Stability results for (log) Sobolev inequalities

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Nonlinear Analysis: Geometric, Variational and Dispersive aspects
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Outline

1. Sobolev and LSI on $\mathbb{R}^d$: optimal dimensional dependence
2. Explicit stability result for the Sobolev inequality: proof
3. Explicit stability result for the logarithmic Sobolev inequality

Explicit stability results for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

Joint papers with M.J. Esteban, A. Figalli, R. Frank, M. Loss

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

arXiv: 2209.08651

A short review on improvements and stability for some interpolation inequalities

arXiv: 2402.08527

An explicit stability result for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\,f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\,\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

Theorem (JD, Esteban, Figalli, Frank, Loss)

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d}\,\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \to +\infty$

A stability result for the logarithmic Sobolev inequality

- Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{1}{4}\,d\,(d-2)\left(\|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$
$$\geq \frac{\beta}{d}\inf_{G\in\mathcal{M}(\mathbb{S}^d)}\left(\|\nabla F-\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\,d\,(d-2)\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

- Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d\to+\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi\,|x|^2}\,dx$

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta>0$ as in the result for the Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi\int_{\mathbb{R}^n} u^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)d\gamma$$
$$\geq \frac{\beta\,\pi}{2}\inf_{a\in\mathbb{R}^d,\ c\in\mathbb{R}}\int_{\mathbb{R}^n}|u-c\,e^{a\cdot x}|^2\,d\gamma$$

Stability for the Sobolev inequality

▷ [Rodemich, 1969], [Aubin, 1976], [Talenti, 1976]
In the inequality $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq S_d\, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, the optimal constant is

$$S_d = \tfrac{1}{4}\, d\, (d-2)\, |\mathbb{S}^d|^{1-2/d}$$

with equality on the manifold $\mathcal{M} = \{g_{a,b,c}\}$ of the *Aubin-Talenti functions*

▷ [Lions] a qualitative stability result

$$\textit{if}\ \lim_{n\to\infty} \|\nabla f_n\|_2^2/\|f_n\|_{2^*}^2 = S_d\,,\ \textit{then}\ \lim_{n\to\infty}\inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2/\|\nabla f_n\|_2^2 = 0$$

▷ [Brezis, Lieb], 1985 a quantitative stability result ?
▷ [Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|_2^2 \geq S_d\,\|f\|_{2^*}^2 + c_{\mathrm{BE}} \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

- The strategy of Bianchi & Egnell involves two steps:

– a local (spectral) analysis: the *neighbourhood* of $\mathcal{M}$
– a local-to-global extension based on concentration-compactness : the *far away regime*

- The constant c_{BE} is not explicit

Stability for the logarithmic Sobolev inequality

▷ [Gross, 1975] *Gaussian logarithmic Sobolev inequality* for $n \geq 1$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$

▷ [Weissler, 1979] scale invariant (but dimension-dependent) version of the Euclidean form of the inequality

▷ [Stam, 1959], [Federbush, 69], [Costa, 85] *Cf.* [Villani, 08]

▷ [Bakry, Emery, 1984], [Carlen, 1991] equality iff

$$u \in \mathscr{M} := \left\{w_{a,c}\,:\,(a,c) \in \mathbb{R}^d \times \mathbb{R}\right\} \quad \text{where} \quad w_{a,c}(x) = c\, e^{a\cdot x} \quad \forall\, x \in \mathbb{R}^n$$

▷ [McKean, 1973], [Beckner, 92] (LSI) as a large d limit of Sobolev

▷ [Carlen, 1991] reinforcement of the inequality (Wiener transform)

▷ [JD, Toscani, 2016] Comparison with Weissler's form, a (dimension dependent) improved inequality

▷ [Bobkov, Gozlan, Roberto, Samson, 2014], [Indrei et al., 2014-23] stability in Wasserstein distance

▷ [Fathi, Indrei, Ledoux, 2016] improved inequality assuming a Poincaré inequality (Mehler formula)

Explicit stability result for the Sobolev inequality

Proof

Sketch of the proof

Goal: prove that there is an ***explicit*** constant $\beta > 0$ such that for all $d \geq 3$ and all $f \in \dot{\mathrm{H}}^1\left(\mathbb{R}^d\right)$

$$\|\nabla f\|_2^2 \geq S_d\, \|f\|_{2^*}^2 + \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

