Relative entropy methods for nonlinear diffusions and applications

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LOGARITHMIC SOBOLEV INEQUALITY AND HEAT EQUATION

Heat equation:

$$u_t = \Delta u \quad \text{in } \mathbb{R}^d$$
$$u_{|t=0} = u_0 \geq 0$$

$$u_0(1+|x|^2) \in L^1$$
$$u_0 \log u_0 \in L^1$$
$$\int u_0 \, dx = 1$$

Time-dependent rescaling:

$$u(t,x) = \frac{1}{R^d(t)} \, v\left(\tau(t), \frac{x}{R(t)}\right)$$

$$R(t) = \sqrt{1+2t}\,, \quad \tau(t) = \log R(t)$$

$$v_\tau = \Delta v + \nabla \cdot (x\, v)\,, \quad v_{|\tau=0} = u_0$$

Stationary solution:

$$v_\infty(x) = (2\pi)^{-d/2}\, e^{-|x|^2/2}$$

A Lyapunov functional: *the free energy*

$$L[v] = \int v \log\left(\frac{v}{v_\infty}\right)\,dx$$

$$i.e. \quad L[v] = \int \left(v \log v + \frac{1}{2}|x|^2 v\right)\,dx + C$$

$$\frac{d}{d\tau}L[v] = -I[v]\,, \quad I[v] = \int v\left|\frac{\nabla v}{v} + x\right|^2\,dx$$

Logarithmic Sobolev inequality [Gross, 1975]

$$\int |g|^2 \log|g|\,d\mu \le \int |\nabla g|^2\,d\mu$$

$d\mu(x) = v_\infty(x)\,dx$ and $\int |g|^2\,d\mu = 1$.
Take $|g|^2 = \frac{v}{v_\infty}$:

$$I[v] \ge 2\,L[v]$$

$$L[v] \le L[u_0]\,e^{-2\tau}$$

Csiszár-Kullback inequality [1967]

$$\|v - v_\infty\|^2_{L^1} \leq 4\, L[v]$$

Intermediate asymptotics

$$u_\infty(t,x) = \frac{1}{R^d(t)}\, v_\infty\left(\frac{x}{R(t)}\right)$$

$$\|u - u_\infty\|_{L^1} = \|v - v_\infty\|_{L^1} \leq \frac{\sqrt{2\, L[u_0]}}{R(t)}$$

$$\|u - u_\infty\|_{L^1} = O\left(\frac{1}{\sqrt{t}}\right) \quad as\ t \to +\infty$$

Probability theory: [Bakry, Emery, Ledoux] Optimality: [Carlen, 1991] A new proof of Log Sobolev: [Toscani, 1997], [AMTU, 1999]

$$\frac{d}{d\tau}(I[v] - 2\, L[v])$$
$$= -4 \int \sum_{i,j=1}^d \left| \frac{\partial^2 w}{\partial x_i \partial x_j} - \frac{1}{w}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} + \frac{w}{2}\,\delta_{ij} \right|^2 \, dx$$

POROUS MEDIUM / FAST DIFFUSION

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^d$$
$$u_{|t=0} = u_0 \geq 0$$
$$u_0(1+|x|^2) \in L^1\,, \quad u_0^m \in L^1$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0\,dx = 1$, the Green function: $G(t) = O(t^{-d/(2-d(1-m))})$ as $t \to +\infty$, [Friedmann, Kamin, 1980]

$$\|u(t,\cdot) - G(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$$

Rescaling: Take $u(t,x) = R^{-d}(t)\, v\,(\tau(t), x/R(t))$ where

$$\dot{R} = R^{d(1-m)-1}\,, \quad R(0) = 1\,, \quad \tau = \log R$$

$$v_\tau = \Delta v^m + \nabla \cdot (x\,v)\,, \quad v_{|\tau=0} = u_0$$

[Ralston, Newman, 1984] Lyapunov functional: *Entropy*

$$L[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v\right)\,dx + C$$

$$\frac{d}{d\tau}L[v] = -I[v] \,, \quad I[v] = \int v \left| \frac{\nabla v^{m-1}}{v} + x \right|^2 \, dx$$

