Asymptotic behaviour for small mass in the two-dimensional parabolic-elliptic Keller-Segel model

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Outline

- Introduction: the two-dimensional parabolic-elliptic Keller-Segel model, the $M < 8\pi$ regime, scalings, etc

- First digression : functional inequalities, rates, etc

- Second digression : relative entropies and linearization

- The asymptotic behaviour of the solutions of the Keller-Segel model

References on the general theory of Keller-Segel systems: [Horstmann]
Disclaimer: many references !

Introduction

The parabolic-elliptic Keller and Segel system

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,,\ t > 0 \\[2em]
-\Delta v = u & x \in \mathbb{R}^2 \,,\ t > 0 \\[2em]
u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2
\end{cases}
$$

We make the choice:

$$
v(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| \, u(t, y) \, dy
$$

and observe that

$$
\nabla v(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \, u(t, y) \, dy
$$

Mass conservation: $\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^2} u(t, x) \, dx = 0$

Blow-up

$M = \int_{\mathbb{R}^2} n_0 \, dx > 8\pi$ and $\int_{\mathbb{R}^2} |x|^2 \, n_0 \, dx < \infty$: blow-up in finite time
a solution u of

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v)$$

satisfies

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \, u(t,x) \, dx$$

$$= -\int_{\mathbb{R}^2} 2x \, \Delta u \, dx + \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \underbrace{\frac{2x \cdot (y-x)}{|x-y|^2} \, u(t,x) \, u(t,y) \, dx \, dy}_{\frac{(x-y) \cdot (y-x)}{|x-y|^2} \, u(t,x) \, u(t,y) \, dx \, dy}$$

$$= 4 \, M - \frac{M^2}{2\pi} < 0 \quad \text{if} \quad M > 8\pi$$

Existence and free energy

$M = \int_{\mathbb{R}^2} n_0 \, dx \leq 8\pi$: global existence [Jäger, Luckhaus], [JD, Perthame], [Blanchet, JD, Perthame], [Blanchet, Carrillo, Masmoudi]

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u \ (\nabla (\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u \, dx - \frac{1}{2} \int_{\mathbb{R}^2} u \, v \, dx$$

satisfies

$$\frac{d}{dt} F[u(t, \cdot)] = - \int_{\mathbb{R}^2} u \left| \nabla (\log u) - \nabla v \right|^2 \, dx$$

Log HLS inequality [Carlen, Loss]: F is bounded from below if $M < 8\pi$

The dimension $d = 2$

- In dimension d, the norm $L^{d/2}(\mathbb{R}^d)$ is critical. If $d = 2$, the mass is critical

- Scale invariance: if (u, v) is a solution in $\mathbb{R}^2$ of the parabolic-elliptic Keller and Segel system, then

$$\left(\lambda^2 \, u(\lambda^2 \, t, \lambda \, x) , \; v(\lambda^2 \, t, \lambda \, x) \right)$$

  is also a solution

- For $M < 8\pi$, the solution vanishes as $t \to \infty$, but saying that "diffusion dominates" is not correct: to see this, study "intermediate asymptotics"

The existence setting

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \; t > 0 \\[2ex] -\Delta v = u & x \in \mathbb{R}^2 \,, \; t > 0 \\[2ex] u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\, dx) \,, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx) \,, \quad M := \int_{\mathbb{R}^2} n_0(x)\, dx < 8\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\, dx$ for any $t \geq 0$, see [Jäger-Luckhaus], [Blanchet, JD, Perthame]

$$v = -\tfrac{1}{2\pi} \log | \cdot | * u$$

Time-dependent rescaling

$$u(x,t) = \frac{1}{R^2(t)}\, n\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad v(x,t) = c\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1+2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\[2mm] c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\[2mm] n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

[Blanchet, JD, Perthame] Convergence in self-similar variables

$$\lim_{t\to\infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

means "intermediate asymptotics" in original variables:

$$\left\| u(x,t) - \tfrac{1}{R^2(t)}\, n_\infty\left(\tfrac{x}{R(t)}, \tau(t)\right) \right\|_{L^1(\mathbb{R}^2)} \searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M \, \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2} \, dx} = -\Delta c_\infty \; , \qquad c_\infty = -\frac{1}{2\pi} \log |\cdot| * n_\infty$$

- Radial symmetry [Naito]

- Uniqueness [Biler, Karch, Laurençot, Nadzieja]

- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon \in (0,1)$ [Blanchet, JD, Perthame]

- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M:

$$\lim_{M \to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[Joseph, Lundgreen] [JD, Stańczy]

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla \left[n \left(\log n - x + \nabla c \right) \right]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \ dx + \int_{\mathbb{R}^2} \frac{1}{2} |x|^2 \, n \ dx - \frac{1}{2} \int_{\mathbb{R}^2} n \, c \ dx$$

satisfies

$$\frac{d}{dt} F[n(t, \cdot)] = - \int_{\mathbb{R}^2} n \left| \nabla \left(\log n \right) + x - \nabla c \right|^2 \ dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2 \, n(\lambda \, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2) \ dx + \int_{\mathbb{R}^2} \frac{\lambda^{-2} - 1}{2} |x|^2 \, n \ dx + \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) \, n(y) \log \frac{1}{\lambda} \ dx \, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi} \right)}_{>0 \ \text{if} \ M < 8\pi} \log \lambda + \frac{\lambda^{-2} - 1}{2} \int_{\mathbb{R}^2} |x|^2 \, n \ dx$$

Main result

Theorem 1. *There exists a positive constant M^* such that, for any initial data $n_0 \in L^2(n_\infty^{-1}\, dx)$ of mass $M < M^*$ satisfying the above assumptions, there is a unique solution $n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau, \infty) \times \mathbb{R}^2)$ for any $\tau > 0$*

Moreover, there are two positive constants, C and δ, such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \, \frac{dx}{n_\infty} \le C\, e^{-\delta\, t} \quad \forall\, t > 0$$

As a function of M, δ is such that $\lim_{M \to 0_+} \delta(M) = 1$

The condition $M \le 8\,\pi$ is necessary and sufficient for the global existence of the solutions, but there are two extra smallness conditions in our proof:

- Uniform estimate: the *method of the trap*
- *Spectral gap* of a linearised operator $\mathcal{L}$

Digression 1: Functional inequalities, rates

Some comments on the Fokker-Planck equation

$$\frac{\partial n}{\partial t} = \Delta n + \nabla(x\, n)$$

L^2 framework

- Spectral decomposition: use Hermite functions. Spectrum is made of all positive integers and

$$n_\infty(x) = \frac{M}{(2\pi)^{d/2}} \, e^{-|x|^2/2}$$

is the unique stationary solution (in the kernel).

- Return to equilibrium: if you are interested only in the rate of return to equilibrium, use spectral gap / Poincaré inequality with gaussian weight n_∞

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \left| \frac{n}{n_\infty} - 1 \right|^2 n_\infty \, dx = - \int_{\mathbb{R}^2} \left| \nabla \left(\frac{n}{n_\infty} \right) \right|^2 n_\infty \, dx \leq - \int_{\mathbb{R}^2} \left| \frac{n}{n_\infty} - 1 \right|^2 n_\infty \, dx$$

$$\|n - n_\infty\|_{L^2}^2 \leq \int_{\mathbb{R}^2} \frac{|n - n_\infty|}{\sqrt{n_\infty}} \sqrt{n_\infty} \left(\|n_0\|_{L^\infty} + \|n_\infty\|_{L^\infty} \right) \leq C \, e^{-t}$$

L^1 **framework**

- Return to equilibrium: use gaussian logarithmic Sobolev inequality

$$\frac{d}{dt} \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx = -\int_{\mathbb{R}^2} \left|\nabla \log\left(\frac{n}{n_\infty}\right)\right|^2 n_\infty \, dx \leq -2 \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx$$

  and conclude by the Csiszár-Kullback inequality

$$\|n - n_\infty\|_{L^1}^2 \leq 4\, M \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx \leq C\, e^{-2t}$$

- Simpler ! If you are only interested in the asymptotic rates, observe that

$$s \log s - s + 1 \leq |s - 1|^2$$

[Bartier, Blanchet, J.D., Escobedo]

Digression 2: Relative entropies and linearization: fast diffusions

Some comments on the fast diffusion ($m < 1$) equation (in self-similar variables)

$$\frac{\partial n}{\partial t} = \Delta n^m + \nabla(x\, n)$$

Convergence towards the stationary solution ?

- $1 - \frac{1}{d} \leq m < 1$: Gagliardo-Nirenberg inequalities [del Pino, J.D.] and many others

- $1 - \frac{2}{d} \leq m < 1$: entropy - entropy production approach (Bakry-Emery method) [Carrillo et al.]