Part 1. We show the inequality for nonnegative functions far from $\mathcal{M}$

… *the far away regime*

Make it *constructive*

Part 2. We show the inequality for nonnegative functions close to $\mathcal{M}$

… *the local problem*

Get *explicit* estimates and remainder terms

Part 3. We show that the inequality for nonnegative functions implies the inequality for functions without a sign restriction, up to an acceptable loss in the constant

… *dealing with sign-changing functions*

Some definitions

What we want to minimize is

$$\mathcal{E}(f) := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\mathsf{d}(f,\mathcal{M})^2} \quad f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$$

where

$$\mathsf{d}(f,\mathcal{M})^2 := \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ up to a *conformal transformation*, we assume that $\mathsf{d}(f,\mathcal{M})^2 = \|\nabla f - \nabla g_*\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ with

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}$$

▷ use the *inverse stereographic* projection

$$F(\omega) = \frac{f(x)}{g_*(x)} \quad x \in \mathbb{R}^d \text{ with } \begin{cases} \omega_j = \frac{2\,x_j}{1+|x|^2} \text{ if } 1 \le j \le d \\ \omega_{d+1} = \frac{1-|x|^2}{1+|x|^2} \end{cases}$$

The problem on the unit sphere

Stability inequality on the unit sphere $\mathbb{S}^d$ for $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$

$$\int_{\mathbb{S}^d} \left(|\nabla F|^2 + \mathsf{A}\,|F|^2\right) d\mu - \mathsf{A} \left(\int_{\mathbb{S}^d} |F|^{2^*}\,d\mu\right)^{2/2^*} \geq \frac{\beta}{d} \inf_{G \in \mathscr{M}} \left\{ \|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathsf{A}\,\|F - G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right\}$$

with $\mathrm{A} = \frac{1}{4}\,d\,(d-2)$ and a manifold $\mathscr{M}$ of optimal functions made of

$$G(\omega) = c\,(a + b \cdot \omega)^{-\frac{d-2}{2}} \quad \omega \in \mathbb{S}^d \quad (a, b, c) \in (0, +\infty) \times \mathbb{R}^d \times \mathbb{R}$$

- make the reduction of a ***far away problem*** to a local problem *constructive*... on $\mathbb{R}^d$
- make the analysis of the ***local problem*** *explicit*... on $\mathbb{S}^d$

Competing symmetries

- ***Rotations on the sphere*** combined with stereographic and inverse stereographic projections. Let $e_d=(0,\ldots,0,1)\in\mathbb{R}^d$

$$(Uf)(x):=\left(\frac{2}{|x-e_d|^2}\right)^{\frac{d-2}{2}} f\left(\frac{x_1}{|x-e_d|^2},\ldots,\frac{x_{d-1}}{|x-e_d|^2},\frac{|x|^2-1}{|x-e_d|^2}\right)$$

$$\mathcal{E}(Uf)=\mathcal{E}(f)$$

- ***Symmetric decreasing rearrangement*** $\mathcal{R}f=f^*$

$$f \text{ and } f^* \text{ are equimeasurable}$$

$$\|\nabla f^*\|_{\mathrm{L}^2(\mathbb{R}^d)}\le\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

The method of ***competing symmetries***

Theorem (Carlen, Loss, 1990)

Let $f\in\mathrm{L}^{2^*}(\mathbb{R}^d)$ be a non-negative function with $\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}=\|g_*\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$. The sequence $f_n=(\mathcal{R}U)^n f$ is such that $\lim_{n\to+\infty}\|f_n-g_*\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}=0$. If $f\in\dot{\mathrm{H}}^1(\mathbb{R}^d)$, then $(\|\nabla f_n\|_{\mathrm{L}^2(\mathbb{R}^d)})_{n\in\mathbb{N}}$ is a non-increasing sequence

Useful preliminary results

- $\lim_{n\to\infty} \|f_n - h_f\|_{2^*} = 0$ where $h_f = \|f\|_{2^*}\, g_* / \|g_*\|_{2^*} \in \mathcal{M}$
- $(\|\nabla f_n\|_2^2)_{n\in\mathbb{N}}$ is a nonincreasing sequence

