Nonlinear flows, entropy methods and applications

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Entropy methods and nonlinear flows with applications to symmetry of optimal functions and stability results in functional inequalities

1. Interpolation inequalities, phase transitions and symmetry
   - Subcritical inequalities, bifurcation results and phase transition
   - Caffarelli-Kohn-Nirenberg inequalities
   - Symmetry results for spinors in dimension $d = 2$
2. Stability results based on entropy methods
   - Subcritical inequalities on $\mathbb{R}^d$ and on $\mathbb{S}^d$
   - Constructive stability results for (log-)Sobolev inequalities
   - More results on LSI and CKN inequalities
3. From parabolic to kinetic equations

Interpolation inequalities, phase transitions and symmetry

▷ Subcritical Gagliardo-Nirenberg-Sobolev inequalities on the sphere and classical bifurcation results

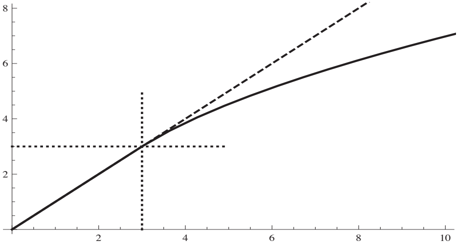
Other mechanisms of phase transition; the *carré du champ* method for the pressure variable

▷ Caffarelli-Kohn-Nirenberg inequalities: a proof of symmetry by the parabolic *carré du champ* method

▷ Magnetic rings, 2d spinors and problems with magnetic fields

Critical and subcritical Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, classical bifurcation results and phase transition

Bifurcation and phase transition in GNS inequalities



$\lambda \mapsto \mu(\lambda)$ *on* $\mathbb{S}^d$ *with* $d=3$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{\mu(\lambda)}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > d$$

▷ The inequality holds with $\mu(\lambda) = \lambda = d$ [Bakry, Emery, 1985] [Beckner, 1993], [Bidaut-Véron, Véron, 1991, Corollary 6.1]

GNS as entropy-entropy production inequalities

- ***(subcritical) Gagliardo-Nirenberg-Sobolev inequality***

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2,2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* = +\infty$ if $d = 1$ or 2

- Limit $p \to 2$: the ***logarithmic Sobolev inequality***

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu \ge \frac{d}{2}\,\mathcal{E}_2[F] := \frac{d}{2}\int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$

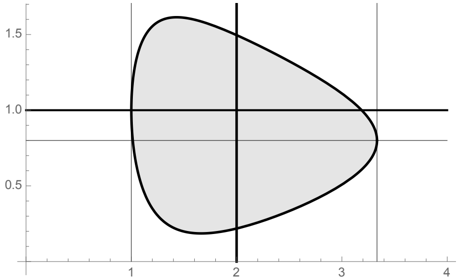
- $p = 1$: ***Poincaré inequality***

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_1[F] := d\left(\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^1(\mathbb{S}^d)}\right)$$

Carré du champ – admissible parameters on $\mathbb{S}^d$

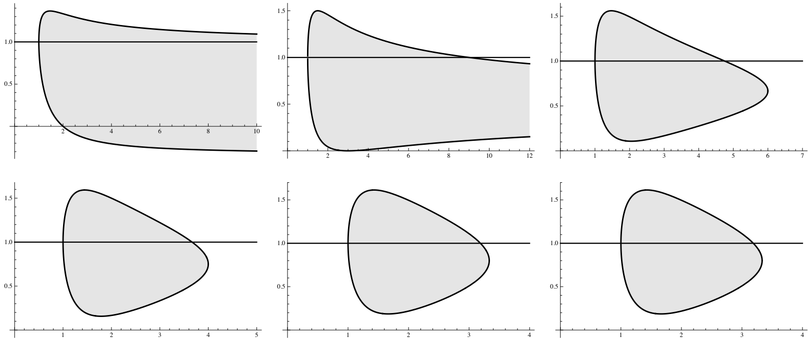
[JD, Esteban, Kowalczyk, Loss, 2014] Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)}\left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u}\right)$$



Case $d = 5$: admissible parameters $1 \le p \le 2^* = 10/3$ and m (horizontal axis: p, vertical axis: m). Improved inequalities inside !