- Linearization [Blanchet, Bonforte, J.D., Grillo, Vázquez]

The free energy for fast diffusions

With $w := u/u_\infty$, we define

- the relative entropy

$$\mathcal{F}[w] := \frac{1}{1-m} \int_{\mathbb{R}^d} \left[(w-1) - \frac{1}{m}(w^m - 1) \right] u_\infty^m \, dx$$

- the relative Fisher information

$$\mathcal{I}[w] := \frac{m}{(m-1)^2} \int_{\mathbb{R}^d} \left| \nabla \left[(w^{m-1} - 1) \, u_\infty^{m-1} \right] \right|^2 w \, u_\infty \, dx$$

$$\frac{d}{dt} \mathcal{F}[w(t)] = -\mathcal{I}[w(t)]$$

- Linearized entropy and Fisher information functionals: $w = 1 + \varepsilon \, \frac{g}{u_\infty^{m-1}}$

$$\mathsf{F}[g] := \frac{1}{2} \int_{\mathbb{R}^d} |g|^2 \, u_\infty^{2-m} \, dx \quad \text{and} \quad \mathsf{I}[g] := m \int_{\mathbb{R}^d} |\nabla g|^2 \, u_\infty \, dx$$

Hardy-Poincaré inequalities and asymptotic behaviour

Hardy-Poincaré inequalities: for any $m \neq m_* = (d-4)/(d-2)$, $m \in (0,1)$, for any $g \in \mathcal{D}(\mathbb{R}^d)$,

$$\mathsf{F}[g] \leq \mathcal{C}_{m,d} \, \mathsf{I}[g]$$

under the condition $\int_{\mathbb{R}^d} g \, u_\infty^{2-m} \, dx = 0$ if $m > m_*$

- Relative entropy

$$C_1 \, \mathsf{F}[g] \leq \mathcal{F}[w] \leq C_2 \, \mathsf{F}[g]$$

- Fisher information

$$\mathsf{I}[g] \leq \beta_1 \, \mathcal{I}[w] + \beta_2 \, \mathsf{F}[g] \quad \text{with} \quad g := (w-1) \, u_\infty^{m-1}$$

- Conclusion

$$\frac{d}{dt} \mathcal{F}[w(t)] = -\mathcal{I}[w(t)] \leq -\frac{1}{\beta_1} \mathsf{I}[g] + \frac{\beta_2}{\beta_1} \, \mathsf{F}[g] \leq -\frac{1}{\beta_1 \, C_2} \left(\frac{1}{\mathcal{C}_{m,d}} - \beta_2 \right) \mathcal{F}[w(t)]$$

Back to the Keller-Segel model

Proof of the main result

- First step: the trap

- Second step: weighted H^1 estimates

- Third step: linearization and spectral gap

- Fourth step: collecting the estimates

The parabolic-elliptic Keller and Segel system

$$
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-\Delta v = u & x \in \mathbb{R}^2 \, , \ t > 0 \\[2ex]
u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2
\end{cases}
$$

Initial conditions

$$
n_0 \in L^1_+(\mathbb{R}^2, (1 + |x|^2) \, dx) \, , \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx) \, , \quad M := \int_{\mathbb{R}^2} n_0(x) \, dx < 8\pi
$$

First step: the trap

Decay Estimates of $u(t)$ in $L^\infty(\mathbb{R}^2)$

Lemma 2. *For any $M < M_1$, there exists $C = C(M)$ such that, for any solution*
$u \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq C\, t^{-1} \quad \forall\, t > 0$$
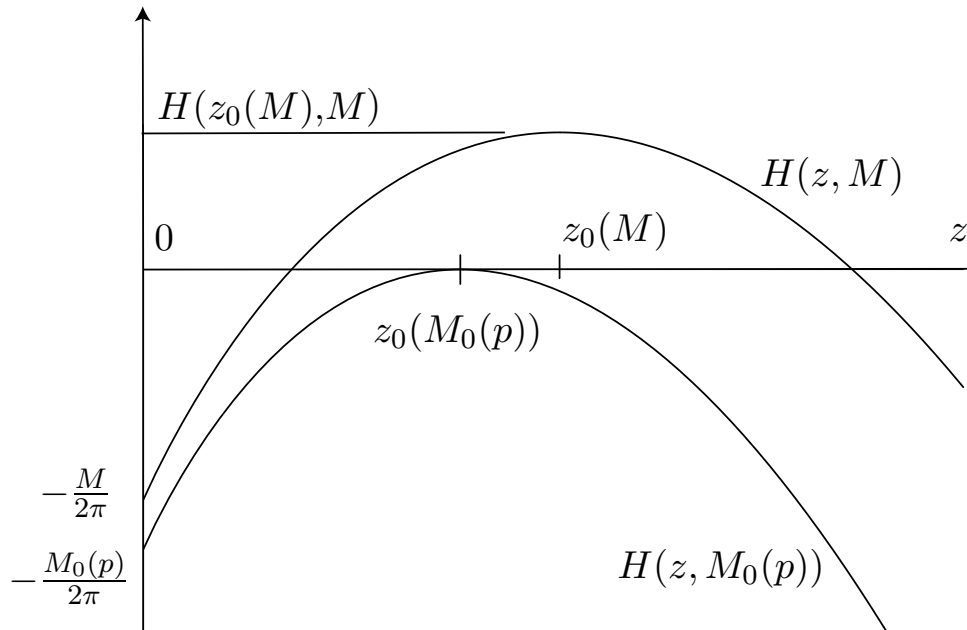
The *method of the trap...* prove that

