Gagliardo-Nirenberg-Sobolev inequalities : Entropy methods and stability

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New Trends in Nonlinear Diffusion:
a Bridge between PDEs, Analysis and Geometry

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Stability in Gagliardo-Nirenberg-Sobolev inequalities

- Lecture 1 [Bruno] A ***variational*** point of view
Variational methods provide good stability results. The deficit functional is estimated by a relative entropy, or a relative Fisher information. However, as in the Bianchi-Egnell method, ***estimates are non-constructive****.*

- Lecture 2 [Nikita] ***Convergence in relative error*** for the FDE
The fast diffusion equation (FDE) has great regularization properties. These quantities are ***constructive****.*

- Lecture 3: Entropy methods and ***stability***
Gagliardo-Nirenberg-Sobolev inequalities can be reformulated as ***entropy – entropy production inequalities****. Entropy methods are fully constructive. Using the FDE as a tool, we obtained* ***improved inequalities*** *that can be reinterpreted as constructive stability estimates.*

Joint work with Matteo Bonforte, Bruno Nazaret, Nikita Simonov

Joint work on ***Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method*** arXiv:2007.03674 (Apr. 29, 2021) with

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Outline

- Chapter 2: Inequalities, ***entropies***, flows

▷ Rényi entropy powers, Gagliardo-Nirenberg-Sobolev inequalities and fast diffusion
▷ The fast diffusion equation in self-similar variables:
relative entropy and the entropy – entropy production inequality
▷ ***Large time asymptotics*** and increased spectral gaps
▷ ***Initial time layer*** and improved entropy – entropy production estimates

- Chapter 5: ***Stability in*** (subcritical) ***Gagliardo-Nirenberg inequalities***

▷ The ***threshold*** time based on regularity results
▷ Gluing the ***initial and asymptotic time layer*** estimates
▷ Form an improved entropy – entropy production inequality to stability

- Chapter 6: ***Stability in Sobolev's inequality*** (critical case)

▷ A constructive stability result
▷ The main ingredient of the proof

From the carré du champ method to stability results

Carré du champ method (adapted from D. Bakry and M. Emery)

$$\frac{\partial u}{\partial t} = \Delta u^m\,, \quad \frac{d\mathscr{F}}{dt} = -\mathscr{I}\,, \quad \frac{d\mathscr{I}}{dt} \le -\Lambda\,\mathscr{I}$$

deduce that $\mathscr{I} - \Lambda\mathscr{F}$ is monotone non-increasing with limit 0

$$\boxed{\mathscr{I}[u] \ge \Lambda\,\mathscr{F}[u]}$$

▷ ***Improved constant*** means ***stability***
Under some restrictions on the functions, there is some $\Lambda_\star \ge \Lambda$ such that

$$\mathscr{I} - \Lambda\,\mathscr{F} \ge (\Lambda_\star - \Lambda)\,\mathscr{F}$$

▷ ***Improved entropy – entropy production inequality*** (weaker form)

$$\mathscr{I} \ge \Lambda\,\psi(\mathscr{F})$$

for some ψ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi'' > 0$

$$\mathscr{I} - \Lambda\,\mathscr{F} \ge \Lambda\big(\psi(\mathscr{F}) - \mathscr{F}\big) \ge 0$$

Fast diffusion equation and entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{FDE}$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy – entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)
- Initial time layer: improved entropy – entropy production estimates

The fast diffusion equation in original variables

Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \ge 1$, $m \in (0,1)$

$$\frac{\partial u}{\partial t} = \Delta u^m$$

with initial datum $u(t=0,x) = u_0(x) \ge 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, dx = \mathcal{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \, dx < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$B(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathscr{B}\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$ and $\mathscr{B}$ is the Barenblatt profile

$$\mathscr{B}(x) := \left(C + |x|^2\right)^{-\frac{1}{1-m}}$$

Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With $m \geq m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x = 0$$