Stationary solution: C s.t. $\|v_\infty\|_{L^1} = \|u\|_{L^1} = 1$

$$v_\infty(x) = \left(C + \frac{1-m}{2m} \, |x|^2 \right)_+^{-1/(1-m)}$$

$$L[v] = \int \sigma \left(\frac{v^m}{v_\infty^m} \right) \, v_\infty^{m-1} \, dx$$

$$\textit{with } \sigma(t) = \tfrac{m t^{1/m} - 1}{1-m} + 1$$

Theorem 1 $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\boxed{I[v] \geq 2 \, L[v]}$$

$p = \frac{1}{2m-1}$, $v = w^{2p}$
$K < 0$ *if* $m < 1$, $K > 0$ *if* $m > 1$

$$\boxed{\tfrac{1}{2}(\tfrac{2m}{2m-1})^2 \int |\nabla w|^2 dx + (\tfrac{1}{1-m} - d) \int |w|^{1+p} dx + K \geq 0}$$

$m = \frac{d-1}{d}$: *Sobolev*, $m \to 1$: *logarithmic Sobolev*
[Del Pino, J.D.], [Carrillo, Toscani], [Otto]

$L[v] \leq L[u_0]\, e^{-2\tau} +$ Ciszár-Kullback inequalities
$\Rightarrow$ Intermediate asymptotics [Del Pino, J.D.]

(i) $\frac{d-1}{d} < m < 1$ if $d \geq 3$, and $\frac{1}{2} < m < 1$ if $d = 2$

$$\limsup_{t\to+\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}} \|u^m - u_\infty^m\|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t\to+\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}} \|\, [u - u_\infty]\, u_\infty^{m-1}\, \|_{L^1} < +\infty$$

Optimal constants for Gagliardo-Nirenberg inequalities [Del Pino, J.D.]

$d \geq 2$, $p > 1$, $p \leq \frac{d}{d-2}$ for $d \geq 3$

$$\|w\|_{2p} \leq A\, \|\nabla w\|_2^\theta\, \|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}$$

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p)},\; y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$. Uses [Serrin-Pucci], [Serrin-Tang].

GENERAL NONLINEAR DIFFUSIONS

[Carrillo, Juengel, Markowich, Toscani, Unterreiter, 2000]

$$\begin{aligned} & u_t = \Delta f(u) + \nabla \cdot (u \nabla V) \\ & with \quad V(x) \sim \alpha\, |x|^2 \end{aligned}$$

Generalized entropy

$$\begin{aligned} L[u] \; & = \int [H(u) + V\, u]\, dx - C \\ & = \int [u\, h(u) - f(u) + V\, u]\, dx - C \\ & with \quad H'(u) = h(u) = \tfrac{f'(u)}{u} \end{aligned}$$

Minimizer with given *mass*

$$\begin{aligned} & h(u_\infty) = C - V(x) \\ & u_\infty(x) = g(C - V(x)) \end{aligned}$$

where g is the generalized inverse of h.

Examples:

$$\begin{aligned} & f(u) = u \quad h(u) = \log u\,, \quad g(t) = e^t \\ & f(u) = u^m\,, \quad h(u) = \tfrac{m}{m-1}\, u^{m-1} \\ & \qquad\qquad\qquad g(t) = \left(\tfrac{m-1}{m}\, t\right)_+^{1/(m-1)} \end{aligned}$$

$$L[u] = \int [u\, h(u) - f(u) + V\, u]\; dx + C$$
$$\textit{such that } L[u_\infty] = 0$$

$$\frac{d}{d\tau} L[u] = -I[u] \;,$$

$$\textit{with } I[u] = \int u \left| h'(u) \nabla u + \nabla V\, u \right|^2 \; dx$$

Generalized Sobolev inequality:

assume that $f' > 0$ *+ technical conditions*

$$\exists K > 0 \quad L[u] \leq K\, I[u]$$

Rate and intermediate asymptotics:
[Biler, J.D., Esteban]

$$u_t = \Delta f(u) \;, \quad u_{|t=0} = u_0 \geq 0$$

Take $u(t,x) = R^{-d}(t)\, v\,(\tau(t), x/R(t))$ where

$$\dot R = R^{d(1-m)-1} \quad \textit{if } f(u) \sim u^m \textit{ as } m \to 0$$
$$v_\tau = e^{md\tau} \Delta f(e^{-d\tau} v) + \nabla \cdot (x\, v)$$