$$H(\psi(t), M) \leq 0 \quad \text{where} \quad \psi(t) := t\, \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^2)}$$

where $z \mapsto H(z, M)$ is a continuous function which is
- negative on $[0, z_1)$
- positive on (z_1, z_2) for some $z_1,\, z_2$ such that $0 < z_1 < z_2 < \infty$

ψ is continuous and $\psi(0) = 0 \implies \psi(t) \leq z_1 \leq z_0(M)$ for any $t \geq 0$ if

$H(z_0(M), M) = \sup_{z \in [z_1, z_2]} H(z, M) \geq 0$

The *method of the trap* amounts to prove that $H(z, M) \leq 0$ implies that

$z = \psi(t)$ is bounded by $z_0(M)$ as long as $H(z_0(M), M) > 0$

Duhamel's formula:

$$u(x, t_0 + t) - \int_{\mathbb{R}^2} N(x - y, t)\, u(y, t_0)\, dy$$

$$= \int_0^t \int_{\mathbb{R}^2} N(x - y, t - s)\, \nabla \cdot [u(y, t_0 + s)\, \nabla v(y, t_0 + s)]\ dy\, ds$$

where $N(x, t) = \frac{1}{4\pi t}\, e^{-|x|^2/(4t)}$. Let $\kappa_\sigma = \|\partial N / \partial x_i\, (\cdot, 1)\|_{L^\sigma(\mathbb{R}^2)}$

$$\|u(\cdot, t_0 + t)\|_{L^\infty(\mathbb{R}^2)} - \frac{1}{4\pi t}\, \|u(\cdot, t_0)\|_{L^1(\mathbb{R}^2)}$$

$$\leq \sum_{i=1,2} \int_0^t \left\| \frac{\partial N}{\partial x_i}(\cdot, t - s) * \left[\left(u\, \frac{\partial v}{\partial x_i} \right)(\cdot, t_0 + s) \right] \right\|_{L^\infty(\mathbb{R}^2)}\, ds$$

$$\leq \sum_{i=1,2} \kappa_\sigma \int_0^t (t - s)^{-(1 - \frac{1}{\sigma}) - \frac{1}{2}} \left\| \left(u\, \frac{\partial v}{\partial x_i} \right)(\cdot, t_0 + s) \right\|_{L^\rho(\mathbb{R}^2)}\, ds$$

HLS inequality + Hölder and take $t_0 = t$

$$2t \, \|u(\cdot, 2t)\|_{L^\infty(\mathbb{R}^2)} - \frac{M}{2\pi}$$

$$\leq \frac{2\,\kappa_\sigma\,C_{\mathrm{HLS}}}{\pi} \, M^{\frac{1}{p}+\frac{1}{r}} \, t \int_0^t (t-s)^{\frac{1}{\sigma}-\frac{3}{2}} \, (t+s)^{\frac{1}{p}+\frac{1}{r}-2} \left[\psi(t)\right]^{2-\frac{1}{p}-\frac{1}{r}} ds$$

with $\psi(t) := \sup_{0 \leq s \leq t} 2s \, \|u(\cdot, 2s)\|_{L^\infty(\mathbb{R}^2)}$ and

$$t \int_0^t (t-s)^{\frac{1}{\sigma}-\frac{3}{2}} \, (t+s)^{\frac{1}{p}+\frac{1}{r}-2} ds \;=\; \frac{\sigma}{2-\sigma}$$

$$\psi(t) \leq \frac{M}{2\pi} + C_0 \left(\psi(t)\right)^\theta \quad \text{with} \quad C_0 = \frac{2\,\kappa_\sigma\,C_{\mathrm{HLS}}}{\pi} \, M^{\frac{1}{p}+\frac{1}{r}} \, \frac{\sigma}{2-\sigma} \, , \quad \theta = 2 - \frac{1}{p} - \frac{1}{r}$$

Choice: $H(z, M) = z - C_0 \, z^\theta - M/(2\pi)$

How small is the mass ?