(ii) *Second moment.* With $m > d/(d+2)$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d,(1+|x|^2)\,\mathrm{d}x)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2\,u(t,x)\,\mathrm{d}x = 2\,d\int_{\mathbb{R}^d} u^m(t,x)\,\mathrm{d}x$$

(iii) *Entropy estimate.* With $m \geq m_1 := (d-1)/d$, $u_0^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d,(1+|x|^2)\,\mathrm{d}x)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u^m(t,x)\,\mathrm{d}x = \frac{m^2}{1-m}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,\mathrm{d}x$$

Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\,\mathrm{d}x \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,\mathrm{d}x$$

Entropy growth rate

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p} \qquad \text{(GNS)}$$

$$p=\frac{1}{2\,m-1}\quad\Longleftrightarrow\quad m=\frac{p+1}{2\,p}\in[m_1,1)$$

$u=f^{2p}$ so that $u^m=f^{p+1}$ and $u\,|\nabla u^{m-1}|^2=(p-1)^2\,|\nabla f|^2$

$$M=\|f\|_{2p}^{2p}\,,\quad \mathsf{E}[u]=\|f\|_{p+1}^{p+1}\,,\quad \mathsf{I}[u]=(p+1)^2\,\|\nabla f\|_2^2$$

If u solves (FDE) $\frac{\partial u}{\partial t}=\Delta u^m$

$$\mathsf{E}'\ge\frac{p-1}{2\,p}\,(p+1)^2\,\Big(\mathscr{C}_{\mathrm{GNS}(p)}\Big)^{\frac{2}{\theta}}\,\|f\|_{2p}^{\frac{2}{\theta}}\,\|f\|_{p+1}^{-\frac{2(1-\theta)}{\theta}}=C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d}u^m(t,x)\,\mathrm{d}x\ge\left(\int_{\mathbb{R}^d}u_0^m\,\mathrm{d}x+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\quad\forall\,t\ge0$$

Equality case: $u(t,x)=\frac{c_1}{R(t)^d}\,\mathscr{B}\left(\frac{c_2\,x}{R(t)}\right)\,,\ \mathscr{B}(x):=\left(1+|x|^2\right)^{\frac{1}{m-1}}$

Pressure variable and decay of the Fisher information

The t-derivative of the *Rényi entropy power* $\mathsf{E}^{\frac{2}{d}\,\frac{1}{1-m}-1}$ is proportional to

$$\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

▷ *Pressure variable*

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

▷ *Fisher information*

$$\mathsf{I}[u] = \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx$$

If u solves (FDE), then

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla \mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta \mathsf{P} + |\nabla \mathsf{P}|^2\Big)\,dx$$

$$= -2\int_{\mathbb{R}^d} u^m\,\Big(\|\mathrm{D}^2\mathsf{P}\|^2 - (1-m)\,(\Delta \mathsf{P})^2\Big)\,dx$$

Rényi entropy powers and interpolation inequalities

▷ Integrations by parts and completion of squares

$$-\frac{\mathsf{I}}{2\theta}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)$$

$$=\int_{\mathbb{R}^d}u^m\left\|\mathrm{D}^2\mathsf{P}-\frac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right\|^2\mathrm{d}x+(m-m_1)\int_{\mathbb{R}^d}u^m\left|\Delta\mathsf{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2\mathrm{d}x$$

▷ Analysis of the asymptotic regime as $t\to+\infty$

$$\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^{\theta}\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{M^{\frac{2\theta}{p}}}=\frac{\mathsf{I}[\mathcal{B}]^{\theta}\,\mathsf{E}[\mathcal{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathcal{B}\|_1^{\frac{2\theta}{p}}}=(p+1)^{2\theta}\,(\mathscr{C}_{\mathrm{GNS}}(p))^{2\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$\mathsf{I}[u]^{\theta}\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\geq(p+1)^{2\theta}\,(\mathscr{C}_{\mathrm{GNS}}(p))^{2\theta}\,M^{\frac{2\theta}{p}}$$