Admissible parameters



$d=1,2,3$ (first line) and $d=4,5$ and 10 (second line)
the curves $p\mapsto m_\pm(p)$ determine the admissible parameters (p,m)
[JD, Esteban, Kowalczyk, Loss, 2014] [JD, Esteban, 2019]

$$m_\pm(d,p):=\frac{1}{(d+2)\,p}\left(d\,p+2\pm\sqrt{d\,(p-1)\,(2\,d-(d-2)\,p)}\right)$$

Another *Gagliardo-Nirenberg-Sobolev* inequality

$$\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^\theta\,\|u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}\geq\left(\frac{\mu(p,\theta,\lambda)}{p-2}\right)^\theta\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

- *Symmetry* holds if $\mu(p,\theta,\lambda)=\lambda$, optimal functions are constant
- *Symmetry breaking* if $\lambda>d\,\theta$: take $u_\varepsilon:=1+\varepsilon\,\varphi$, $\Delta\varphi+d\,\varphi=0$

Bakry-Emery exponent : $2^\#:=+\infty$ if $d=1$, $2^\#:=(2\,d^2+1)/(d-1)^2$ if $d\geq 2$

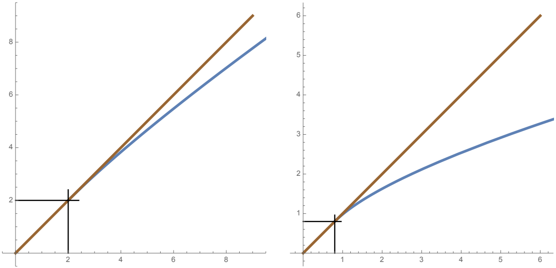
and take $p\in(2,2^\#]$

$$\theta^\#:=3\,\frac{p-2}{4\,p-7}\quad\text{if } d=1\,,\qquad \frac{1}{\theta^\#}:=1+\frac{(p-1)\,(2^\#-p)}{p-2}\,\left(\frac{d-1}{d+2}\right)^2\ \text{if } d\geq 2$$

Proposition (JD, Esteban)

Let $d\geq 1$, $p\in(2,2^\#)$, and $\theta^\#\leq\theta\leq 1$. The function $\lambda\mapsto\mu(p,\theta,\lambda)$ is monotone increasing, concave and $\mu(p,\theta,\lambda)<\lambda$ if and only if $\lambda>d\,\theta$

Second order phase transition



$d=1$, $p=5$: $\theta=2$ (left) $\theta=0.8$ (right). Bifurcation at $\lambda=\mu=d\,\theta$

Parameter range

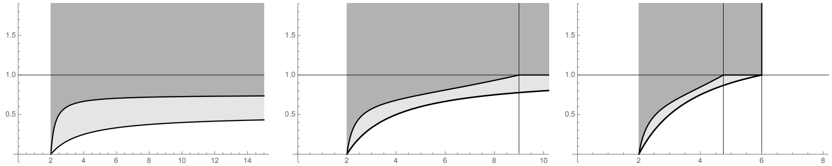
Theorem (Bou Dagher, JD)

Let $d \geq 1$, $p \in (2, 2^*)$ and $\theta_\star := d\,(p-2)/(2\,p) < \theta < \infty$
The function $\lambda \mapsto \mu(p,\theta,\lambda)$ is monotone increasing, concave

$$\mu(p,\theta,\lambda) \sim \kappa\,\lambda^{1-\theta_\star/\theta} \quad \text{as} \quad \lambda \to +\infty$$

