non-compact case**
  ▷ Without confinement: Nash inequality
  ▷ With very weak confinement: Caffarelli-Kohn-Nirenberg inequality
  ▷ With sub-exponential equilibria: weighted Poincaré inequality
- **The Vlasov-Poisson-Fokker-Planck system**
  ▷ Linearized system and hypocoercivity
  ▷ Results in the diffusion limit and in the non-linear case

Abstract method and motivation

- ▷ Abstract statement
- ▷ Diffusion limit
- ▷ A toy model

Collaboration with C. Mouhot and C. Schmeiser

An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$

$$\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$$

* denotes the adjoint with respect to $\langle\cdot,\cdot\rangle$

Π is the orthogonal projection onto the null space of L

The assumptions

λ_m, λ_M, and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity:*

$$-\langle \mathsf{L} F, F\rangle \geq \lambda_m \,\|(1-\Pi)F\|^2 \tag{H1}$$

▷ *macroscopic coercivity:*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \,\|\Pi F\|^2 \tag{H2}$$

▷ *parabolic macroscopic dynamics:*

$$\Pi \mathsf{T} \Pi \, F = 0 \tag{H3}$$

▷ *bounded auxiliary operators:*

$$\|\mathsf{A}\mathsf{T}(1-\Pi)F\| + \|\mathsf{A}\mathsf{L}F\| \leq C_M \,\|(1-\Pi)F\| \tag{H4}$$

The estimate

$$\frac{1}{2}\,\frac{d}{dt}\|F\|^2 = \langle \mathsf{L}F, F\rangle \leq -\,\lambda_m\,\|(1-\Pi)F\|^2$$

is not enough to conclude that $\|F(t,\cdot)\|^2$ decays exponentially

Equivalence and entropy decay

For some $\delta > 0$ to be determined later, the L^2 entropy / Lyapunov functional is defined by

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

so that $\langle \mathsf{AT}\Pi F, F\rangle \sim \|\Pi F\|^2$ and

$$\begin{aligned} -\frac{d}{dt}\mathsf{H}[F] &=: \mathsf{D}[F] \\ &= -\,\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT}\Pi F, F\rangle \\ &\quad -\,\delta\,\mathrm{Re}\langle \mathsf{TA}F, F\rangle + \delta\,\mathrm{Re}\langle \mathsf{AT}(1-\Pi)F, F\rangle - \delta\,\mathrm{Re}\langle \mathsf{AL}F, F\rangle \end{aligned}$$

▷ *entropy decay rate:* for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$\lambda\,\mathsf{H}[F] \le \mathsf{D}[F]$$

▷ *norm equivalence* of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \frac{2+\delta}{4}\,\|F\|^2$$

Exponential decay of the entropy

$$\lambda = \tfrac{\lambda_M}{3\,(1+\lambda_M)}\,\min\left\{1,\lambda_m,\tfrac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\},\;\delta=\tfrac{1}{2}\,\min\left\{1,\lambda_m,\tfrac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\}$$

$$h_1(\delta,\lambda) := (\delta\,C_M)^2 - 4\left(\lambda_m - \delta - \frac{2+\delta}{4}\,\lambda\right)\left(\frac{\delta\,\lambda_M}{1+\lambda_M} - \frac{2+\delta}{4}\,\lambda\right)$$

Theorem

Let L *and* T *be closed linear operators (respectively Hermitian and anti-Hermitian) on* $\mathcal{H}$. *Under* (H1)–(H4), *for any* $t \ge 0$

$$\mathsf{H}[F(t,\cdot)] \le \mathsf{H}[F_0]\,e^{-\lambda_\star t}$$

where $\lambda_\star$ *is characterized by*

$$\lambda_\star := \sup\left\{\lambda>0\,:\,\exists\,\delta>0\;s.t.\;h_1(\delta,\lambda)=0\,,\;\lambda_m-\delta-\tfrac{1}{4}\,(2+\delta)\,\lambda>0\right\}$$

Sketch of the proof

- Since $\mathsf{AT\Pi} = \big(1 + (\mathsf{T\Pi})^* \mathsf{T\Pi}\big)^{-1} (\mathsf{T\Pi})^* \mathsf{T\Pi}$, from (H1) and (H2)

$$-\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT\Pi}F, F\rangle \ge \lambda_m \,\|(1-\Pi)F\|^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,\|\Pi F\|^2$$

- By (H4), we know that

$$|\mathrm{Re}\langle \mathsf{AT}(1-\Pi)F, F\rangle + \mathrm{Re}\langle \mathsf{AL}F, F\rangle| \le C_M\,\|\Pi F\|\,\|(1-\Pi)F\|$$

- The equation $G = \mathsf{A}F$ is equivalent to $(\mathsf{T\Pi})^* F = G + (\mathsf{T\Pi})^*\,\mathsf{T\Pi}\,G$

$$\langle \mathsf{TA}F, F\rangle = \langle G, (\mathsf{T\Pi})^*\,F\rangle = \|G\|^2 + \|\mathsf{T\Pi}G\|^2 = \|\mathsf{A}F\|^2 + \|\mathsf{TA}F\|^2$$

$$\langle G, (\mathsf{T\Pi})^*\,F\rangle \le \|\mathsf{TA}F\|\,\|(1-\Pi)F\| \le \frac{1}{2\,\mu}\,\|\mathsf{TA}F\|^2 + \frac{\mu}{2}\,\|(1-\Pi)F\|^2$$

$$\|\mathsf{A}F\| \le \frac{1}{2}\,\|(1-\Pi)F\|\,,\ \ \|\mathsf{TA}F\| \le \|(1-\Pi)F\|\,,\ \ |\langle \mathsf{TA}F, F\rangle| \le \|(1-\Pi)F\|^2$$

- With $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$

$$\mathsf{D}[F] - \lambda\,\mathsf{H}[F] \ge (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{2+\delta}{4}\,\lambda\,(X^2+Y^2)$$

Hypocoercivity

Corollary

For any $\delta \in (0,2)$, if $\lambda(\delta)$ is the largest positive root of $h_1(\delta,\lambda)=0$ for which $\lambda_m - \delta - \frac{1}{4}(2+\delta)\,\lambda > 0$, then for any solution F of the evolution equation

$$\|F(t)\|^2 \le \frac{2+\delta}{2-\delta}\, e^{-\lambda(\delta)\,t}\, \|F(0)\|^2 \quad \forall\, t \ge 0$$

From the norm equivalence of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \frac{2+\delta}{4}\,\|F\|^2$$

We use $\frac{2-\delta}{4}\,\|F_0\|^2 \le \mathsf{H}[F_0]$ so that $\lambda_\star \ge \sup_{\delta\in(0,2)} \lambda(\delta)$

Formal macroscopic (diffusion) limit

Scaled evolution equation

$$\varepsilon \frac{dF}{dt} + \mathsf{T}F = \frac{1}{\varepsilon}\mathsf{L}F$$

on the Hilbert space $\mathcal{H}$. $F_\varepsilon = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$

$$\begin{aligned} \varepsilon^{-1} : \quad & \mathsf{L}F_0 = 0\,, \\ \varepsilon^{0} : \quad & \mathsf{T}F_0 = \mathsf{L}F_1\,, \\ \varepsilon^{1} : \quad & \tfrac{dF_0}{dt} + \mathsf{T}F_1 = \mathsf{L}F_2 \end{aligned}$$

The first equation reads as $u = F_0 = \Pi F_0$
The second equation is simply solved by $F_1 = -\,(\mathsf{T}\Pi)\,F_0$
After projection, the third equation is

$$\tfrac{d}{dt}\,(\Pi F_0) - \,\Pi\mathsf{T}\,(\mathsf{T}\Pi)\,F_0 = \Pi\mathsf{L}F_2 = 0$$

$$\partial_t u + (\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)\,u = 0$$

is such that $\frac{d}{dt}\|u\|^2 = -\,2\,\|(\mathsf{T}\Pi)\,u\|^2 \le -\,2\,\lambda_M\,\|u\|^2$