The exponents σ, ρ, p, q and r are related by

$$\begin{cases} \frac{1}{\sigma} + \frac{1}{\rho} = 1 \, , & 1 < \sigma < 2 \\[2mm] \frac{1}{p} + \frac{1}{q} = \frac{1}{\rho} \, , & p \, , \ q > 2 \\[2mm] \frac{1}{r} - \frac{1}{q} = \frac{1}{2} \, , & r > 1 \end{cases}$$

For the choice $r = 4/3$, $q = 4$, $C_{\mathrm{HLS}} = 2\sqrt{\pi}$

$$C_0 = \frac{4\kappa_\sigma}{\sqrt{\pi}} \, M^{\frac{1}{p} + \frac{1}{4}} \, \frac{\sigma}{2 - \sigma} \ \text{with} \ \sigma = \frac{4p}{3p-4}$$

... there exists $M_0(p)$ such that $H(z_0(M), M) > 0$ if and only if $M < M_0(p)$

and $\sup_{p \in (4, +\infty)} M_0(p) = \lim_{p \to +\infty} M_0(p) \approx 0.822663 < 8\pi \approx 25.1327$

Other norms: interpolation

Corollary 3. *For any mass $M < M_1$ and all $p \in [1, \infty]$, there exists a positive constant $C = C(p, M)$ with $\lim_{M \to 0_+} C(p, M) = 0$, such that*

$$\|u(t)\|_{L^p(\mathbb{R}^2)} \leq C\, t^{-\left(1 - \frac{1}{p}\right)} \quad \forall\, t > 0$$

Remark 1. The above rates are optimal as can easily be checked using the self-similar solutions (n_∞, c_∞)

Second step: weighted H^1 estimates

L^p and H^1 estimates in the self-similar variables

Consider the solution of

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2 \,,\ t > 0 \\[2mm] c = -\dfrac{1}{2\pi} \log |\cdot| * n & x \in \mathbb{R}^2 \,,\ t > 0 \\[2mm] n(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

For any $p \in (1, \infty]$

$$\|n(t)\|_{L^p(\mathbb{R}^2)} \leq C_1 \quad \forall\, t > 0$$

for some positive constant C_1, and for $p > 2$

$$2\pi\,\|\nabla c(t)\|_{L^\infty} \leq \underbrace{\sup_{x \in \mathbb{R}^2} \int_{|x-y| \geq 1} \frac{n(t,y)}{|x-y|}\, dy}_{\leq M} + \underbrace{\sup_{x \in \mathbb{R}^2} \int_{|x-y| \leq 1} \frac{n(t,y)}{|x-y|}\, dy}_{\leq \left(2\pi\, \frac{p-1}{p-2}\right)^{\frac{p}{p-1}} \|n\|_{L^p(\mathbb{R}^2)}}$$

$$\|n(t)\|_{L^p(\mathbb{R}^2)} \leq C_1 \quad \text{and} \quad \|\nabla c(t)\|_{L^\infty(\mathbb{R}^2)} \leq C_2 \quad \forall\, t > 0$$

Lemma 4. *The constants C_1 and C_2 depend on M and are such that*

$$\lim_{M \to 0_+} C_i(M) = 0 \quad i = 1,\, 2$$

Exponential weights

With $K = K(x) = e^{|x|^2/2}$, let us rewrite the equation for n as

$$\frac{\partial n}{\partial t} - \frac{1}{K} \nabla \cdot (K \nabla n) = -\nabla c \cdot \nabla n + 2n + n^2$$

Proposition 5. *For any mass $M \in (0, M_1)$, there is a positive constant C such that*

$$\|n(t)\|_{H^1(K)} \leq C \quad \forall \, t > 0$$

First ingredient [M. Escobedo and O. Kavian]: for any $q > 2$ and $\varepsilon > 0$, there exists a positive constant $C(\varepsilon, q)$ such that

$$\int_{\mathbb{R}^2} n^2 \, K \, dx \leq \varepsilon \int_{\mathbb{R}^2} |\nabla n|^2 \, K \, dx + C(\varepsilon, q) \, \|n\|_{L^q(\mathbb{R}^2)}^2$$