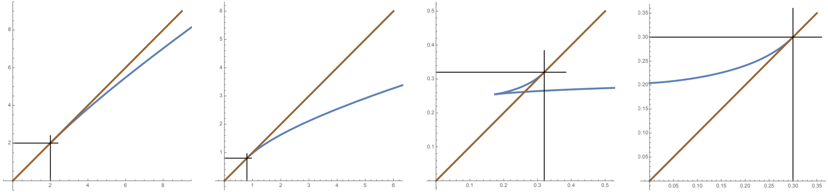
$$\mu(p,\theta,\lambda) \leq \lambda \text{ and } \mu(p,\theta,\lambda) < \lambda \text{ if } \lambda > d\,\theta$$

$$\mu(p,\theta,\lambda) = \lambda \text{ if } \lambda \leq d\,\theta\,,\ \theta^\# \leq \theta \leq 1\,,\ p \in (2, 2^\#] \text{ or } p > 2 \text{ if } d = 1$$



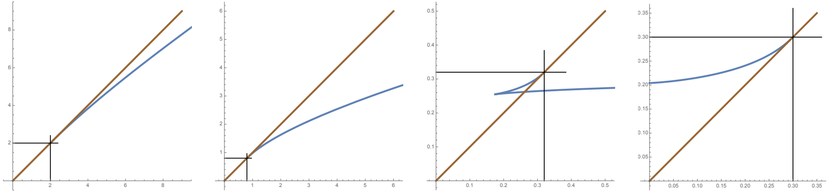
horizontal axis: p, vertical axis: θ
in dimensions $d = 1$, $d = 2$ and $d = 3$ (from left to right)

Second and first order phase transitions

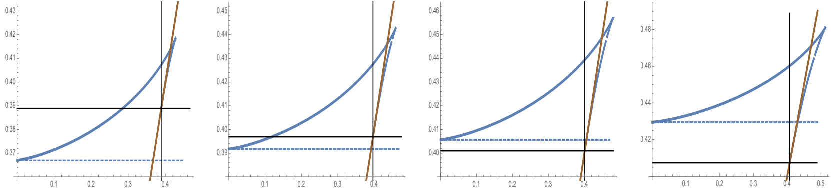


$d = 1$, $p = 5$, $\theta = 2$: $\theta = 0.8$, $\theta = 0.32$ and $\theta = \theta_\star = 0.3$

Second and first order phase transitions



$d=1$, $p=5$, $\theta=2$: $\theta=0.8$, $\theta=0.32$ and $\theta=\theta_\star=0.3$



Critical case: $d=1$, $\theta=\theta_\star$, for $p=9.0$, 9.7, 10.1 and 10.8

Reparametrization and consequences

Euler-Lagrange equation for an optimal function (with $\theta = 1$)

$$-\Delta u + \frac{\Lambda}{p-2}\,u = u^{p-1} \qquad (\mathrm{EL}_{1,\Lambda})$$

Theorem (Bou Dagher, JD)

Let $d \ge 1$, $p \in (2, 2^*)$, $\theta \ge \theta_\star$
A solution u of $(\mathrm{EL}_{1,\Lambda})$ also solves $(\mathrm{EL}_{\theta,\lambda})$ for $\lambda = \lambda(\theta,\Lambda)$ with

$$\lambda(\theta,\Lambda) := \tfrac{1}{\theta}\left(\Lambda + (1-\theta)\,(p-2)\,\frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)$$

- For $\lambda > 0$ small enough, we have $\mu(\theta,\lambda) = \lambda$
- For $\theta - \theta_\star > 0$ small enough, symmetry breaking occurs for $\lambda < d\,\theta$

Symmetry breaking with $\lambda < d\,\theta$ means *first order phase transition*