A toy problem

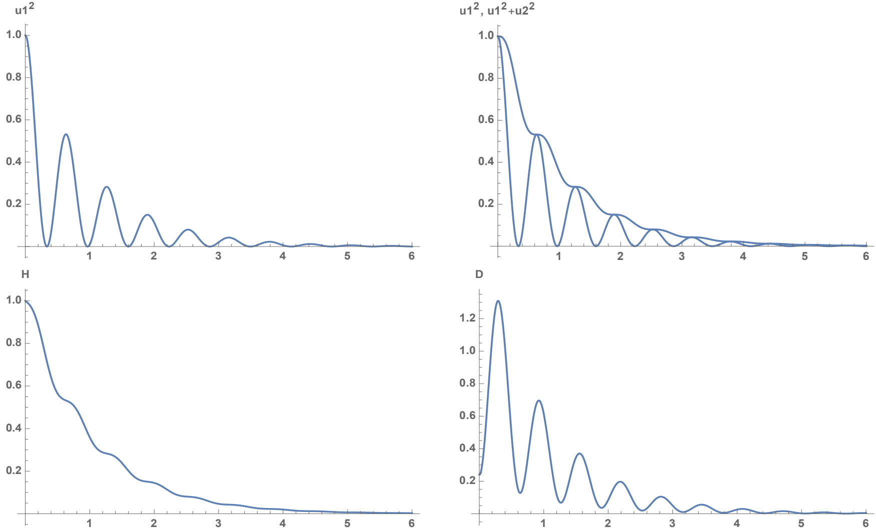
$$\frac{du}{dt} = (\mathsf{L}-\mathsf{T})\, u\,, \quad \mathsf{L} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{T} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad k^2 \geq \Lambda > 0$$

Non-monotone decay, a well known picture:
see for instance (Filbet, Mouhot, Pareschi, 2006)

- H-theorem: $\frac{d}{dt}|u|^2 = \frac{d}{dt}\left(u_1^2 + u_2^2\right) = -\,2\,u_2^2$
- macroscopic/diffusion limit: $\left(\frac{du_1}{dt} = -\,k^2\,u_1\right)$
- generalized entropy: $\mathsf{H}(u) = |u|^2 - \frac{\delta\,k}{1+k^2}\,u_1\,u_2$

$$\begin{aligned} \frac{d\mathsf{H}}{dt} &= -\left(2 - \frac{\delta\,k^2}{1+k^2}\right) u_2^2 - \frac{\delta\,k^2}{1+k^2}\,u_1^2 + \frac{\delta\,k}{1+k^2}\,u_1\,u_2 \\ &\leq -(2-\delta)\,u_2^2 - \frac{\delta\Lambda}{1+\Lambda}\,u_1^2 + \frac{\delta}{2}\,u_1 u_2 \end{aligned}$$

Plots for the toy problem



Some references

- C. Mouhot and L. Neumann. Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus. Nonlinearity, 19(4):969-998, 2006
- F. Hérau. Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation. Asymptot. Anal., 46(3-4):349-359, 2006
- J. Dolbeault, P. Markowich, D. Oelz, and C. Schmeiser. Non linear diffusions as limit of kinetic equations with relaxation collision kernels. Arch. Ration. Mech. Anal., 186(1):133-158, 2007.
- J. Dolbeault, C. Mouhot, and C. Schmeiser. Hypocoercivity for kinetic equations with linear relaxation terms. Comptes Rendus Mathématique, 347(9-10):511 - 516, 2009
- J. Dolbeault, C. Mouhot, and C. Schmeiser. Hypocoercivity for linear kinetic equations conserving mass. Transactions of the American Mathematical Society, 367(6):3807-3828, 2015

The compact case

- Fokker-Planck equation and scattering collision operators

▷ A mode-by-mode (Fourier) hypocoercivity result

▷ Enlargement of the space by factorization

▷ Application to the torus and numerical improvements

- Further results: Euclidean space with strong confinement

Collaboration with E. Bouin, S. Mischler, C. Mouhot, C. Schmeiser

Fokker-Planck and scattering collision operators

Two basic examples:

- Linear *Fokker-Planck* collision operator

$$\mathsf{L} f = \Delta_v f + \nabla_v \cdot (v\, f)$$

- Linear relaxation operator (linear BGK)

$$\mathsf{L} f = \rho\, (2\pi)^{-d/2}\, \exp(-|v|^2/2) - f$$

with $\rho = \int f\, dv$

Fokker-Planck equation with general equilibria

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f \,, \quad f(0,x,v) = f_0(x,v)$$

for a distribution function $f(t,x,v)$, with *position* variable $x \in \mathbb{R}^d$ or $x \in \mathbb{T}^d$ the flat d-dimensional torus

Fokker-Planck collision operator with a general equilibrium M

$$\mathsf{L} f = \nabla_v \cdot \Big[M \, \nabla_v \left(M^{-1} f \right) \Big]$$

Notation and assumptions: an *admissible local equilibrium* M is positive, radially symmetric and

$$\int_{\mathbb{R}^d} M(v)\,dv = 1 \,, \quad d\gamma = \gamma(v)\,dv := \frac{dv}{M(v)}$$

γ is an *exponential weight* if

$$\lim_{|v|\to\infty} \frac{|v|^k}{\gamma(v)} = \lim_{|v|\to\infty} M(v)\,|v|^k = 0 \quad \forall\, k \in (d,\infty)$$

Definitions

$$\Theta = \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 \, M(v) \, dv = \int_{\mathbb{R}^d} (v \cdot \mathsf{e})^2 \, M(v) \, dv$$

for an arbitrary $\mathsf{e} \in \mathbb{S}^{d-1}$

$$\int_{\mathbb{R}^d} v \otimes v \, M(v) \, dv = \Theta \, \mathrm{Id}$$

Then

$$\theta = \frac{1}{d} \, \|\nabla_v M\|^2_{\mathrm{L}^2(d\gamma)} = \frac{4}{d} \int_{\mathbb{R}^d} \big|\nabla_v \sqrt{M}\big|^2 \, dv < \infty$$

If $M(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}}$, then $\Theta = 1$ and $\theta = 1$

$$\overline{\sigma} := \frac{1}{2} \sqrt{\theta/\Theta}$$

Microscopic coercivity property (Poincaré inequality): for all $u = M^{-1} F \in \mathrm{H}^1(M \, dv)$

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, M \, dv \geq \lambda_m \int_{\mathbb{R}^d} \left(u - \int_{\mathbb{R}^d} u \, M \, dv \right)^2 M \, dv$$

Scattering collision operators

Scattering collision operator

$$\mathsf{L} f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \left(f(v')\, M(\cdot) - f(\cdot)\, M(v') \right) dv'$$

Main assumption on the *scattering rate* σ: for some positive, finite $\overline{\sigma}$

$$1 \le \sigma(v, v') \le \overline{\sigma} \quad \forall\, v,\, v' \in \mathbb{R}^d$$

Example: linear BGK operator

$$\mathsf{L} f = M \rho_f - f\,, \quad \rho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv$$

Local mass conservation

$$\int_{\mathbb{R}^d} \mathsf{L} f \, dv = 0$$

and we have

$$\int_{\mathbb{R}^d} |\mathsf{L} f|^2 \, d\gamma \le 4\, \overline{\sigma}^2 \int_{\mathbb{R}^d} |M \rho_f - f|^2 \, d\gamma$$

The symmetry condition

$$\int_{\mathbb{R}^d} \big(\sigma(v,v') - \sigma(v',v)\big)\, M(v')\, dv' = 0 \quad \forall\, v \in \mathbb{R}^d$$

implies the *local mass conservation* $\int_{\mathbb{R}^d} \mathsf{L} f\, dv = 0$

Micro-reversibility, *i.e.*, the symmetry of σ, is not required

The null space of L is spanned by the local equilibrium M
L only acts on the velocity variable

Microscopic coercivity property: for some $\lambda_m > 0$

$$\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \sigma(v,v')\, M(v)\, M(v')\, \big(u(v) - u(v')\big)^2\, dv'\, dv \ge \lambda_m \int_{\mathbb{R}^d} \big(u - \rho_u\, M\big)^2\, M\, dv$$

holds according to Proposition 2.2 of (Degond, Goudon, Poupaud, 2000) for all $u = M^{-1}\, F \in \mathrm{L}^2(M\, dv)$. If $\sigma \equiv 1$, then $\lambda_m = 1$

Mode-by-mode decomposition

▷ Spectral decomposition (Hermite functions): linear Fokker-Planck operator

$$\mathsf{L} f = \Delta_v f + \nabla_v \cdot (v\, f)$$

or the linear relaxation operator (linear BGK)

$$\mathsf{L} f = \rho\,(2\pi)^{-d/2}\,\exp(-|v|^2/2) - f\,,$$

with $\rho = \int f\,dv$: (Arnold, Erb), (Achleitner, Arnold, Stürzer), (Achleitner, Arnold, Carlen), (Arnold, Einav, Wöhrer)

▷ Decomposition in Fourier modes

Fourier modes

In order to perform a *mode-by-mode hypocoercivity* analysis, we introduce the Fourier representation with respect to x,

$$f(t,x,v)=\int_{\mathbb{R}^d}\hat f(t,\xi,v)\,e^{-i\,x\cdot\xi}\,d\mu(\xi)$$

$d\mu(\xi)=(2\pi)^{-d}\,d\xi$ and $d\xi$ is the Lesbesgue measure if $x\in\mathbb{R}^d$
$d\mu(\xi)=(2\pi)^{-d}\sum_{z\in\mathbb{Z}^d}\delta(\xi-z)$ is discrete for $x\in\mathbb{T}^d$