Lemma

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - S_d \sup_{g\in\mathcal{M},\, \|g\|_{2^*}=1} \left(f, g^{2^*-1}\right)^2$$

Corollary

$\big(\mathrm{d}(f_n,\mathcal{M})\big)_{n\in\mathbb{N}}$ is strictly decreasing, $n \mapsto \sup_{g\in\mathcal{M}_1} \left(f_n, g^{2^*-1}\right)$ is strictly increasing, and

$$\lim_{n\to\infty} \mathrm{d}(f_n,\mathcal{M})^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\, \|h_f\|_{2^*}^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\, \|f\|_{2^*}^2$$

but no monotonicity for $n \mapsto \mathcal{E}(f_n) = \frac{\|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\, \|f_n\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\mathrm{d}(f_n,\mathcal{M})^2}$

Part 1: Global to local reduction

The ***local problem***

$$\mathscr{I}(\delta) := \inf \left\{ \mathcal{E}(f) \,:\, f \geq 0\,,\ \mathsf{d}(f,\mathcal{M})^2 \leq \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right\}$$

Assume that $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a nonnegative function in the ***far away regime***

$$\mathsf{d}(f,\mathcal{M})^2 = \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

for some $\delta \in (0,1)$

Let $f_n = (\mathcal{R}U)^n f$. There are two cases:

- (Case 1) $\mathsf{d}(f_n,\mathcal{M})^2 \geq \delta\, \|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ for all $n \in \mathbb{N}$
- (Case 2) for some $n \in \mathbb{N}$, $\mathsf{d}(f_n,\mathcal{M})^2 < \delta\, \|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

Global to local reduction – Case 1

Assume that $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a nonnegative function in the far away regime

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

Lemma

Let $f_n = (\mathcal{R}U)^n f$ and $\delta \in (0,1)$. If $\mathrm{d}(f_n,\mathcal{M})^2 \ge \delta \, \|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ for all $n \in \mathbb{N}$, then

$$\mathcal{E}(f) \ge \delta$$

$$\lim_{n\to+\infty} \|\nabla f_n\|_2^2 \le \frac{1}{\delta} \lim_{n\to+\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \frac{1}{\delta}\left(\lim_{n\to+\infty} \|\nabla f_n\|_2^2 - S_d \, \|f\|_{2^*}^2\right)$$

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - S_d \, \|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2} \ge \frac{\|\nabla f\|_2^2 - S_d \, \|f\|_{2^*}^2}{\|\nabla f\|_2^2} \ge \frac{\|\nabla f_n\|_2^2 - S_d \, \|f\|_{2^*}^2}{\|\nabla f_n\|_2^2} \ge \delta$$

Global to local reduction – Case 2

$$\mathscr{I}(\delta) := \inf \left\{ \mathcal{E}(f) \,:\, f \ge 0\,,\ \mathsf{d}(f,\mathcal{M})^2 \le \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right\}$$

Lemma

$$\mathcal{E}(f) \ge \delta\, \mathscr{I}(\delta)$$

if $\inf_{g\in\mathcal{M}} \|\nabla f_{n_0} - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > \delta\, \|\nabla f_{n_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

and $\inf_{g\in\mathcal{M}} \|\nabla f_{n_0+1} - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} < \delta\, \|\nabla f_{n_0+1}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

Adapt a strategy due to Christ: build a (semi-)*continuous* rearrangement *flow* $(\mathsf{f}_\tau)_{n_0\le\tau<n_0+1}$ with $\mathsf{f}_{n_0} = U f_n$ such that $\|\mathsf{f}_\tau\|_{2^*} = \|f\|_2$, $\tau \mapsto \|\nabla \mathsf{f}_\tau\|_2$ is nonincreasing, and $\lim_{\tau\to n_0+1} \mathsf{f}_\tau = f_{n_0+1}$

$$\mathcal{E}(f) \ge 1 - S_d\, \frac{\|f\|^2_{2^*}}{\|\nabla f\|^2_2} \ge 1 - S_d\, \frac{\|\mathsf{f}_{\tau_0}\|^2_{2^*}}{\|\nabla \mathsf{f}_{\tau_0}\|^2_2} = \delta\, \mathcal{E}(f_{\tau_0}) \ge \delta\, \mathscr{I}(\delta)$$