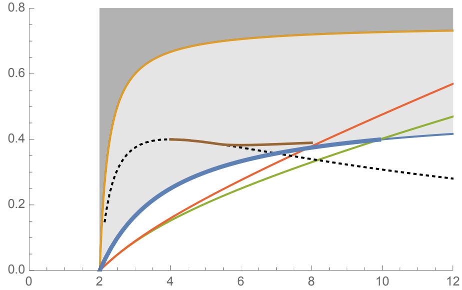
More qualitative properties

Proposition (Bou Dagher, JD)

Let $\theta_0 := \frac{(d+2)\,(d+3)\,(p-2)}{2\,(p^2+2\,p-6)+d\,(p^2+6\,p-12)-d^2\,(p-2)^2}$

Assuming that the curve $\mathcal{C}:[d,d+\epsilon)\to(\mathbb{R}^+)^2$ *is smooth enough:*

- *If* $\theta\neq\theta_0$*, the curve* $\mathcal{C}$ *bifurcates from* $(d\,\theta,d\,\theta)$ *tangentially to* $\mu=\lambda$
- *The curve* $\mathcal{C}$ *is concave and below* $\mu=\lambda$ *(on the right) if* $\theta>\theta_0$
- *The curve* $\mathcal{C}$ *is convex and above the line* $\mu=\lambda$ *(on the left) if* $\theta<\theta_0$



- blue curve: $p\mapsto\theta_\star(p)$
- yellow curve: if $\theta\geq\theta^\#(p)$, the phase transition is of second order
- red curve: if it is below $p\mapsto\theta_\star(p)$, the phase transition is of first order for $\theta-\theta_\star(p)>0$ small (Gaussian test functions)
- green curve: if $\kappa(p,\theta_\star)<\theta_\star(p)$, the phase transition is of first order for $\theta-\theta_\star(p)>0$ small (comparison with GNS on $\mathbb{R}^d$)
- black, dotted curve: $p\mapsto\theta_0(p)$ (at the bifurcation point)
- brown curve $p\mapsto\theta_\bullet(p)$: a numerical approximation of the threshold between first / second order phase transitions

References

[Gidas, Spruck], [Bidaut-Véron, Véron], *etc.*
[Bakry, Emery], [Demange], *etc.*
[Obata]

Joint paper with E. Bou Dagher
Interpolation inequalities on the sphere and phase transition: rigidity, symmetry and symmetry breaking
arXiv:2210.16878

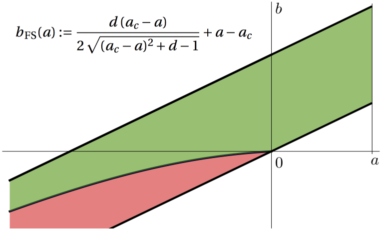
Caffarelli-Kohn-Nirenberg inequalities

Joint paper with E. Bou Dagher

Caffarelli-Kohn-Nirenberg inequalities, parabolic carré du champ estimates and symmetry results for weighted interpolation inequalities on the sphere

(work in progress)

The critical Caffarelli-Kohn-Nirenberg inequality



$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

$a \le b \le a+1$, $a < a_c$, $d \ge 3$

$p = \frac{2\,d}{d-2+2\,(b-a)} > 0$, $a_c = \frac{1}{2}\,(d-2)$

▷ *A radial optimal function:*
$v_\star(x) = \left(1+|x|^{(p-2)\,(a_c-a)}\right)^{-2/(p-2)}$
among radially symmetric functions

Theorem (JD, Esteban, Loss, 2015)

There is *symmetry*, *i.e.*, $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$, and all optimal functions are radially symmetric if $b_{\rm FS(a)} \le b < a+1$. If $a < b < b_{\rm FS}(a)$, then there is *symmetry breaking*, $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$, and optimal functions are not radially symmetric.