Parseval's identity if $\xi\in\mathbb{Z}^d$ and Plancherel's formula if $x\in\mathbb{R}^d$ read

$$\|f(t,\cdot,v)\|_{\mathrm{L}^2(dx)}=\left\|\hat f(t,\cdot,v)\right\|_{\mathrm{L}^2(d\mu(\xi))}$$

The Cauchy problem is now decoupled in the ξ-direction

$$\partial_t\hat f+\mathsf{T}\hat f=\mathsf{L}\hat f\,,\quad \hat f(0,\xi,v)=\hat f_0(\xi,v)$$

$$\mathsf{T}\hat f=i\,(v\cdot\xi)\,\hat f$$

For any fixed $\xi \in \mathbb{R}^d$, let us apply the abstract result with

$$\mathcal{H} = \mathrm{L}^2\,(d\gamma)\;, \quad \|F\|^2 = \int_{\mathbb{R}^d} |F|^2\,d\gamma\,, \quad \Pi F = M \int_{\mathbb{R}^d} F\,dv = M\,\rho_F$$

and $\mathsf{T}\hat{f} = i\,(v\cdot\xi)\,\hat{f}$, $\mathsf{T}\Pi F = i\,(v\cdot\xi)\,\rho_F\,M$,

$$\|\mathsf{T}\Pi F\|^2 = |\rho_F|^2 \int_{\mathbb{R}^d} |v\cdot\xi|^2\,M(v)\,dv \;= \Theta\,|\xi|^2\,|\rho_F|^2 = \Theta\,|\xi|^2\,\|\Pi F\|^2$$

(H2) *Macroscopic coercivity* $\|\mathsf{T}\Pi F\|^2 \ge \lambda_M\,\|\Pi F\|^2 \,:\, \lambda_M = \Theta\,|\xi|^2$

(H3) $\int_{\mathbb{R}^d} v\,M(v)\,dv = 0$

The operator A is given by

$$\mathsf{A}F = \frac{-\,i\,\xi\cdot\int_{\mathbb{R}^d} v'\,F(v')\,dv'}{1+\Theta\,|\xi|^2}\,M$$

A mode-by-mode hypocoercivity result

$$\begin{aligned}\|\mathsf{A}F\| = \|\mathsf{A}(1-\Pi)F\| &\le \frac{1}{1+\Theta\,|\xi|^2}\int_{\mathbb{R}^d}\frac{|(1-\Pi)F|}{\sqrt{M}}\,|v\cdot\xi|\,\sqrt{M}\,dv\\ &\le \frac{1}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|\left(\int_{\mathbb{R}^d}(v\cdot\xi)^2\,M\,dv\right)^{1/2}\\ &= \frac{\sqrt{\Theta}\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|\end{aligned}$$

- Scattering operator $\|\mathsf{L}F\|^2 \le 4\,\overline{\sigma}^2\,\|(1-\Pi)F\|^2$
- Fokker-Planck (FP) operator

$$\|\mathsf{AL}F\| \le \frac{2}{1+\Theta\,|\xi|^2}\int_{\mathbb{R}^d}\frac{|(1-\Pi)F|}{\sqrt{M}}\,|\xi\cdot\nabla_v\sqrt{M}|\,dv \le \frac{\sqrt{\theta}\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

In both cases with $\kappa = \sqrt{\theta}$ (FP) or $\kappa = 2\,\overline{\sigma}\,\sqrt{\Theta}$ we obtain

$$\|\mathsf{AL}F\| \le \frac{\kappa\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

$$\mathsf{TA}F(v) = -\frac{(v\cdot\xi)\,M}{1+\Theta\,|\xi|^2}\int_{\mathbb{R}^d}(v'\cdot\xi)\,(1-\Pi)F(v')\,dv'$$

is estimated by

$$\|\mathsf{TA}F\| \le \frac{\Theta\,|\xi|^2}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

(H4) holds with $C_M = \frac{\kappa\,|\xi|+\Theta\,|\xi|^2}{1+\Theta\,|\xi|^2}$

The two "good" terms

$$-\langle\mathsf{L}F,F\rangle \ge \lambda_m\,\|(1-\Pi)F\|^2$$

$$\langle\mathsf{AT}\Pi F,F\rangle = \mathsf{A}F = \frac{\Theta\,|\xi|^2}{1+\Theta\,|\xi|^2}\,\|\Pi F\|^2$$

Two elementary estimates

$$\frac{\Theta\,|\xi|^2}{1+\Theta\,|\xi|^2} \ge \frac{\Theta}{\max\{1,\Theta\}}\,\frac{|\xi|^2}{1+|\xi|^2}\,, \quad \frac{\lambda_M}{(1+\lambda_M)\,C_M^2} = \frac{\Theta\left(1+\Theta\,|\xi|^2\right)}{\left(\kappa+\Theta\,|\xi|\right)^2} \ge \frac{\Theta}{\kappa^2+\Theta}$$

Mode-by-mode hypocoercivity with exponential weights

Theorem

Let us consider an admissible M and a collision operator L satisfying the assumptions, and take $\xi \in \mathbb{R}^d$. If $\hat{f}$ is a solution such that $\hat{f}_0(\xi,\cdot) \in \mathrm{L}^2(d\gamma)$, then for any $t \ge 0$, we have

$$\left\| \hat{f}(t,\xi,\cdot) \right\|^2_{\mathrm{L}^2(d\gamma)} \le 3\, e^{-\mu_\xi\, t} \left\| \hat{f}_0(\xi,\cdot) \right\|^2_{\mathrm{L}^2(d\gamma)}$$

where

$$\mu_\xi := \frac{\Lambda\,|\xi|^2}{1+|\xi|^2} \quad \text{and} \quad \Lambda = \frac{\Theta}{3\,\max\{1,\Theta\}}\,\min\left\{1, \frac{\lambda_m\,\Theta}{\kappa^2+\Theta}\right\}$$

with $\kappa = 2\,\overline{\sigma}\,\sqrt{\Theta}$ for scattering operators
and $\kappa = \sqrt{\theta}$ for (FP) operators

Exponential convergence to equilibrium in $\mathbb{T}^d$

The unique global equilibrium in the case $x \in \mathbb{T}^d$ is given by

$$f_\infty(x,v) = \rho_\infty \, M(v) \quad \text{with} \quad \rho_\infty = \frac{1}{|\mathbb{T}^d|} \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dx \, dv$$

Theorem

Assume that γ has an exponential growth. We consider an admissible M, a collision operator L satisfying the assumptions. There exists a positive constant C such that the solution f of the Cauchy problem on $\mathbb{T}^d \times \mathbb{R}^d$ with initial datum $f_0 \in \mathrm{L}^2(dx\,d\gamma)$ satisfies

$$\|f(t,\cdot,\cdot) - f_\infty\|_{\mathrm{L}^2(dx\,d\gamma)} \le C \, \|f_0 - f_\infty\|_{\mathrm{L}^2(dx\,d\gamma)} \, e^{-\frac{1}{4}\Lambda t} \quad \forall\, t \ge 0$$

Enlargement of the space by factorization

A simple case (factorization of order 1) of the *factorization method* of (Gualdani, Mischler, Mouhot)

Theorem

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be Banach spaces and let $\mathcal{B}_2$ be continuously imbedded in $\mathcal{B}_1$, i.e., $\|\cdot\|_1 \le c_1 \|\cdot\|_2$. Let $\mathfrak{B}$ and $\mathfrak{A}+\mathfrak{B}$ be the generators of the strongly continuous semigroups $e^{\mathfrak{B}\,t}$ and $e^{(\mathfrak{A}+\mathfrak{B})\,t}$ on $\mathcal{B}_1$. If for all $t \ge 0$,

$$\left\|e^{(\mathfrak{A}+\mathfrak{B})\,t}\right\|_{2\to2} \le c_2\, e^{-\lambda_2\,t}\,, \quad \left\|e^{\mathfrak{B}t}\right\|_{1\to1} \le c_3\, e^{-\lambda_1\,t}\,, \quad \|\mathfrak{A}\|_{1\to2} \le c_4$$

where $\|\cdot\|_{i\to j}$ denotes the operator norm for linear mappings from $\mathcal{B}_i$ to $\mathcal{B}_j$. Then there exists a positive constant $C = C(c_1, c_2, c_3, c_4)$ such that, for all $t \ge 0$,

$$\left\|e^{(\mathfrak{A}+\mathfrak{B})\,t}\right\|_{1\to1} \le \begin{cases} C\left(1+|\lambda_1-\lambda_2|^{-1}\right)\, e^{-\min\{\lambda_1,\lambda_2\}\,t} & \text{for } \lambda_1 \ne \lambda_2 \\ C\,(1+t)\, e^{-\lambda_1\,t} & \text{for } \lambda_1 = \lambda_2 \end{cases}$$