Altogether: $\boxed{\text{if } \mathsf{d}(f,\mathcal{M})^2 > \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)},\ \text{then } \mathcal{E}(f) \ge \min\{\delta, \delta\, \mathscr{I}(\delta)\}}$

Part 2: The (simple) Taylor expansion

Proposition

Let $(X, d\mu)$ *be a measure space and* u, $r \in \mathrm{L}^q(X, d\mu)$ *for some* $q \geq 2$ *with* $u \geq 0$, $u + r \geq 0$ *and* $\int_X u^{q-1}\, r\, d\mu = 0$

▷ *If* $q = 6$, *then*

$$\|u+r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \Big(5\int_X u^{q-2}\, r^2\, d\mu + \tfrac{20}{3}\int_X u^{q-3}\, r^3\, d\mu + 5\int_X u^{q-4}\, r^4\, d\mu + 2\int_X u^{q-5}\, r^5\, d\mu + \tfrac{1}{3}\int_X r^6\, d\mu\Big)$$

▷ *If* $3 \leq q \leq 4$, *then*

$$\|u+r\|_q^2 - \|u\|_q^2$$

$$\leq \|u\|_q^{2-q} \left((q-1)\int_X u^{q-2}\, r^2\, d\mu + \frac{(q-1)\,(q-2)}{3}\int_X u^{q-3}\, r^3\, d\mu + \frac{2}{q}\int_X |r|^q\, d\mu\right)$$

▷ *If* $2 \leq q \leq 3$, *then*

$$\|u+r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \left((q-1)\int_X u^{q-2}\, r^2\, d\mu + \frac{2}{q}\int_X r_+^q\, d\mu\right)$$

Corollary

For all $\nu > 0$ *and for all* $r \in \mathrm{H}^1(\mathbb{S}^d)$ *satisfying* $r \ge -1$,

$$\left(\int_{\mathbb{S}^d} |r|^q \, d\mu\right)^{2/q} \le \nu^2 \quad \text{and} \quad \int_{\mathbb{S}^d} r \, d\mu = 0 = \int_{\mathbb{S}^d} \omega_j \, r \, d\mu \quad \forall j = 1, \dots d+1$$

if $d\mu$ *is the uniform probability measure on* $\mathbb{S}^d$, *then*

$$\int_{\mathbb{S}^d} \left(|\nabla r|^2 + \mathrm{A}\,(1+r)^2\right) d\mu - \mathrm{A}\left(\int_{\mathbb{S}^d} (1+r)^q \, d\mu\right)^{2/q} \ge \mathrm{m}(\nu) \int_{\mathbb{S}^d} \left(|\nabla r|^2 + \mathrm{A}\, r^2\right) d\mu$$

$$\begin{aligned} \mathrm{m}(\nu) &:= \tfrac{4}{d+4} - \tfrac{2}{q}\,\nu^{q-2} && \text{if} \quad d \ge 6 \\ \mathrm{m}(\nu) &:= \tfrac{4}{d+4} - \tfrac{1}{3}\,(q-1)\,(q-2)\,\nu - \tfrac{2}{q}\,\nu^{q-2} && \text{if} \quad d = 4,\, 5 \\ \mathrm{m}(\nu) &:= \tfrac{4}{7} - \tfrac{20}{3}\,\nu - 5\,\nu^2 - 2\,\nu^3 - \tfrac{1}{3}\,\nu^4 && \text{if} \quad d = 3 \end{aligned}$$

An explicit expression of $\mathscr{I}(\delta)$ if $\nu > 0$ is small enough so that $\mathrm{m}(\nu) > 0$