$L^2(K)$ **estimate**

Multiply the equation by $n\,K$ and integrate by parts

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|n|^2\,K\,dx + \int_{\mathbb{R}^2}|\nabla n|^2\,K\,dx$$

$$= -\int_{\mathbb{R}^2}n\nabla c\cdot\nabla n\,K\,dx + 2\int_{\mathbb{R}^2}n^2\,K\,dx + \int_{\mathbb{R}^2}n^3\,K\,dx$$

$$\leq \varepsilon\int_{\mathbb{R}^2}|\nabla n|^2\,K\,dx + C$$

and so

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|n|^2\,K\,dx + (1-\varepsilon)\underbrace{\int_{\mathbb{R}^2}|\nabla n|^2\,K\,dx}_{\geq\int_{\mathbb{R}^2}|n|^2\,K\,dx}\leq C$$

(expand the square in $\int_{\mathbb{R}^2}|\nabla(n\,K)|^2\,K^{-1}\,dx \geq 0$)

$H^1(K)$ **estimate (1/2)**

Let $S(t)$ be the semi-group generated by $-K^{-1}\nabla\cdot(K\,\nabla\cdot)$ on $L^2(K)$

$$n(t,x) = S(t)\,n_0(x) - \int_0^t S(t-s)\,(\nabla c\cdot\nabla n)(s)\,ds + \int_0^t S(t-s)\,(2n+n^2)(s)\,ds$$

$$\|n(t)\|_{H^1(K)} - \|S(t)\,n_0\|_{H^1(K)}$$

$$\leq \int_0^t \|S(t-s)\,(\nabla c\cdot\nabla n)(s)\|_{H^1(K)}\,ds + \int_0^t \|S(t-s)\,(2n+n^2)(s)\|_{H^1(K)}\,ds$$

Second ingredient: $\|S(t)\,h\|_{H^1(K)} \leq \kappa\,(1 + t^{-1/2})\,\|h\|_{L^2(K)}$

$$\frac{1}{\kappa}\left(\|n(t)\|_{H^1(K)} - \|S(t)\,n_0\|_{H^1(K)}\right)$$

$$\leq C_2 \int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right)\|\nabla n(s)\|_{L^2(K)}\,ds + (2+C_1)\int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right)\|n(s)\|_{L^2(K}$$

$H^1(K)$ **estimate (2/2)**

$$\frac{1}{\kappa}\,\|n(t+\tau)\|_{H^1(K)} \le \left(1 + \frac{1}{\sqrt{t}}\right) C_1 + C_3 \int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right) \|n(s+\tau)\|_{H^1(K)}\, ds$$

Let $H(T) = \sup_{t \in (0,T)} \int_0^t \left(1 + \frac{1}{\sqrt{t-s}}\right) \|n(s+\tau)\|_{H^1(K)}\, ds$ and choose $T > 0$

such that $\frac{1}{2\kappa} = C_3 \int_0^T \left(1 + \frac{1}{\sqrt{T-s}}\right) ds = C_3\left(T + 2\sqrt{T}\right)$

$$\frac{1}{\kappa}\,H(T) \le \left(\pi + 4\sqrt{T} + T\right) C_1 + \frac{1}{2\kappa}\,H(T) \implies H(T) \le 2\left(\pi + 4\sqrt{T} + T\right)\kappa\,C_1$$

For any $t \in (0,T)$

$$\frac{1}{\kappa}\,\|n(t+\tau)\|_{H^1(K)} \le \left(1 + \frac{1}{\sqrt{t}}\right) C_1 + C_3\,H(T) \le \left(1 + \frac{1}{\sqrt{t}}\right) C_1 + 2\left(\pi + 4\sqrt{T} + T\right)$$

Conclusion: bound in $H^1(K)$

The estimate

$$\frac{1}{\kappa} \|n(t+\tau)\|_{H^1(K)} \leq \left(1 + \tfrac{1}{\sqrt{t}}\right) C_1 + C_3\, H(T) \leq \left(1 + \tfrac{1}{\sqrt{t}}\right) C_1 + 2\left(\pi + 4\sqrt{T} + T\right)$$

for any $t \in (0, T)$ gives a bound on $\|n(T+\tau)\|_{H^1(K)}$ for any $\tau > 0$

Lemma 6.

$$\|n(t)\|_{H^1(K)} \leq C \,\max\left\{1, \tfrac{\sqrt{T}}{\sqrt{t}}\right\} \quad \forall\, t > 0$$

Actually $n(t)$ can be bounded also in $H^1(n_\infty^{-1})$ but further estimates are needed...