Exponential convergence to equilibrium in $\mathbb{T}^d$

The unique global equilibrium in the case $x \in \mathbb{T}^d$ is given by

$$f_\infty(x,v) = \rho_\infty\, M(v) \quad \text{with} \quad \rho_\infty = \frac{1}{|\mathbb{T}^d|} \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0\, dx\, dv$$

Theorem

Assume that $k \in (d, \infty]$ and

$$d\gamma_k := \gamma_k(v)\, dv \quad \textit{where} \quad \gamma_k(v) = \left(1 + |v|^2\right)^{k/2} \quad \textit{and} \quad k > d$$

We consider an admissible M, a collision operator L satisfying the assumptions
There exists a positive constant C_k such that the solution f of the Cauchy problem on $\mathbb{T}^d \times \mathbb{R}^d$ with initial datum $f_0 \in \mathrm{L}^2(dx\, d\gamma_k)$ satisfies

$$\|f(t,\cdot,\cdot) - f_\infty\|_{\mathrm{L}^2(dx\, d\gamma_k)} \le C_k\, \|f_0 - f_\infty\|_{\mathrm{L}^2(dx\, d\gamma_k)}\, e^{-\frac{1}{4}\Lambda\, t} \quad \forall\, t \ge 0$$

Numerical improvements

Where do we have space for ***numerical improvements*** ?

- With $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$, we wrote

$$\mathsf{D}[F] - \lambda\,\mathsf{H}[F]$$

$$\geq (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{\lambda}{2}\left(X^2 + Y^2 + \delta\,X\,Y\right)$$

$$\geq (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{2+\delta}{4}\,\lambda\,(X^2 + Y^2)$$

- We can directly study the positivity condition for the quadratic form

$$(\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{\lambda}{2}\left(X^2 + Y^2 + \delta\,X\,Y\right)$$

$\lambda_m = 1$, $\lambda_M = |\xi|^2$ and $C_M = |\xi|\,(1+|\xi|)/(1+|\xi|^2)$

- Look for the optimal value of ε

$$\mathsf{A}_\varepsilon F = \frac{-\,i\,\xi\cdot\int_{\mathbb{R}^d} v'\,F(v')\,dv'}{\varepsilon + |\xi|^2}\,M$$

Euclidean space, confinement, Poincaré inequality

(H1) *Regularity & Normalization:* $V \in W^{2,\infty}_{\rm loc}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} e^{-V}\,dx = 1$

(H2) *Spectral gap condition:* for some $\Lambda > 0$, $\forall\, u \in H^1(e^{-V}dx)$ such that $\int_{\mathbb{R}^d} u\, e^{-V}\,dx = 0$

$$\int_{\mathbb{R}^d} |u|^2\, e^{-V}\,dx \le \Lambda \int_{\mathbb{R}^d} |\nabla_x u|^2\, e^{-V}\,dx$$

(H3) *Pointwise conditions:*

there exists $c_0 > 0$, $c_1 > 0$ and $\theta \in (0,1)$ s.t.

$$\Delta V \le \frac{\theta}{2}\,|\nabla_x V(x)|^2 + c_0\,, \quad |\nabla_x^2 V(x)| \le c_1\,(1 + |\nabla_x V(x)|)\ \forall\, x \in \mathbb{R}^d$$

(H4) *Growth condition:* $\int_{\mathbb{R}^d} |\nabla_x V|^2\, e^{-V}\,dx < \infty$

Theorem (D., Mouhot, Schmeiser)

Let L *be either a Fokker-Planck operator or a linear relaxation operator with a local equilibrium* $F(v) = (2\pi)^{-d/2}\exp(-|v|^2/2)$*. If* f *solves*

$$\partial_t f + v\cdot\nabla_x f - \nabla_x V\cdot\nabla_v f = \mathsf{L} f$$

then

$$\forall\, t \ge 0\,, \quad \|f(t) - F\|^2 \le (1+\eta)\,\|f_0 - F\|^2\, e^{-\lambda t}$$

Further references

- Exponential rates in kinetic equations: (Talay 2001), (Wu 2001)
- in presence of a strongly confining potential (Hérau 2006 & 2007), (Mouhot, Neumann, 2006)
- hypo-elliptic methods (Hérau, Nier 2004), (Eckmann, Hairer, 2003), (Hörmander, 1967), (Kolmogorov, 1934), (Il'in, Has'min'ski, 1964) with applications to the Vlasov-Poisson-Fokker-Planck equation: (Victory, O'Dwyer, 1990), (Bouchut, 1993)
- Related topics:

▷ H^1-hypocoercivity (Villani...), (Gallay)
▷ Diffusion limits (Degond, Poupaud, Schmeiser, Goudon,...)
▷ Poincaré inequalities and Lyapunov functions
▷ Harris type methods, use of coupling

The non-compact case

- Nash's inequality and a decay rate when $V = 0$
- The global picture ▷ by what can we replace the Poincaré inequality ?
- Very weak confinement: Caffarelli-Kohn-Nirenberg inequalities and moments
- With sub-exponential equilibria: weighted Poincaré / Hardy-Poincaré

A result based on Nash's inequality

$$\mathsf{D}[f] = -\frac{d}{dt}\mathsf{H}[f] \geq \mathsf{a}\left(\|(1-\Pi)f\|^2 + 2\,\langle \mathsf{AT\Pi}f, f\rangle\right)$$

$$g = \mathsf{A}f = \left(1 + (\mathsf{T\Pi})^*\mathsf{T\Pi}\right)^{-1}(\mathsf{T\Pi})^* f \quad\Longleftrightarrow\quad g = u_f\, M$$

where

$$u_f - \Theta\,\Delta u_f = -\nabla_x \cdot \left(\int_{\mathbb{R}^d} v\, f\, dv\right)$$

We observe that, for any $t \geq 0$,

$$\|u_f(t,\cdot)\|_{\mathrm{L}^1(dx)} = \|\rho_f(t,\cdot)\|_{\mathrm{L}^1(dx)} = \|f_0\|_{\mathrm{L}^1(dx\,dv)}$$

$$\|\nabla_x u_f\|^2_{\mathrm{L}^2(dx)} \leq \frac{1}{\Theta}\,\langle \mathsf{AT\Pi}f, f\rangle$$

$$\|\Pi f\|^2 \leq \|u_f\|^2_{\mathrm{L}^2(dx)} + 2\,\langle \mathsf{AT\Pi}f, f\rangle$$

Nash's inequality

$$\|u\|^2_{\mathrm{L}^2(dx)} \le \mathcal{C}_{Nash}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(dx)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(dx)} \quad \forall\, u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)$$

Use $\|\Pi f\|^2 \le \Phi^{-1}\big(2\,\langle \mathsf{AT}\Pi f, f\rangle\big)$ with $\Phi^{-1}(y) := y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}}$ to get

$$\|(1-\Pi)f\|^2 + 2\,\langle \mathsf{AT}\Pi f, f\rangle \ge \Phi(\|f\|^2) \ge \Phi\Big(\tfrac{2}{1+\delta}\,\mathsf{H}[f]\Big)$$

$$\mathsf{D}[f] = -\frac{d}{dt}\mathsf{H}[f] \ge \mathsf{a}\,\Phi\Big(\tfrac{2}{1+\delta}\,\mathsf{H}[f]\Big)$$

Algebraic decay rates in $\mathbb{R}^d$

$V = 0$: On the whole Euclidean space, we can define the entropy

$$\mathsf{H}[f] := \frac{1}{2}\,\|f\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} + \delta\,\langle \mathsf{A}f, f\rangle_{dx\,d\gamma_k}$$

Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|^2_{\mathrm{L}^2(dx)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(dx)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(dx)}$$

proves that

$$\mathsf{H}[f] \le C\left(\mathsf{H}[f_0] + \|f_0\|^2_{\mathrm{L}^1(dx\,dv)}\right)(1+t)^{-\frac{d}{2}}$$

Theorem

Assume that γ_k has an exponential growth ($k=\infty$) or a polynomial growth of order $k>d$

There exists a constant $C>0$ such that, for any $t\ge 0$

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} \le C\left(\|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} + \|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(dx))}\right)(1+t)^{-\frac{d}{2}}$$