The fast diffusion equation in self-similar variables

- ▷ Rescaling and self-similar variables
- ▷ Relative entropy and the entropy – entropy production inequality
- ▷ Large time asymptotics and spectral gaps

Entropy – entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x) = \frac{1}{\kappa^d R^d} v\left(\tau, \frac{x}{\kappa R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \tfrac{1}{2} \log R(t)$$

$\frac{\partial u}{\partial t} = \Delta u^m$ is changed into *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \left[v \left(\nabla u^{m-1} - 2\,x \right) \right] = 0 \qquad (r\,\mathrm{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathscr{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathscr{B}^m - m\,\mathscr{B}^{m-1} (v - \mathscr{B}) \right) \mathrm{d}x$$

$$\mathscr{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2 \mathrm{d}x$$

are such that $\mathscr{I}[v] \geq 4\,\mathscr{F}[v]$ by (GNS) [del Pino, JD, 2002] so that

$$\mathscr{F}[v(t,\cdot)] \leq \mathscr{F}[v_0]\, e^{-4\,t}$$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}}\le v_0\le\left(C_1+|x|^2\right)^{-\frac{1}{1-m}}\qquad\text{(H)}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d}f^2\,\mathrm{d}\mu_{\alpha-1}\le\int_{\mathbb{R}^d}|\nabla f|^2\,\mathrm{d}\mu_\alpha\quad\forall\,f\in\mathrm{H}^1(\mathrm{d}\mu_\alpha)\,,\quad\int_{\mathbb{R}^d}f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha:=(1+|x|^2)^\alpha\,dx$, for $\alpha<0$

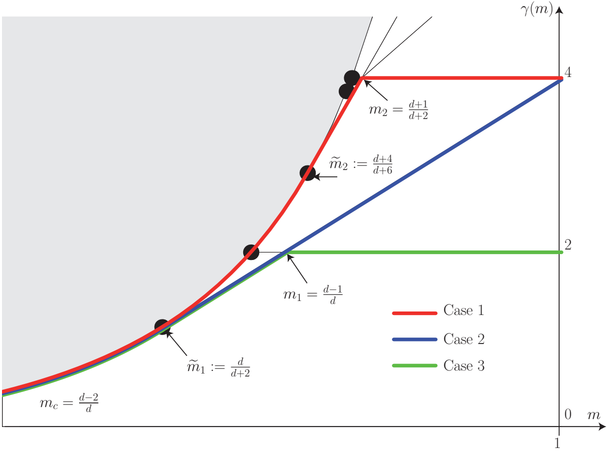
Lemma

Under assumption (H),

$$\mathcal{F}[v(t,\cdot)]\le C\,e^{-2\gamma(m)\,t}\quad\forall\,t\ge0\,,\quad\gamma(m):=(1-m)\,\Lambda_{1/(m-1),d}$$

Moreover $\gamma(m):=2$ *if* $1-1/d\le m<1$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Initial and asymptotic time layers

▷ Asymptotic time layer: constraint, spectral gap and improved entropy – entropy production inequality

▷ Initial time layer: the carré du champ inequality and a backward estimate

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathscr{B}^{2-m}\,\mathrm{d}x \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,\mathrm{d}x$$

Hardy-Poincaré inequality. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}^{2-m}\,dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,\mathrm{d}x = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,\mathrm{d}x = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2\,(d\,m - d + 1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\,\mathrm{d}x = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,\mathrm{d}x = 0$ and

$$(1-\varepsilon)\,\mathscr{B} \le v \le (1+\varepsilon)\,\mathscr{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then

$$\mathscr{I}[v] \ge (4+\eta)\,\mathscr{F}[v]$$

The initial time layer improvement: backward estimate

A hint: for some strictly convex function ψ with $\psi(0) = \psi'(0) = 0$, we have

$$\mathscr{I} - 4\mathscr{F} \geq 4\big(\psi(\mathscr{F}) - \mathscr{F}\big) \geq 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions) for (FDE), we expect some improvement