Part 3: Removing the positivity assumption

Take $f = f_+ - f_-$ with $\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = 1$ and define $m := \|f_-\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ and $1 - m = \|f_+\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} > 1/2$. The positive concave function

$$h_d(m) := m^{\frac{d-2}{d}} + (1-m)^{\frac{d-2}{d}} - 1$$

satisfies

$$2\,h_d(1/2)\,m \le h_d(m)\,, \quad h_d(1/2) = 2^{2/d} - 1$$

With $\delta(f) = \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, one finds $g_+ \in \mathcal{M}$ such that

$$\delta(f) \ge C^{d,\mathrm{pos}}_{\mathrm{BE}}\,\|\nabla f_+ - \nabla g_+\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{2\,h_d(1/2)}{h_d(1/2)+1}\,\|\nabla f_-\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

and therefore

$$C^d_{\mathrm{BE}} \ge \tfrac{1}{2}\,\min\left\{\max_{0<\delta<1/2}\,\delta\,\mathscr{I}(\delta), \frac{2\,h_d(1/2)}{h_d(1/2)+1}\right\}$$

Part 2, refined: The (complicated) Taylor expansion

To get a dimensionally sharp estimate, we expand $(1+r)^{2^*}-1-2^*r$ with an accurate remainder term for all $r\geq -1$

$$r_1:=\min\{r,\gamma\}\,,\quad r_2:=\min\left\{(r-\gamma)_+,M-\gamma\right\}\quad\text{and}\quad r_3:=(r-M)_+$$

with $0<\gamma<M$. Let $\theta=4/(d-2)$

Lemma

Given $d\geq 6$, $r\in[-1,\infty)$, and $\overline{M}\in[\sqrt{e},+\infty)$, we have

$$\begin{aligned}(1+r)^{2^*}&-1-2^*r\\ &\leq\tfrac12\,2^*\,(2^*-1)\,(r_1+r_2)^2+2\,(r_1+r_2)\,r_3+\left(1+C_M\,\theta\,\overline{M}^{-1}\ln\overline{M}\right)r_3^{2^*}\\ &\qquad+\left(\tfrac32\,\gamma\,\theta\,r_1^2+C_{M,\overline{M}}\,\theta\,r_2^2\right)\mathbb{1}_{\{r\leq M\}}+C_{M,\overline{M}}\,\theta\,M^2\,\mathbb{1}_{\{r>M\}}\end{aligned}$$

where all the constants in the above inequality are explicit

There are constants ϵ_1, ϵ_2, k_0, and $\epsilon_0 \in (0,1/\theta)$, such that

$$\begin{aligned}\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mathrm{A}\,\|1+r\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)}&\\ \ge \frac{4\,\epsilon_0}{d-2}\left(\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) + \sum_{k=1}^{3} I_k&\end{aligned}$$

$$I_1 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_1|^2 + \mathrm{A}\,r_1^2\right)d\mu - \mathrm{A}\left(2^*-1+\epsilon_1\,\theta\right)\int_{\mathbb{S}^d} r_1^2\,d\mu + \mathrm{A}\,k_0\,\theta\int_{\mathbb{S}^d}(r_2^2 +$$

$$I_2 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_2|^2 + \mathrm{A}\,r_2^2\right)d\mu - \mathrm{A}\left(2^*-1+(k_0+C_{\epsilon_1,\epsilon_2})\,\theta\right)\int_{\mathbb{S}^d} r_2^2\,d\mu$$

$$I_3 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_3|^2 + \mathrm{A}\,r_3^2\right)d\mu - \tfrac{2}{2^*}\,\mathrm{A}\left(1+\epsilon_2\,\theta\right)\int_{\mathbb{S}^d} r_3^{2^*}\,d\mu - \mathrm{A}\,k_0\,\theta\int_{\mathbb{S}^d} r_3^2\,d\mu$$

- spectral gap estimates : $I_1 \ge 0$
- Sobolev inequality : $I_3 \ge 0$
- improved spectral gap inequality using that $\mu\big(\{r_2>0\}\big)$ is small: $I_2 \ge 0$

Explicit stability result for the logarithmic Sobolev inequality

Subcritical interpolation inequalities on the sphere

- ***Gagliardo-Nirenberg-Sobolev inequality***

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2,2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* = +\infty$ if $d = 1$ or 2

- Limit $p \to 2$: the ***logarithmic Sobolev inequality***

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu \ge \frac{d}{2}\int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

Gagliardo-Nirenberg inequalities: a result by R. Frank

[Frank, 2022]: if $p \in (2, 2^*)$, there is $\mathsf{c}(d,p) > 0$ such that

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \geq \mathsf{c}(d,p)\,\frac{\left(\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F - \overline{F}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^2}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

where $\overline{F} := \int_{\mathbb{S}^d} F\,d\mu$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \geq \mathsf{c}(d,p)\,\frac{\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