Improved decay rate for zero average solutions

Theorem

Assume that $f_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ *with* $\iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x,v)\,dx\,dv = 0$ *and*
$\mathcal{C}_0 := \|f_0\|^2_{\mathrm{L}^2(d\gamma_{k+2};\,\mathrm{L}^1(dx))} + \|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(|x|\,dx))} + \|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} < \infty$

Then there exists a constant $c_k > 0$ *such that*

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} \le c_k\,\mathcal{C}_0\,(1+t)^{-\left(1+\frac{d}{2}\right)}$$

The global picture

- Depending on the local equilibria and on the external potential (H1) and (H2) (which are Poincaré type inequalities) can be replaced by other functional inequalities:

▷ *microscopic coercivity* (H1)

$$-\langle \mathsf{L} F, F\rangle \geq \lambda_m \left\|(1-\Pi)F\right\|^2$$

$\Longrightarrow$ ***weak Poincaré inequalities*** or ***Hardy-Poincaré inequalities***

▷ *macroscopic coercivity* (H2)

$$\left\|\mathsf{T}\Pi F\right\|^2 \geq \lambda_M \left\|\Pi F\right\|^2$$

$\Longrightarrow$ ***Nash inequality***, ***weighted Nash*** or ***Caffarelli-Kohn-Nirenberg inequalities***

- This can be done at the level of the *diffusion equation* (homogeneous case) or at the level of the *kinetic equation* (non-homogeneous case)

Diffusion (Fokker-Planck) equations

Potential	$V=0$	$V(x)=\gamma\,\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\ge 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2\,(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 1: $\partial_t u=\Delta u+\nabla\cdot(n\,\nabla V)$

Kinetic Fokker-Planck equations

B = Bouin, L = Lafleche, M = Mouhot, MM = Mischler, Mouhot
S = Schmeiser

Potential	$V = 0$	$V(x) = \gamma \log\lvert x\rvert$ $\gamma < d$	$V(x) = \lvert x\rvert^\alpha$ $\alpha \in (0,1)$	$V(x) = \lvert x\rvert^\alpha$ $\alpha \geq 1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v) = e^{-\langle v\rangle^\beta}$, $\beta \geq 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b < 1$, $\beta = 2$ convergence	DMS, Mischler-Mouhot $e^{-\lambda t}$ convergence
$F(v) = e^{-\langle v\rangle^\beta}$, $\beta \in (0,1)$	BDLS: $t^{-\zeta}$, $\zeta =$ $\min\left\{\frac{d}{2}, \frac{k}{\beta}\right\}$ decay			
$F(v) = \langle v\rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 1: $\partial_t f + v \cdot \nabla_x f = F\, \nabla_v \big(F^{-1}\, \nabla_v f\big)$. Notation: $\langle v\rangle = \sqrt{1+|v|^2}$

Further references

- *Weak Poincaré inequality*: (Röckner & Wang, 2001), (Kavian, Mischler), (Cao, PhD thesis), (Hu, Wang, 2019) + (Ben-Artzi, Einav) for recent spectral considerations
- *Weighted Nash inequalities*: (Bakry, Bolley, Gentil, Maheux, 2012), (Wang, 2000, 2002, 2010)
- Related topics:
  ▷ fractional diffusion (Cattiaux, Puel, Fournier, Tardif,...)

Very weak confinement: Caffarelli-Kohn-Nirenberg

In collaboration with Emeric Bouin and Christian Schmeiser

The macroscopic Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \Delta_x u + \nabla_x \cdot (\nabla_x V\, u) = \nabla_x \left(e^{-V}\, \nabla_x \left(e^V\, u\right)\right)$$

Here $x \in \mathbb{R}^d$, $d \geq 3$, and V is a potential such that $e^{-V} \notin \mathrm{L}^1(\mathbb{R}^d)$ corresponding to a *very weak confinement*

Two examples

$$V_1(x) = \gamma\, \log|x| \quad \text{and} \quad V_2(x) = \gamma\, \log\langle x\rangle$$

with $\gamma < d$ and $\langle x\rangle := \sqrt{1+|x|^2}$ for any $x \in \mathbb{R}^d$

Potential	$V = 0$	$V(x) = \gamma \log \lvert x \rvert$ $\gamma < d$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \in (0,1)$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \geq 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 2: $\partial_t u = \Delta u + \nabla \cdot (n \nabla V)$

A first decay result

Theorem

Assume that $d \geq 3$, $\gamma < (d-2)/2$ and $V = V_1$ or $V = V_2$
F any solution u with initial datum $u_0 \in \mathrm{L}^1_+ \cap \mathrm{L}^2(\mathbb{R}^d)$,

$$\|u(t,\cdot)\|_2^2 \leq \frac{\|u_0\|_2^2}{(1+c\,t)^{\frac{d}{2}}} \quad \textit{with} \quad c := \frac{4}{d} \min\left\{1, 1-\tfrac{2\,\gamma}{d-2}\right\} \mathcal{C}_{\mathrm{Nash}}^{-1} \frac{\|u_0\|_2^{4/d}}{\|u_0\|_1^{4/d}}$$

Here $\mathcal{C}_{\mathrm{Nash}}$ denotes the optimal constant in Nash's inequality

$$\|u\|_2^{2+\frac{4}{d}} \leq \mathcal{C}_{\mathrm{Nash}} \, \|u\|_1^{\frac{4}{d}} \, \|\nabla u\|_2^2 \quad \forall\, u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)$$

An extended range of exponents: with moments

Theorem

Let $d \ge 1$, $0 < \gamma < d$, $V = V_1$ *or* $V = V_2$, *and* $u_0 \in \mathrm{L}^1_+ \cap \mathrm{L}^2(e^V)$ *with* $\big\| |x|^k u_0 \big\|_1 < \infty$ *for some* $k \ge \max\{2, \gamma/2\}$

$$\forall\, t \ge 0\,, \quad \|u(t,\cdot)\|^2_{\mathrm{L}^2(e^V\,dx)} \le \|u_0\|^2_{\mathrm{L}^2(e^V\,dx)}\,(1 + c\,t)^{-\frac{d-\gamma}{2}}$$

for some c *depending on* d, γ, k, $\|u_0\|_{\mathrm{L}^2(e^V\,dx)}$, $\|u_0\|_1$, *and* $\big\| |x|^k u_0 \big\|_1$

An extended range of exponents: in self-similar variables

$$u_\star(t,x) = \frac{c_\star}{(1+2\,t)^{\frac{d-\gamma}{2}}}\,|x|^{-\gamma}\exp\left(-\frac{|x|^2}{2\,(1+2\,t)}\right)$$

Here the initial data need have a sufficient decay...
$c_\star$ is chosen such that $\|u_\star\|_1 = \|u_0\|_1$

Theorem

Let $d \ge 1$, $\gamma \in (0,d)$, $V = V_1$ assume that

$$\forall\, x \in \mathbb{R}^d\,, \quad 0 \le u_0(x) \le K\,u_\star(0,x)$$

for some constant $K > 1$

$$\forall\, t \ge 0\,, \quad \|u(t,\cdot) - u_\star(t,\cdot)\|_p \le K\,c_\star^{1-\frac{1}{p}}\,\|u_0\|_1^{\frac{1}{p}}\left(\tfrac{e}{2\,|\gamma|}\right)^{\frac{\gamma}{2}\left(1-\frac{1}{p}\right)}(1+2\,t)^{-\zeta_p}$$

for any $p \in [1,+\infty)$, where $\zeta_p := \frac{d}{2}\left(1-\frac{1}{p}\right) + \frac{1}{2\,p}\,\min\left\{2, \frac{d}{d-\gamma}\right\}$

Proofs: basic case

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2\,dx = -\,2\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} \Delta V\,|u|^2\,dx$$

with either $V = V_1$ or $V = V_2$ and

$$\Delta V_1(x) = \gamma\,\frac{d-2}{|x|^2} \quad \text{and} \quad \Delta V_2(x) = \gamma\,\frac{d-2}{1+|x|^2} + \frac{2\,\gamma}{(1+|x|^2)^2}$$

For $\gamma \le 0$: apply Nash's inequality

$$\frac{d}{dt}\,\|u\|_2^2 \le -\,2\,\|\nabla u\|_2^2 \le -\,\frac{2}{\mathcal{C}_{\mathrm{Nash}}}\,\|u_0\|_1^{-4/d}\,\|u\|_2^{2+4/d}$$

For $0 < \gamma < (d-2)/2$: Hardy-Nash inequalities

Lemma

Let $d \ge 3$ and $\delta < (d-2)^2/4$

$$\|u\|_2^{2+\frac{4}{d}} \le \mathcal{C}_\delta\left(\|\nabla u\|_2^2 - \delta\int_{\mathbb{R}^d}\frac{u^2}{|x|^2}\,dx\right)\|u\|_1^{\frac{4}{d}} \quad \forall\,u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)$$