Rephrasing the *carré du champ* method, $\mathscr{Q}[v] := \frac{\mathscr{I}[v]}{\mathscr{F}[v]}$ is such that

$$\frac{d\mathscr{Q}}{dt} \leq \mathscr{Q}\,(\mathscr{Q} - 4)$$

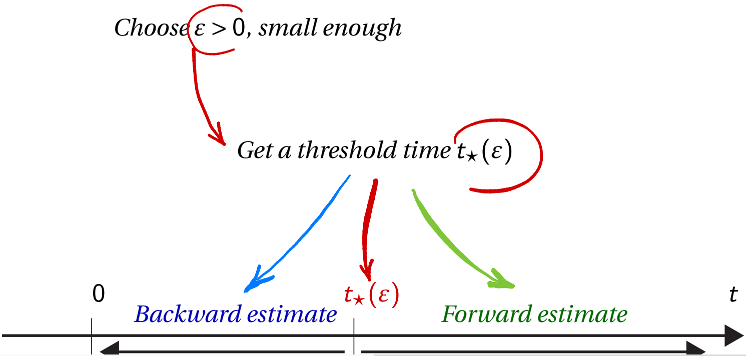
Lemma

Assume that $m > m_1$ and v is a solution to $(r\,\mathrm{FDE})$ with nonnegative initial datum v_0. If for some $\eta > 0$ and $T > 0$, we have $\mathscr{Q}[v(T,\cdot)] \geq 4 + \eta$, then

$$\mathscr{Q}[v(t,\cdot)] \geq 4 + \frac{4\,\eta\,e^{-4\,T}}{4 + \eta - \eta\,e^{-4\,T}} \quad \forall\, t \in [0, T]$$

Stability in (subcritical) Gagliardo-Nirenberg inequalities

Our strategy



The threshold time and the uniform convergence in relative error

▷ The regularity results (Lecture 2) allow us to glue the initial time layer estimates with the asymptotic time layer estimates:

The improved entropy – entropy production inequality holds for any time along the evolution along (r FDE)

(and in particular for the initial datum)

Uniform convergence in relative error

Theorem

Assume that $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$ *and let* $\varepsilon \in (0, 1/2)$, *small enough,* $A > 0$, *and* $G > 0$ *be given. There exists an explicit threshold time* $T \geq 0$ *such that, if* u *is a solution of*

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *satisfying*

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} u_0 \, dx \leq A < \infty \tag{H_A}$$

$\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B} \, \mathrm{d}x = \mathscr{M}$ *and* $\mathscr{F}[u_0] \leq G$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall \, t \geq T$$

The threshold time

Proposition

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/3,1)$ if $d=1$, $\varepsilon\in(0,\varepsilon_{m,d})$, $A>0$ and $G>0$

$$T=\mathsf{c}_\star\,\frac{1+A^{1-m}+G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a}=\frac{\alpha}{\vartheta}\,\frac{2-m}{1-m}$, $\alpha=d\,(m-m_c)$ and $\vartheta=\nu/(d+\nu)$

$$\mathsf{c}_\star=\mathsf{c}_\star(m,d)=\sup_{\varepsilon\in(0,\varepsilon_{m,d})}\max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\,\varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\,\varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m):=\max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1},\frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m):=\frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}}\quad\text{and}\quad\kappa_3(\varepsilon,m):=\frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

Improved entropy – entropy production inequality

Theorem

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. Then there is a positive number ζ such that

$$\mathscr{I}[v]\ge(4+\zeta)\,\mathscr{F}[v]$$

for any nonnegative function $v\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v]=G$, $\int_{\mathbb{R}^d}v\,\mathrm{d}x=\mathscr{M}$, $\int_{\mathbb{R}^d}x\,v\,\mathrm{d}x=0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon\in(0,2\varepsilon_\star)\,,\quad \varepsilon_\star:=\frac12\min\{\varepsilon_{m,d},\chi\,\eta\}\quad\text{with}\quad T=\frac12\log R(T)$$

$$(1-\varepsilon)\,\mathscr{B}\le v(t,\cdot)\le(1+\varepsilon)\,\mathscr{B}\quad\forall\,t\ge T$$

and, as a consequence, the *initial time layer estimate*

$$\mathscr{I}[v(t,.)]\ge(4+\zeta)\,\mathscr{F}[v(t,.)]\quad\forall\,t\in[0,T],\quad\text{where}\quad \zeta=\frac{4\,\eta\,e^{-4\,T}}{4+\eta-\eta\,e^{-4\,T}}$$