- a compactness method
- the exponent 4 in the r.h.s. is optimal
- the (generalized) entropy is

$$\mathcal{E}_p[u] := \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Gagliardo-Nirenberg inequalities: stability

An improved inequality under orthogonality constraint and the stability inequality arising from the *carré du champ* method can be combined as follows

Theorem

Let $d \ge 1$ and $p \in (1,2) \cup (2,2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\,\mathcal{E}_p[F] \ge \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

Carré du champ – admissible parameters on $\mathbb{S}^d$

[JD, Esteban, Kowalczyk, Loss] Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u} \right)$$

$$m_\pm(d,p) := \tfrac{1}{(d+2)\,p} \left(d\,p + 2 \pm \sqrt{d\,(p-1)\,\big(2\,d-(d-2)\,p\big)} \right)$$

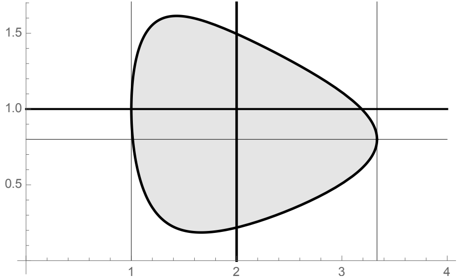

Figure: Case $d=5$: admissible parameters $1 \le p \le 2^* = 10/3$ and m (horizontal axis: p, vertical axis: m). Improved inequalities inside !

Gaussian carré du champ and nonlinear diffusion

$$\frac{\partial v}{\partial t} = v^{-p\,(1-m)} \left(\mathcal{L} v + (m\,p-1)\,\frac{|\nabla v|^2}{v} \right) \quad \text{on} \quad \mathbb{R}^n$$

[JD, Brigati, Simonov] *Ornstein-Uhlenbeck operator:* $\mathcal{L} = \Delta - x \cdot \nabla$

$$m_\pm(p) := \lim_{d \to +\infty} m_\pm(d,p) = 1 \pm \frac{1}{p}\sqrt{(p-1)\,(2-p)}$$

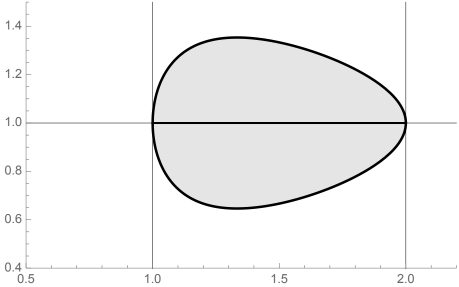

Figure: The admissible parameters $1 \le p \le 2$ and m are independent of n

Large dimensional limit

Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)$$

Theorem

Let $v \in \mathrm{H}^1(\mathbb{R}^n, dx)$ *with compact support,* $d \ge n$ *and*

$$u_d(\omega) = v\Big(\omega_1/r_d, \omega_2/r_d, \ldots, \omega_n/r_d\Big), \quad r_d = \sqrt{\frac{d}{2\,\pi}}$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. *With* $d\gamma(y) := (2\,\pi)^{-n/2}\, e^{-\frac{1}{2}\,|y|^2}\, dy$,

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

Stability of LSI: some comments

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma \\ \ge \frac{\beta\,\pi}{2} \inf_{a\in\mathbb{R}^d,\,c\in\mathbb{R}} \int_{\mathbb{R}^n} |u - c\,e^{a\cdot x}|^2\,d\gamma$$

- The $\dot{\mathrm{H}}^1(\mathbb{R}^n)$ does not appear, it gets lost in the limit $d \to +\infty$
- Two proofs. Taking the limit is difficult because of the lack of compactness
- One dimension is lost (for the manifold of invariant functions) in the limiting process
- Euclidean forms of the stability
- $\int_{\mathbb{R}^n} |\nabla(u - c\,e^{a\cdot x})|^2\,d\gamma$? False, but makes sense under additioal assumptions. Some results based on the Ornstein-Uhlenbeck flow and entropy methods: [Fathi, Indrei, Ledoux, 2016], [JD, Brigati, Simonov, 2023-24]

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