Proofs: moments

Growth of the moment

$$M_k(t) := \int_{\mathbb{R}^d} |x|^k u \, dx$$

From the equation

$$M_k' = k\left(d + k - 2 - \gamma\right) \int_{\mathbb{R}^d} u \, |x|^{k-2} \, dx \le k\left(d + k - 2 - \gamma\right) M_0^{\frac{2}{k}} \, M_k^{1-\frac{2}{k}}$$

then use the Caffarelli-Kohn-Nirenberg inequality

$$\int_{\mathbb{R}^d} |x|^\gamma \, u^2 \, dx \le \mathcal{C} \left(\int_{\mathbb{R}^d} |x|^{-\gamma} \, |\nabla\left(|x|^\gamma u\right)|^2 \, dx \right)^a \left(\int_{\mathbb{R}^d} |x|^k \, |u| \, dx \right)^{2(1-a)}$$

Proofs: self-similar solutions

The proof relies on *uniform decay estimates* + Poincaré inequality in self-similar variables

Proposition

Let $\gamma \in (0,d)$ *and assume that*

$$0 \le u(0,x) \le c_\star \left(\sigma + |x|^2\right)^{-\gamma/2} \exp\left(-\frac{|x|^2}{2}\right) \quad \forall\, x \in \mathbb{R}^d$$

with $\sigma = 0$ *if* $V = V_1$ *and* $\sigma = 1$ *if* $V = V_2$*. Then*

$$0 \le u(t,x) \le \frac{c_\star}{(1+2\,t)^{\frac{d-\gamma}{2}}} \left(\sigma + |x|^2\right)^{-\gamma/2} \exp\left(-\frac{|x|^2}{2\,(1+2\,t)}\right)$$

for any $x \in \mathbb{R}^d$ *and* $t \ge 0$

The kinetic Fokker-Planck equation

Let us consider the kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \mathsf{L} f$$

where $\mathsf{L} f$ is one of the two following collision operators

(a) a Fokker-Planck operator

$$\mathsf{L} f = \nabla_v \cdot \Big(M \, \nabla_v \left(M^{-1} \, f \right) \Big)$$

(b) a scattering collision operator

$$\mathsf{L} f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \left(f(v') \, M(\cdot) - f(\cdot) \, M(v') \right) dv'$$

Potential	$V=0$	$V(x)=\gamma\,\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\ge1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v)=e^{-\langle v\rangle^\beta}$, $\beta\ge1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b<1$, $\beta=2$ convergence	DMS, Mischler- Mouhot $e^{-\lambda t}$ convergence
$F(v)=e^{-\langle v\rangle^\beta}$, $\beta\in(0,1)$	BDLS: $t^{-\zeta}$, $\zeta=$ $\min\left\{\frac{d}{2},\frac{k}{\beta}\right\}$ decay			
$F(v)=\langle v\rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 2: $\partial_t f+v\cdot\nabla_x f=F\,\nabla_v\big(F^{-1}\,\nabla_v f\big)$. Notation: $\langle v\rangle=\sqrt{1+\lvert v\rvert^2}$

Decay rates

$$\forall\,(x,v)\in\mathbb{R}^d\times\mathbb{R}^d\,,\quad \mathcal{M}(x,v)=M(v)\,e^{-V(x)}\,,\quad M(v)=(2\pi)^{-\frac{d}{2}}\,e^{-\frac{1}{2}\,|v|^2}$$

$$\textbf{(H1)}\quad 1\le\sigma(v,v')\le\overline{\sigma}\,,\quad \forall\,v\,,\,v'\in\mathbb{R}^d\,,\quad \text{for some}\quad \overline{\sigma}\ge1$$

$$\textbf{(H2)}\quad \int_{\mathbb{R}^d}\big(\sigma(v,v')-\sigma(v',v)\big)\,M(v')\,dv'=0\quad\forall\,v\in\mathbb{R}^d$$

Theorem

Let $d\ge1$, $V=V_2$ *with* $\gamma\in[0,d)$, $k>\max\{2,\gamma/2\}$ *and* $f_0\in\mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)$ *such that*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^k\,f_0\,dx\,dv+\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^k\,f_0\,dx\,dv<+\infty$$

If (H1)–(H2) *hold, then there exists* $C>0$ *such that*

$$\forall\,t\ge0\,,\quad \|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)}\le C\,(1+t)^{-\frac{d-\gamma}{2}}$$

With sub-exponential equilibria

▷ The homogeneous Fokker-Planck equation with sub-exponential equilibria $F(v) = C_\alpha \, e^{-\langle v \rangle^\alpha}$, $\alpha \in (0,1)$
– decay rates based on the weak Poincaré inequality (Kavian, Mischler)
– decay rates based on a weighted Poincaré / Hardy-Poincaré inequality

▷ The kinetic Fokker-Planck equation with sub-exponential local equilibria and ***no confinement***, the equation with linear scattering

In collaboration with Emeric Bouin, Laurent Lafleche and Christian Schmeiser

Fokker-Planck with sub-exponential equilibria

We consider the ***homogeneous Fokker-Planck equation***

$$\partial_t g = \nabla_v \cdot \Big(F \, \nabla_v \big(F^{-1} \, g \big) \Big)$$

associated with ***sub-exponential equilibria***

$$F(v) = C_\alpha \, e^{-\langle v \rangle^\alpha} \,, \quad \alpha \in (0,1)$$

The corresponding Ornstein-Uhlenbeck equation for $h = g/F$ is

$$\partial_t h = F^{-1} \, \nabla_v \cdot \Big(F \, \nabla_v h \Big)$$

Potential	$V = 0$	$V(x) = \gamma \log\lvert x\rvert$ $\gamma < d$	$V(x) = \lvert x\rvert^\alpha$ $\alpha \in (0,1)$	$V(x) = \lvert x\rvert^\alpha$ $\alpha \ge 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 3: $\partial_t u = \Delta u + \nabla \cdot (n \nabla V)$

Weak Poincaré inequality

$$\int_{\mathbb{R}^d} \left|h - \widetilde{h}\right|^2 d\xi \le \mathcal{C}_{\alpha,\tau} \left(\int_{\mathbb{R}^d} |\nabla h|^2 \, d\xi\right)^{\frac{\tau}{1+\tau}} \left\|h - \widetilde{h}\right\|_{L^\infty(\mathbb{R}^d)}^{\frac{2}{1+\tau}}$$

for some explicit positive constant $\mathcal{C}_{\alpha,\tau}$, $\widetilde{h} := \int_{\mathbb{R}^d} h \, d\xi$. Using

$$\frac{d}{dt} \int_{\mathbb{R}^d} \left|h(t,\cdot) - \widetilde{h}\right|^2 d\xi = -\,2 \int_{\mathbb{R}^d} |\nabla_v h|^2 \, d\xi$$

where $h = g/F$ and $d\xi = F \, dv$ + Hölder's inequality

$$\int_{\mathbb{R}^d} \left|h - \widetilde{h}\right|^2 d\xi \le \left(\int_{\mathbb{R}^d} \left|h - \widetilde{h}\right|^2 \langle v\rangle^{-\beta} \, d\xi\right)^{\frac{\tau}{\tau+1}} \left(\int_{\mathbb{R}^d} \left\|h - \widetilde{h}\right\|_{L^\infty(\mathbb{R}^d)}^2 \langle v\rangle^{\beta\,\tau} \, d\xi\right)^{\frac{1}{1+\tau}}$$

with $(\tau+1)/\tau = \beta/\eta$, then for with $\mathcal{M} = \sup_{s\in(0,t)} \left\|h(s,\cdot) - \widetilde{h}\right\|_{L^\infty(\mathbb{R}^d)}^{2/\tau}$

$$\int_{\mathbb{R}^d} \left|h(t,\cdot) - \widetilde{h}\right|^2 d\xi \le \left(\left(\int_{\mathbb{R}^d} \left|h(0,\cdot) - \widetilde{h}\right|^2 d\xi\right)^{-\frac{1}{\tau}} + \frac{2\,\tau^{-1}}{\mathcal{C}_{\alpha,\tau}^{1+1/\tau}\,\mathcal{M}}\, t\right)^{-\tau}$$

Weighted Poincaré inequality

There exists a constant $\mathcal{C} > 0$ such that

$$\int_{\mathbb{R}^d} |\nabla h|^2 \, F \, \mathrm{d}v \geq \mathcal{C} \int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \langle v \rangle^{-\beta} \, F \, \mathrm{d}v$$

with $\beta = 2\,(1-\alpha)$, $\widetilde{h} := \int_{\mathbb{R}^d} h\, F \,\mathrm{d}v$ and $F(v) = C_\alpha \, e^{-\langle v \rangle^\alpha}$ and $\alpha \in (0,1)$