Assumptions:

$$f(s) = s^m F(s)$$
$$F \in C^0(\mathbb{R}^+) \cap C^1(0, +\infty)$$
$$F > 0 \,, \quad F(0) = 1$$
$$F'(s) = O(s^k) \text{ with } k > -1 \text{ as } s \to 0_+$$
$$(m-1)s\,h(s) - m\,f(s) \le 0$$

Example: $f(s) = s^m + s^q, \quad q > m$

Entropy

$$L[\tau, v] = e^{md\tau} \int [H(e^{-d\tau} v) - H(e^{-d\tau} v_\infty^\tau)]\, dx + \frac{1}{2} \int |x|^2 \,(v - v_\infty^\tau)\, dx$$

Theorem 2 *Decay rate of the entropy*

$$0 \le L[\tau, v] \le K e^{-\beta\,\tau}, \ \beta = \min\,(d(k+1), 2)$$

Corollary 3 *Intermediate asymptotics*

$$\|H(u) - H(u_\infty)\|_{L^1} \le C\, R^{-d(m-1)-\beta/2}$$

A DRIFT-DIFFUSION-POISSON MODEL
[Gajewski], [Arnold, Markowich, Toscani], [Arnold, Markowich, Toscani, Unterreiter] – use Holley-Stroock's perturbation lemma

$m = 1$: [Biler, J.D.]
$m \neq 1$: [Biler, J.D., Markowich]
Debye-Hückel model for electrolytes, Drift-diffusion model for semi-conductors.

$$\begin{aligned} u_t &= \nabla \cdot (\nabla u^m + u \nabla V + u \nabla \phi) \\ v_t &= \nabla \cdot (\nabla v^m + v \nabla V - v \nabla \phi) \\ \Delta \phi &= v - u \end{aligned}$$

After a time-dependent rescaling (use abusively same notations):

$$\begin{aligned} u_\tau &= \nabla \cdot (\nabla u^m + u \nabla V + \beta(\tau) u \nabla \phi) \\ v_\tau &= \nabla \cdot (\nabla v^m + v \nabla V - \beta(\tau) v \nabla \phi) \\ \Delta \phi &= v - u \,, \quad \beta(\tau) = e^{(2-d)\tau} \end{aligned}$$

Entropy

$$W[u] = \int (u\, h(u) - f(u) + \frac{1}{2}|x|^2 u) \, dx$$

$$L[u,v] = W[u] - W[u_\infty] + W[v] - W[v_\infty]$$
$$+\frac{\beta(\tau)}{2}\int (u-v)\phi \, dx$$

Proposition 4 *Entropy decay*

$$\frac{dL}{d\tau} = -2\, I[u] - 2\, I[v] - \beta^2 \int (u+v)|\nabla\phi|^2 \, dx$$

Theorem 5 *Rates, $d=3$*

1) L has an exponential decay

2) $m=1$: $L(\tau) \leq C\, e^{-\tau}$
If $\int u_0 \, dx = \int v_0 \, dx$, $L(\tau) \leq C\, e^{-2\,\tau}$

More general nonlinearities – Example:
$F(\sigma) := \int_{\mathbb{R}^3_v} \frac{dv}{\epsilon + \exp(|v|^2/2-\sigma)}$
$f(s) = sF^{-1}(s) - \int_0^s F^{-1}(x)\, dx$

Stationary solutions, singular limit (dielectric constant $\to 0$)
[Caffarelli, J.D., Markowich, Schmeiser]
[J.D., Markowich, Unterreiter]

$$\begin{aligned}
&\Delta f(u) + \nabla \cdot (u\nabla V + u\nabla\phi) = 0\\
&\Delta f(v) + \nabla \cdot (v\nabla V - v\nabla\phi) = 0\\
&\Delta\phi = v - u - C(x)
\end{aligned}$$

Ω bounded domain, zero flux + insulator boundary conditions
$\int v\,dx = N$, $\int u\,dx = P$ given

Global electroneutrality: $N - P - \int_\Omega C(x)\,dx = 0$
Entropy: $L = \int H(u)\,dx + \int H(v)\,dx + \frac{1}{2}\int |\nabla\phi|^2\,dx$
$H'(u) = h(u) = \frac{f'(u)}{u}$, $g = h^{-1}$,

$$u = g(\alpha[\phi] + \phi)\ , \quad v = g(\beta[\phi] - \phi)$$

$$\Delta\phi = g(\alpha[\phi] + \phi) - g(\beta[\phi] - \phi) - C(x)$$

A convex functional

$$\begin{aligned}
J[\phi] = \ &\tfrac{1}{2}\int |\nabla\phi|^2\,dx + \int G(\alpha[\phi] + \phi)\,dx\\
&+ \int G(\beta[\phi] - \phi)\,dx - N\beta[\phi] - P\alpha[\phi]
\end{aligned}$$

which plays a crucial role in the study of singular limits.