Two consequences

$$\zeta = \mathsf{Z}\big(A, \mathscr{F}[u_0]\big)\,,\quad \mathsf{Z}(A,G) := \frac{\zeta_\star}{1 + A^{(1-m)\frac{2}{\alpha}} + G}\,,\quad \zeta_\star := \frac{4\,\eta\,c_\alpha}{4+\eta}\left(\frac{\varepsilon_\star^a}{2\,\alpha\,c_\star}\right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (r FDE) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ and v_0 satisfies (H_A), then

$$\mathscr{F}[v(t,.)] \leq \mathscr{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\, t \geq 0$$

▷ The ***stability in the entropy - entropy production estimate*** $\mathscr{I}[v] - 4\,\mathscr{F}[v] \geq \zeta\,\mathscr{F}[v]$ also holds in a stronger sense

$$\mathscr{I}[v] - 4\,\mathscr{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathscr{I}[v]$$

Stability results

▷ We rephrase the results obtained by entropy methods in the language of stability *à la* Bianchi-Egnell (as in Lecture 1)

Subcritical range

$$p^* = +\infty \text{ if } d = 1 \text{ or } 2,\ p^* = \frac{d}{d-2} \text{ if } d \geq 3$$

$$\lambda[f] := \left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2p}{d-p(d-4)}}\,,\quad \kappa[f] := \frac{\mathscr{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

$$\mathsf{A}[f] := \frac{\mathscr{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}}\,\|f\|_{2p}^{2p}}\,\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}}\int_{|x|>r}|f(x+x_f)|^{2p}\,dx$$

$$\mathsf{E}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}}\,f^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p}\,\mathrm{g}^{1-p}\left(\frac{\kappa[f]^{2p}}{\lambda[f]^2}\,f^{2p} - \mathrm{g}^{2p}\right)\right)\mathrm{d}x$$

$$\mathfrak{S}[f] := \frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf{Z}(\mathsf{A}[f],\mathsf{E}[f])$$

Theorem

Let $d \ge 1$, $p \in (1,p^*)$

If $f \in \mathscr{W}_p(\mathbb{R}^d) := \left\{f \in \mathrm{L}^{2p}(\mathbb{R}^d) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\ |x|\,f^p \in \mathrm{L}^2(\mathbb{R}^d)\right\}$,

$$\left(\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\right)^{2p\gamma} - \left(\mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p}\right)^{2p\gamma} \ge \mathfrak{S}[f]\,\|f\|_{2p}^{2p\gamma}\,\mathsf{E}[f]$$

With $\mathscr{K}_{\mathrm{GNS}} = C(p,d)\,\mathscr{C}_{\mathrm{GNS}}^{2p\gamma}$, $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GNS}}\,\|f\|_{2p}^{2p\gamma}$$

Theorem

Let $d \ge 1$ and $p \in (1,p^*)$. There is an explicit $\mathscr{C} = \mathscr{C}[f]$ such that, for any $f \in \mathrm{L}^{2p}(\mathbb{R}^d,(1+|x|^2)\,dx)$ such that $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathsf{A}[f^{2p}] < \infty$,

$$\delta[f] \ge \mathscr{C}[f] \inf_{\varphi\in\mathscr{M}} \int_{\mathbb{R}^d} \left|(p-1)\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 \mathrm{d}x$$

▷ The dependence of $\mathscr{C}[f]$ on $\mathsf{A}[f^{2p}]$ and $\mathscr{F}[f^{2p}]$ is explicit and does not degenerate if $f \in \mathscr{M}$

▷ Can we remove the condition $\mathsf{A}[f^{2p}] < \infty$?