Written in terms of $g = h\,F$, the inequality becomes

$$\int_{\mathbb{R}^d} \left| \nabla_v \big(F^{-1} \, g \big) \right|^2 \, F^2 \, \mathrm{d}\mu \geq \mathcal{C} \int_{\mathbb{R}^d} |g - \overline{g}|^2 \, \langle v \rangle^{-2\,(1-\alpha)} \, \mathrm{d}\mu$$

where $\mathrm{d}\mu = F \, \mathrm{d}v$

$$\frac{d}{dt}\int_{\mathbb{R}^d} |h(t,v)|^2 \langle v\rangle^k F\,\mathrm{d}v + 2\int_{\mathbb{R}^d} |\nabla_v h|^2 \langle v\rangle^k F\,\mathrm{d}v = -\int_{\mathbb{R}^d} \nabla_v(h^2)\cdot\left(\nabla_v\langle v\rangle^k\right) F\,\mathrm{d}v$$

With $\ell = 2-\alpha$, $a\in\mathbb{R}$, $b\in(0,+\infty)$

$$\nabla_v\cdot\left(F\,\nabla_v\langle v\rangle^k\right) = \frac{k}{\langle v\rangle^4}\left(d+(k+d-2)\,|v|^2-\alpha\,\langle v\rangle^\alpha\,|v|^2\right) \le a-b\,\langle v\rangle^{-\ell}$$

Proposition (Weighted L^2 norm)

There exists a constant $\mathcal{K}_k>0$ such that, if h solves the Ornstein-Uhlenbeck equation, then

$$\forall\, t\ge 0 \quad \|h(t,\cdot)\|_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}\xi)} \le \mathcal{K}_k\,\left\|h^{\rm in}\right\|_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}\xi)}$$

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left|h(t,\cdot)-\widetilde{h}\right|^2 \mathrm{d}\xi = -\,2\int_{\mathbb{R}^d}|\nabla_v h|^2\,\mathrm{d}\xi \le -\,2\,\mathcal{C}\int_{\mathbb{R}^d}\left|h-\widetilde{h}\right|^2\,\langle v\rangle^{-\beta}\,\mathrm{d}\xi$$

+ Hölder

Theorem

Assume that $\alpha\in(0,1)$. *Let* $g^{\mathrm{in}}\in\mathrm{L}^1_+(\mathrm{d}\mu)\cap\mathrm{L}^2(\langle v\rangle^k\mathrm{d}\mu)$ *for some* $k>0$ *and consider the solution* g *to the homogeneous Fokker-Planck equation with initial datum* g^{in}. *If* $\overline{g}=\left(\int_{\mathbb{R}^d} g\,\mathrm{d}v\right)F$, *then*

$$\int_{\mathbb{R}^d}|g(t,\cdot)-\overline{g}|^2\,\mathrm{d}\mu \le \left(\left(\int_{\mathbb{R}^d}\left|g^{\mathrm{in}}-\overline{g}\right|^2\mathrm{d}\mu\right)^{-\beta/k}+\frac{2\,\beta\,\mathcal{C}}{k\,\mathcal{K}^{\beta/k}}\,t\right)^{-k/\beta}$$

with $\beta=2\,(1-\alpha)$ *and* $\mathcal{K}:=\mathcal{K}_k^2\,\left\|g^{\mathrm{in}}\right\|^2_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}\mu)}+\Theta_k\,\left(\int_{\mathbb{R}^d} g^{\mathrm{in}}\,\mathrm{d}v\right)^2$

The kinetic equation with sub-exponential local equilibria

- the *Fokker-Planck* operator

$$\mathsf{L}_1 f = \nabla_v \cdot \Big(F \, \nabla_v \big(F^{-1} \, f \big) \Big)$$

- the *scattering* collision operator

$$\mathsf{L}_2 f = \int_{\mathbb{R}^d} \mathrm{b}(\cdot, v') \Big(f(v') \, F(\cdot) - f(\cdot) \, F(v') \Big) \, \mathrm{d}v'$$

Potential	$V=0$	$V(x)=\gamma\,\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\geq 1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v)=e^{-\langle v\rangle^\beta}$, $\beta\geq 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b<1$, $\beta=2$ convergence	DMS, Mischler- Mouhot $e^{-\lambda t}$ convergence
$F(v)=e^{-\langle v\rangle^\beta}$, $\beta\in(0,1)$	BDLS: $t^{-\zeta}$, $\zeta=$ $\min\left\{\frac{d}{2},\frac{k}{\beta}\right\}$ decay			
$F(v)=\langle v\rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 3: $\partial_t f+v\cdot\nabla_x f=F\,\nabla_v\big(F^{-1}\,\nabla_v f\big)$. Notation: $\langle v\rangle=\sqrt{1+|v|^2}$

The decay rate with sub-exponential local equilibria

Theorem

Let $\alpha \in (0,1)$, $\beta > 0$, $k > 0$ *and let* $F(v) = C_\alpha\, e^{-\langle v \rangle^\alpha}$. *Assume that either* $\mathsf{L} = \mathsf{L}_1$ *and* $\beta = 2\,(1-\alpha)$, *or* $\mathsf{L} = \mathsf{L}_2$ + *Assumptions. There exists a numerical constant* $\mathcal{C} > 0$ *such that any solution* f *of*

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad f(0,\cdot,\cdot) = f^{\rm in} \in \mathrm{L}^2(\langle v \rangle^k \mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^1_+(\mathrm{d}x\,\mathrm{d}v)$$

satisfies

$$\forall\, t \geq 0\,, \quad \|f(t,\cdot,\cdot)\|^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \big|f(t,x,v)\big|^2\,\mathrm{d}x\,\mathrm{d}\mu \leq \mathcal{C}\, \frac{\big\|f^{\rm in}\big\|^2}{(1+\kappa\, t)^{\zeta}}$$

with rate $\zeta = \min\{d/2, k/\beta\}$, *for some positive* κ *which is an explicit function of the two quotients,* $\big\|f^{\rm in}\big\| \,/\, \big\|f^{\rm in}\big\|_k$ *and* $\big\|f^{\rm in}\big\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)} \,/\, \big\|f^{\rm in}\big\|$

Preliminaries

$$\begin{aligned}\mathsf{D}[f] := &- \langle \mathsf{L} f, f\rangle + \delta\,\langle \mathsf{AT\Pi} f, \mathsf{\Pi} f\rangle \\ &+ \delta\,\langle \mathsf{AT}(\mathrm{Id}-\mathsf{\Pi})f, \mathsf{\Pi} f\rangle - \delta\,\langle \mathsf{TA}(\mathrm{Id}-\mathsf{\Pi})f, (\mathrm{Id}-\mathsf{\Pi})f\rangle \\ &- \delta\,\langle \mathsf{AL}(\mathrm{Id}-\mathsf{\Pi})f, \mathsf{\Pi} f\rangle\end{aligned}$$

- ***microscopic coercivity***. If $\mathsf{L} = \mathsf{L}_1$, we rely on the weighted Poincaré inequality

$$\langle \mathsf{L} f, f\rangle \le -\,\mathcal{C}\,\|(\mathrm{Id}-\Pi)f\|^2_{-\beta}$$

If $\mathsf{L} = \mathsf{L}_2$, we assume that there exists a constant $\mathcal{C} > 0$ such that

$$\int_{\mathbb{R}^d} \big|h - \widetilde{h}\big|^2\,\langle v\rangle^{-\beta}\,F\,\mathrm{d}v \le \mathcal{C} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{b}(v,v')\,\big|h'-h\big|^2\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'$$

- ***Weighted*** L^2 ***norms*** Let $k>0$, $f^{\rm in} \in \mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$ a solution. $\exists\,\mathcal{K}_k > 1$ such that

$$\forall\,t \ge 0\,,\quad \|f(t,\cdot,\cdot)\|_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)} \le \mathcal{K}_k\,\big\|f^{\rm in}\big\|_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)}$$

Proof

$$\mathsf{H}_\delta[f] := \frac{1}{2}\,\|f\|^2 + \delta\,\langle \mathsf{A} f, f\rangle\,, \quad \frac{d}{dt}\mathsf{H}_\delta[f] = -\,\mathsf{D}[f]$$

- There exists $\kappa > 0$ such that $\forall\, f \in \mathrm{L}^2\left(\langle v\rangle^{-\beta}\,\mathrm{d}x\,\mathrm{d}\mu\right) \cap \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$,

$$\mathsf{D}[f] \geq \kappa \left(\|(\mathrm{Id} - \Pi)f\|^2_{-\beta} + \langle \mathsf{AT\Pi} f, \Pi f\rangle\right)$$