ENERGY-TRANSPORT: STREATER'S MODELS

$$\begin{cases} u_t = \nabla \cdot \left[\kappa(\nabla u + \frac{u}{\theta}(\epsilon\nabla\phi + \zeta\nabla\phi_0))\right] \\ (u\theta)_t = \nabla \cdot (\lambda\nabla\theta) \\ \qquad\qquad +\nabla \cdot [\kappa(\theta\nabla u + \epsilon u\nabla\phi + \zeta u\nabla\phi_0)] \\ \qquad\qquad +(\epsilon\nabla\phi + \zeta\nabla\phi_0) \\ \qquad\qquad\qquad \cdot\left[\kappa(\nabla u + \frac{u}{\theta}(\epsilon\nabla\phi + \zeta\nabla\phi_0))\right] \\ \pm\Delta\phi = u \end{cases}$$

with the boundary conditions

$$\begin{cases} \partial_n u + \frac{u}{\theta}(\epsilon\partial_n\phi + \zeta\partial_n\phi_0) = 0 & \text{(no mass flux)} \\ \partial_n\theta = 0 & \text{(no heat flux)} \end{cases}$$

Mass: $M = \int u\, dx$

Energy: $\int (u\,\theta + \phi_0 + \frac{1}{2}\phi)\, dx$

Entropy: $\int u \log(\frac{u}{\theta})\, dx$

$$\frac{dL}{dt} = -\int \frac{|\nabla\theta|^2}{\theta^2}\, dx - \int u|\frac{\nabla u}{u} + \frac{1}{\theta}(\nabla\phi + \nabla\phi_0)|^2\, dx$$

Electrostatic case: [Biler, Esteban, J.D., Karch]
Stationary solutions: existence, uniqueness in some cases

$$L[u,\theta] - L[u_\infty,\theta_\infty]$$
$$= \int s_1(\tfrac{u}{u_\infty})\, u_\infty\, dx + \int s_2(\tfrac{\theta}{\theta_\infty})\, u\, dx$$
$$s_1(t) = t \log t + 1 - t$$
$$s_2(t) = t - 1 - \log t$$

For uniqueness, use [Desvillettes, J.D.]:

Theorem 6 *Ω of class C^1, bounded, $\psi \in H^1_0(\Omega)$*
$\sigma(t) = (-1 + \sqrt{1+\chi t^2})/t^2$, $\chi = 2E/M^2$

$$-\sigma(\|\nabla\psi\|_{L^2})\,\Delta\psi = \frac{e^{-\psi}}{\int_\Omega e^{-\psi}\,dx}\,, \quad \psi_{|\partial\Omega} = 0$$

has a unique solution for any $M > 0$ and $E > 0$.

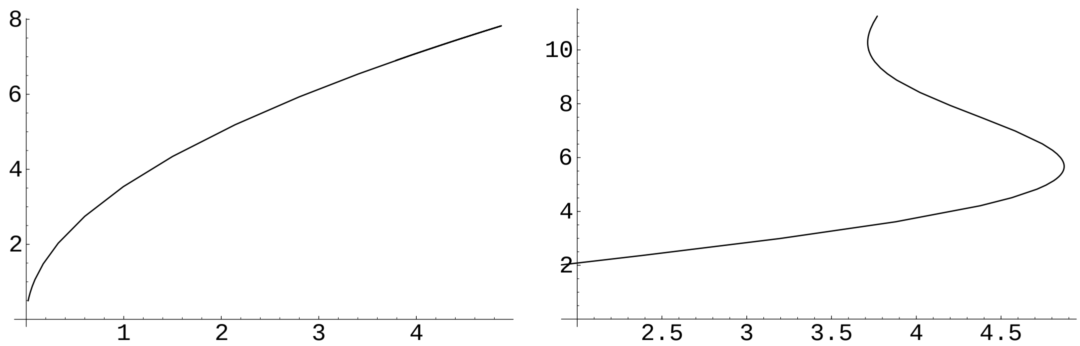
Lemma 7 *Ω of class C^1, bounded, $\psi \in H^1_0(\Omega)$*

$$\log\left(\int_\Omega e^{-\psi}\,dx\right) \geq C - 2\log\left(\|\nabla\psi\|_{L^2}\right)(1+o(1))$$

as $\|\nabla\psi\|_{L^2} \to \infty$.