Third step: linearization and spectral gap

A linearized operator

Introduce f and g defined by

$$n(x,t) = n_\infty(x)(1 + f(x,t)) \qquad \text{and} \qquad c(x,t) = c_\infty(x)(1 + g(x,t))$$

(f, g) is solution of the non-linear problem

$$\begin{cases} \dfrac{\partial f}{\partial t} - \mathcal{L}(t, x, f, g) = -\dfrac{1}{n_\infty} \nabla \cdot [f \, n_\infty \, \nabla \, (g \, c_\infty)] & x \in \mathbb{R}^2 \, , \ t > 0 \\[2mm] -\Delta(c_\infty \, g) = f \, n_\infty & x \in \mathbb{R}^2 \, , \ t > 0 \end{cases}$$

where $\mathcal{L}$ is the linear operator given by

$$\mathcal{L}(t, x, f, g) = \dfrac{1}{n_\infty} \nabla \cdot [n_\infty \, \nabla \, (f - g \, c_\infty)]$$

The conservation of mass is replaced here by $\int_{\mathbb{R}^2} f \, n_\infty \, dx = 0$

A spectral gap estimate

Proposition 7. *For any $M \in (0, M_2)$, for any $f \in H^1(n_\infty \, dx)$ such that*

$$\int_{\mathbb{R}^2} f \, n_\infty \, dx = 0 \quad \implies \quad \int_{\mathbb{R}^2} |\nabla f|^2 \, n_\infty \, dx \geq \Lambda(M) \int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx$$

for some $\Lambda(M) > 0$ and $\lim_{M \to 0_+} \Lambda(M) = 1$

Let $h = \sqrt{n_\infty} \, f = \sqrt{\lambda} \, e^{-|x|^2/4 + c_\infty/2} \, f$

$$\lambda \, |\nabla f|^2 \, n_\infty = |\nabla h|^2 + \frac{|x|^2}{4} \, h^2 + \frac{1}{4} \, |\nabla c_\infty|^2 \, h^2 + h \, \nabla h \cdot (x - \nabla c_\infty) - \frac{1}{2} \, x \cdot \nabla c_\infty \, h^2$$

(integrations by parts)

$\int_{\mathbb{R}^2} h \, \nabla h \cdot x \, dx = -\int_{\mathbb{R}^2} h^2 \, dx$

$\int_{\mathbb{R}^2} h \, \nabla h \cdot \nabla c_\infty \, dx = \frac{1}{2} \int_{\mathbb{R}^2} h^2 \, (-\Delta c_\infty) \, dx \leq \frac{1}{2} \, \|n_\infty\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} h^2 \, dx$

$\frac{1}{2} \int_{\mathbb{R}^2} x \cdot \nabla c_\infty \, h^2 \, dx \leq \frac{\sigma^2 - 1}{\sigma^2} \int_{\mathbb{R}^2} \frac{|x|^2}{4} \, h^2 \, dx + \frac{1}{4} \frac{\sigma^2}{\sigma^2 - 1} \int_{\mathbb{R}^2} |\nabla c_\infty|^2 \, h^2 \, dx$

$H^1(n_\infty^{-1})$ **estimate**

Assume that $n_0/n_\infty \in L^2(n_\infty)$

There exists a constant $C > 0$ such that

$$|x| > 1 \quad \Longrightarrow \quad \left| c_\infty + M/(2\pi) \log |x| \right| \le C$$

$n_\infty\, K = e^{c_\infty}$ behaves like $O(|x|^{-M/(2\pi)})$ as $|x| \to \infty$

$$\frac{\partial n}{\partial t} - n_\infty \nabla \cdot \left(\frac{1}{n_\infty} \nabla n \right) = (\nabla c_\infty - \nabla c) \cdot \nabla n + 2n + n^2$$

Corollary 8. *If $M < M_2$, then any solution n is bounded in*

$$L^\infty(\mathbb{R}^+, L^2(n_\infty^{-1}\, dx)) \cap L^\infty((\tau, \infty), H^1(n_\infty^{-1}\, dx))$$

for any $\tau > 0$

Fourth step: collecting the estimates

Proof of the exponential rate of convergence

$$
\begin{cases}
\dfrac{\partial f}{\partial t} - \mathcal{L}(t, x, f, g) = -\dfrac{1}{n_\infty} \nabla \cdot [f\, n_\infty\, \nabla\,(g\, c_\infty)] & x \in \mathbb{R}^2,\ t > 0 \\[2em]
-\Delta(c_\infty\, g) = f\, n_\infty & x \in \mathbb{R}^2,\ t > 0
\end{cases}
$$