- For any $f \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^2(\mathrm{d}x\,\mathrm{d}v)$,

$$\langle \mathsf{AT\Pi} f, \Pi f\rangle \geq \Phi\left(\|\Pi f\|^2\right)$$

$$\Phi^{-1}(y) := 2\,y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}}\,, \quad \mathsf{c} = \Theta\,\mathcal{C}_{\mathrm{Nash}}^{-\frac{d+2}{d}}\,\|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}^{-\frac{4}{d}}$$

- For any $f \in \mathrm{L}^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$,

$$\|(\mathrm{Id} - \Pi)f\|^2_{-\beta} \geq \Psi\left(\|(\mathrm{Id} - \Pi)f\|^2\right)$$

$$\Psi(y) := C_0\,y^{1+\beta/k}\,, \quad C_0 := \left(\mathcal{K}_k\,\left(1 + \Theta_k\right)\|f^{\mathrm{in}}\|_k\right)^{-\frac{2\,\beta}{k}}$$

More references

- E. Bouin, J. Dolbeault, S. Mischler, C. Mouhot, and C. Schmeiser. Hypocoercivity without confinement. Preprint hal-01575501 and arxiv: 1708.06180, Oct. 2017, to appear. Nash
- E. Bouin, J. Dolbeault, and C. Schmeiser. Diffusion and kinetic transport with very weak confinement. Preprint hal-01991665 and arxiv: 1901.08323, to appear in Kinetic Rel. Models. Nash / Caffarelli-Kohn-Nirenberg inequalities
- M. P. Gualdani, S. Mischler, and C. Mouhot. Factorization of non-symmetric operators and exponential H-theorem. Mém. Soc. Math. Fr. (N.S.), (153):137, 2017.
- E. Bouin, J. Dolbeault, and C. Schmeiser. A variational proof of Nash's inequality. Preprint hal-01940110 and arxiv: 1811.12770, to appear in Atti della Accademia Nazionale dei Lincei. Rendiconti Lincei. Matematica e Applicazioni, 2018. Nash
- E. Bouin, J. Dolbeault, L. Lafleche, and C. Schmeiser. Hypocoercivity and sub-exponential local equilibria, soon. Weighted Poincaré inequalities

The Vlasov-Poisson-Fokker-Planck system: linearization and hypocoercivity

▷ Linearized Vlasov-Poisson-Fokker-Planck system

▷ A result in the non-linear case, $d = 1$

In collaboration with Lanoir Addala, Xingyu Li and Lazhar M. Tayeb

L^2-Hypocoercivity and large time asymptotics of the linearized Vlasov-Poisson-Fokker-Planck system. Preprint hal-02299535 and arxiv: 1909.12762
(Hérau, Thomann, 2016), (Herda, Rodrigues, 2018)

Linearized Vlasov-Poisson-Fokker-Planck system

The *Vlasov-Poisson-Fokker-Planck system* in presence of an external potential V is

$$\begin{aligned} \partial_t f + v\cdot\nabla_x f - (\nabla_x V + \nabla_x \phi)\cdot\nabla_v f &= \Delta_v f + \nabla_v\cdot(v\,f) \\ -\Delta_x \phi = \rho_f &= \int_{\mathbb{R}^d} f\,dv \end{aligned} \tag{VPFP}$$

Linearized problem around $f_\star$: $f = f_\star\,(1+\eta\,h)$, $\iint_{\mathbb{R}^d\times\mathbb{R}^d} h\,f_\star\,dx\,dv = 0$

$$\partial_t h + v\cdot\nabla_x h - (\nabla_x V + \nabla_x \phi_\star)\cdot\nabla_v h + v\cdot\nabla_x\psi_h - \Delta_v h + v\cdot\nabla_v h = \eta\,\nabla_x\psi_h\cdot\nabla_v h$$

$$-\Delta_x\psi_h = \int_{\mathbb{R}^d} h\,f_\star\,dv$$

Drop the $\mathcal{O}(\eta)$ term : *linearized Vlasov-Poisson-Fokker-Planck* system

$$\begin{aligned} \partial_t h + v\cdot\nabla_x h - (\nabla_x V + \nabla_x \phi_\star)\cdot\nabla_v h + v\cdot\nabla_x\psi_h - \Delta_v h + v\cdot\nabla_v h = 0 \\ -\Delta_x\psi_h = \int_{\mathbb{R}^d} h\,f_\star\,dv\,, \quad \iint_{\mathbb{R}^d\times\mathbb{R}^d} h\,f_\star\,dx\,dv = 0 \end{aligned} \tag{VPFPlin}$$

Hypocoercivity

Let us define the norm

$$\|h\|^2 := \iint_{\mathbb{R}^d\times\mathbb{R}^d} h^2\, f_\star\, dx\, dv + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2\, dx$$

Theorem

Let us assume that $d \geq 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. Then there exist two positive constants $\mathcal{C}$ and λ such that any solution h of (VPFPlin) with an initial datum h_0 of zero average with $\|h_0\|^2 < \infty$ is such that

$$\|h(t,\cdot,\cdot)\|^2 \leq \mathcal{C}\, \|h_0\|^2\, e^{-\lambda t} \quad \forall\, t \geq 0$$

Diffusion limit

Linearized problem in the parabolic scaling

$$\varepsilon\,\partial_t h + v\cdot\nabla_x h - (\nabla_x V + \nabla_x \phi_\star)\cdot\nabla_v h + v\cdot\nabla_x\psi_h - \frac{1}{\varepsilon}\left(\Delta_v h - v\cdot\nabla_v h\right) = 0$$

$$-\Delta_x\psi_h = \int_{\mathbb{R}^d} h\,f_\star\,dv\,,\quad \iint_{\mathbb{R}^d\times\mathbb{R}^d} h\,f_\star\,dx\,dv = 0 \tag{VPFPscal}$$

Expand $h_\varepsilon = h_0 + \varepsilon\,h_1 + \varepsilon^2\,h_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$. With $W_\star = V + \phi_\star$

$$\varepsilon^{-1}: \qquad \Delta_v h_0 - v\cdot\nabla_v h_0 = 0$$

$$\varepsilon^{0}: \qquad v\cdot\nabla_x h_0 - \nabla_x W_\star\cdot\nabla_v h_0 + v\cdot\nabla_x\psi_{h_0} = \Delta_v h_1 - v\cdot\nabla_v h_1$$

$$\varepsilon^{1}: \qquad \partial_t h_0 + v\cdot\nabla_x h_1 - \nabla_x W_\star\cdot\nabla_v h_1 = \Delta_v h_2 - v\cdot\nabla_v h_2$$

With $u = \Pi h_0$, $-\Delta\psi = u\,\rho_\star$, $w = u + \psi$, equations simply mean

$$u = h_0\,,\quad v\cdot\nabla_x w = \Delta_v h_1 - v\cdot\nabla_v h_1$$

from which we deduce that $h_1 = -\,v\cdot\nabla_x w$ and

$$\partial_t u - \Delta w + \nabla_x W_\star\cdot\nabla u = 0$$

Results in the diffusion limit / in the non-linear case

Theorem

Let us assume that $d \geq 1$, $V(x) = |x|^\alpha$ *for some* $\alpha > 1$ *and* $M > 0$*. For any* $\varepsilon > 0$ *small enough, there exist two positive constants* $\mathcal{C}$ *and* λ*, which do not depend on* ε*, such that any solution* h *of* (VPFPscal) *with an initial datum* h_0 *of zero average and such that* $\|h_0\|^2 < \infty$ *satisfies*

$$\|h(t,\cdot,\cdot)\|^2 \leq \mathcal{C}\, \|h_0\|^2\, e^{-\lambda t} \quad \forall\, t \geq 0$$

Corollary

Assume that $d = 1$, $V(x) = |x|^\alpha$ *for some* $\alpha > 1$ *and* $M > 0$*. If* f *solves* (VPFP) *with initial datum* $f_0 = (1 + h_0)\, f_\star$ *such that* h_0 *has zero average,* $\|h_0\|^2 < \infty$ *and* $(1 + h_0) \geq 0$*, then*

$$\|h(t,\cdot,\cdot)\|^2 \leq \mathcal{C}\, \|h_0\|^2\, e^{-\lambda t} \quad \forall\, t \geq 0$$

holds with $h = f/f_\star - 1$ *for some positive constants* $\mathcal{C}$ *and* λ

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/
▷ Lectures

The papers can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/
▷ Preprints / papers

For final versions, use `Dolbeault` as login and `Jean` as password

Thank you for your attention !