Gravitational case: [Biler, Esteban, J.D., Markowich, Nadzieja]

Theorem 8 Ω *bounded star-shaped in* $\mathbb{R}^d$
$d \geq 3$, *of class* C^1
$E/M^2 > \ell_1$*: existence of bounded solutions*
$E/M^2 < \ell_0$, *no nontrivial bounded solution*
If $d \geq 15$, $E/M^2 < \tilde{\ell}_0$*: no solution in* $H^1_0(\Omega)$



Solutions of $-\Delta\psi = m\,\frac{e^\psi}{\int e^\psi\,dx}$ in the ball with zero Dirichlet boundary conditions:
left: $(m, \|\psi\|_{H^1})$, right: $(m, \|\psi\|_{L^\infty})$

FURTHER RESULTS ON NONLINEAR DIFFUSIONS

$$u_t = \Delta_p u^m$$

Optimal constants for Gagliardo-Nirenberg ineq. [Del Pino, J.D.]

Theorem 9 $1 < p < d$, $1 < a \leq \frac{p(d-1)}{d-p}$, $b = p\,\frac{a-1}{p-1}$

$$\|w\|_b \leq \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_a^{1-\theta} \quad \textit{if } a > p$$
$$\|w\|_a \leq \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_b^{1-\theta} \quad \textit{if } a < p$$

Equality if $w(x) = A\,(1 + B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$

$a > p$: $\theta = \frac{(q-p)\,d}{(q-1)\,(d\,p-(d-p)\,q)}$

$a < p$: $\theta = \frac{(p-q)\,d}{q\,(d\,(p-q)+p\,(q-1))}$

Proof based on [Serrin, Tang]

A new logarithmic Sobolev inequality, with optimal constant [Del Pino, J.D.]

Theorem 10 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \leq \frac{d}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx \right]$$

$$\mathcal{L}_p = \frac{p}{d} \left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d\,\frac{p-1}{p}+1)} \right]^{\frac{p}{d}}$$

Equality: $u = \pi^{-\frac{d}{2}} \sigma^{-d\,\frac{p-1}{p}} \dfrac{\Gamma(\frac{d}{2}+1)}{\Gamma(d\,\frac{p-1}{p}+1)} \, e^{-\frac{1}{\sigma}|x-\bar{x}|^{\frac{p}{p-1}}}$

$p = 2$: Gross' logaritmic Sobolev inequality
$p = 1$: [Beckner]

[Del Pino, J.D.] Intermediate asymptotics of

$$u_t = \Delta_p u^m$$

Theorem 11 $d \geq 2,\ 1 < p < d$
$\frac{d-(p-1)}{d(p-1)} \leq m \leq \frac{p}{p-1}$ *and* $q = 1 + m - \frac{1}{p-1}$

$$\text{(i)} \quad \|u(t,\cdot) - u_\infty(t,\cdot)\|_q \leq K\, R^{-(\frac{\alpha}{2}+d(1-\frac{1}{q}))}$$

$$\text{(ii)} \quad \|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}}$$

(i): $\frac{1}{p-1} \leq m \leq \frac{p}{p-1}$, *(ii):* $\frac{d-(p-1)}{d(p-1)} \leq m \leq \frac{1}{p-1}$
$\alpha = (1 - \frac{1}{p}\,(p-1)^{\frac{p-1}{p}})\,\frac{p}{p-1}$, $R = (1 + \gamma\, t)^{1/\gamma}$
$\gamma = (md+1)(p-1) - (d-1)$

$$u_\infty(t,x) = \frac{1}{R^d}\, v_\infty(\log R, \frac{x}{R})$$

$v_\infty(x) = (C - \frac{p-1}{mp}\,(q-1)\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ $m \neq \frac{1}{p-1}$
$v_\infty(x) = C\, e^{-(p-1)^2|x|^{p/(p-1)}/p}$ *if* $m = (p-1)^{-1}$.

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