Multiply by $f\, n_\infty$ and integrate by parts

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx + \int_{\mathbb{R}^2} |\nabla f|^2\, n_\infty\, dx
$$

$$
= \underbrace{\int_{\mathbb{R}^2} \nabla f \cdot \nabla\,(g\, c_\infty)\, n_\infty\, dx}_{=\mathrm{I}} + \underbrace{\int_{\mathbb{R}^2} \nabla f \cdot \nabla(g\, c_\infty)\, f\, n_\infty\, dx}_{=\mathrm{II}}
$$

Cauchy-Schwarz' inequality

$$
\mathrm{I} = \int_{\mathbb{R}^2} \nabla f \cdot \nabla\,(g\, c_\infty)\, n_\infty\, dx \leq \|\nabla f\|_{L^2(n_\infty\, dx)}\, \|\nabla(g\, c_\infty)\|_{L^2(n_\infty\, dx)}
$$

first term

Hölder's inequality (with $q > 2$)

$$\|\nabla(g\, c_\infty)\|_{L^2(n_\infty\, dx)} \leq M^{1/2-1/q} \|n_\infty\|_{L^\infty(\mathbb{R}^2)}^{1/q} \|\nabla(g\, c_\infty)\|_{L^q(\mathbb{R}^2)}$$

HLS inequality (with $1/p = 1/2 + 1/q$)

$$\|\nabla(g\, c_\infty)\|_{L^q(\mathbb{R}^2)} \leq \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \left| (f\, n_\infty) * \frac{1}{|\cdot|} \right|^q dx \right)^{\frac{1}{q}} \leq \frac{C_{\mathrm{HLS}}}{2\pi} \|f\, n_\infty\|_{L^p(\mathbb{R}^2)}$$

Hölder's inequality: $\|f\, n_\infty\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^2(n_\infty\, dx)} \|n_\infty\|_{L^{q/2}(\mathbb{R}^2)}^{1/2}$

$$\mathrm{I} = \int_{\mathbb{R}^2} \nabla f \cdot \nabla(g\, c_\infty)\, f\, n_\infty\, dx \leq C_*(M) \|f\|_{L^2(n_\infty\, dx)} \|\nabla f\|_{L^2(n_\infty\, dx)}$$

$$C_*(M) := C_{\mathrm{HLS}}\, (2\pi)^{-1}\, M^{1/2-1/q} \|n_\infty\|_{L^{q/2}(\mathbb{R}^2)}^{1/2} \|n_\infty\|_{L^\infty(\mathbb{R}^2)}^{1/q} \to 0 \text{ as } M \to 0$$

second term and conclusion

Use $g\,c_\infty = c - c_\infty$ and the Cauchy-Schwarz inequality

$$\int_{\mathbb{R}^2} \nabla f \cdot \nabla(g\,c_\infty)\, f\, n_\infty\, dx \leq \underbrace{\|\nabla c - \nabla c_\infty\|_{L^\infty(\mathbb{R}^2)}}_{\leq 2\,C_2(M) \searrow 0} \|f\|_{L^2(n_\infty\, dx)} \|\nabla f\|_{L^2(n_\infty\, dx)}$$

Spectral gap estimate

$$\underbrace{\sqrt{\Lambda(M)}}_{\to 1} \|f\|_{L^2(n_\infty\, dx)} \leq \|\nabla f\|_{L^2(n_\infty\, dx)}$$

With $\gamma(M) := \dfrac{C_*(M) + 2\,C_2(M)}{\sqrt{\Lambda(M)}} \searrow 0$,

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx \leq -\left[1 - \gamma(M)\right] \int_{\mathbb{R}^2} |\nabla f|^2\, n_\infty\, dx$$

Uniqueness

If n_1 and n_2 are two solutions in $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau, \infty) \times \mathbb{R}^2)$ for any $\tau > 0$, with $f = (n_2 - n_1)/n_\infty$ we also get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx \leq -\left[1 - \gamma(M)\right] \Lambda(M) \int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx$$

As a consequence, if the initial condition is the same, then $n_1 = n_2$

Thank you for your attention !