Stability in Sobolev's inequality (critical case)

- ▷ A constructive stability result
- ▷ The main ingredient of the proof

A constructive stability result

Let $2\,p^\star=2d/(d-2)=2^*$, $d\ge3$ and

$$\mathscr{W}_{p^\star}(\mathbb{R}^d)=\left\{f\in\mathrm{L}^{p^\star+1}(\mathbb{R}^d)\,:\,\nabla f\in\mathrm{L}^2(\mathbb{R}^d),\;|x|\,f^{p^\star}\in\mathrm{L}^2(\mathbb{R}^d)\right\}$$

Theorem

Let $d\ge3$ and $A>0$. Then for any nonnegative $f\in\mathscr{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d}(1,x,|x|^2)\,f^{2^*}\,\mathrm{d}x=\int_{\mathbb{R}^d}(1,x,|x|^2)\,\mathrm{g}\,\mathrm{d}x\quad\text{and}\quad\sup_{r>0}r^d\int_{|x|>r}f^{2^*}\,\mathrm{d}x\le A$$

we have

$$\delta[f]:=\|\nabla f\|_2^2-\mathsf{S}_d^2\,\|f\|_{2^*}^2\ge\frac{\mathscr{C}_\star(A)}{4+\mathscr{C}_\star(A)}\int_{\mathbb{R}^d}\left|\nabla f+\tfrac{d-2}{2}\,f^{\frac{d}{d-2}}\,\nabla\mathrm{g}^{-\frac{2}{d-2}}\right|^2\mathrm{d}x$$

$\mathscr{C}_\star(A)=\mathfrak{C}_\star\left(1+A^{1/(2d)}\right)^{-1}$ *and $\mathfrak{C}_\star>0$ depends only on d*

We can remove the normalization of f, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x+x_f) \quad \text{and} \quad Z[f] := \left(1+\mu[f]^{-d}\,\lambda[f]^d\,A[f]\right)$$

the *Bianchi-Egnell type result*

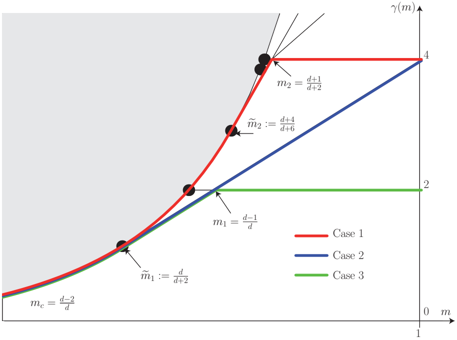
$$\delta[f] \ge \frac{\mathfrak{C}_\star\,Z[f]}{4+Z[f]} \inf_{g\in\mathscr{M}} \mathscr{J}[f|g]$$

with x_f, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathscr{B}_\lambda(x) = \lambda^{-d/2}\,\mathscr{B}\big(x/\sqrt{\lambda}\big)$ in order to match $\int_{\mathbb{R}^d} |x|^2\, v\, dx$ where the function v solves (r FDE) or to further rescale v according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\, w\left(t + \tau(t), \frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{d\tau}{dt} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d} |x|^2\, v\, dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,, \quad \tau(0) = 0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\tau(t)}$$

Lemma

$t \mapsto \tau(t)$ is bounded on $\mathbb{R}^+$

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Thank you for your attention !

1. Fast diffusion equation and entropy methods
   - Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities
   - The fast diffusion equation in self-similar variables
   - Initial and asymptotic time layers
2. Stability in (subcritical) Gagliardo-Nirenberg inequalities
   - The threshold time
   - Improved entropy – entropy production inequality
   - First stability results
3. Stability in Sobolev's inequality (critical case)
   - A constructive stability result
   - The main ingredient of the proof