[Caffarelli, Kohn, Nirenberg (1984)], [F. Catrina, Z.-Q. Wang (2001)]
[Smets, Willem], [Catrina, Wang], [Felli, Schneider]
[Bonforte, JD, Nazaret, Muratori]

A new proof: rewriting of CKN

1) Change of variables: $v(r,\omega)=u\big(r^\alpha,\omega\big)$, $\mathsf{D}_\alpha u=(\alpha\,\partial_r u,\nabla_\omega u)$

$$\int_{\mathbb{R}^d}|\mathsf{D}_\alpha u|^2\,|x|^{n-d}\,dx\ge\mathcal{C}_{\alpha,n}\left(\int_{\mathbb{R}^d}|u|^p\,|x|^{n-d}\,dx\right)^{2/p}$$

with $n=2\,p/(p-2)$. Symmetry means that the Aubin-Talenti function $u_*(x):=(1+|x|^2)^{-(n-2)/2}$ realizes the equality case

2) Relative measure: with $w=u/u_*$ and $d\mu_q(x)=|u_*(x)|^q\,|x|^{n-d}\,dx$

$$\int_{\mathbb{R}^d}|\mathsf{D}_\alpha w|^2\,d\mu_2\,dx+\tfrac14\,\alpha^2\,n\,(n-2)\int_{\mathbb{R}^d}|w|^2\,d\mu_p\,dx\ge\mathcal{C}_{\alpha,n}\left(\int_{\mathbb{R}^d}|w|^p\,d\mu_p\,dx\right)^{2/p}$$

3) Stereographic projection: $w(x)=f(z,\omega)$ with $z=\frac{1-|x|^2}{1+|x|^2}$, $\omega=\frac{2\,x}{1+|x|^2}$

$$\int_{\mathbb{S}^d}\left(\alpha^2\,(1-z^2)\,|f'|^2+\frac{|\nabla_\omega f|^2}{1-z^2}\right)d\sigma_n+\frac{\alpha^2}{4}\,n\,(n-2)\int_{\mathbb{S}^d}|f|^2\,d\sigma_n$$
$$\ge\mathcal{K}_{\alpha,n}\left(\int_{\mathbb{S}^d}|f|^p\,d\sigma_n\right)^{2/p}$$

$d\sigma_n=Z_n^{-1}\,\big(1-z^2\big)^{(n-2)/2}\,dz\,d\omega$, $z\in[-1,+1]$, $\omega\in\mathbb{S}^{d-1}$

A new proof: fast diffusion equation and *carré du champ*

Let $'$ and ∇ denote the derivatives with respect to $z\in[-1,1]$ and $\omega\in\mathbb{S}^{d-1}$, $\Delta=\nabla\cdot\nabla$ and

$$\mathsf{D}v:=\left(\alpha\,\sqrt{1-z^2}\,v',\frac{1}{\sqrt{1-z^2}}\nabla v\right)\,,\quad \mathsf{L}v:=\mathsf{D}\cdot\mathsf{D}v$$

$$\mathsf{L}v=\alpha^2\,\mathcal{L}v+\frac{1}{1-z^2}\,\Delta v\,,\quad \mathcal{L}v:=\left(1-z^2\right)v''-n\,z\,v'$$

Weighted fast diffusion equation

$$\frac{\partial v}{\partial t}=\mathsf{L}v^m=-\,\mathsf{D}\cdot\left(v\,\mathsf{D}\mathrm{P}\right),\quad \mathrm{P}=\frac{m}{1-m}\,v^{m-1}\,,\quad m=\frac{n-1}{n}\,,\quad p=\frac{2\,n}{n-2}$$

$$v=u^p\quad\text{and}\quad\mathcal{D}(t):=\int_{\mathbb{S}^d}|\mathsf{D}u(t,\cdot)|^2\,d\sigma_n+\frac{n\,\alpha^2}{p-2}\int_{\mathbb{S}^d}|u(t,\cdot)|^2\,d\sigma_n$$

Proposition (Bou Dagher, JD)

$$\mathcal{D}'(t)\le 0\ \textit{if}\ \alpha\le\alpha_{\mathrm{FS}}:=\sqrt{\tfrac{d-1}{n-1}}$$

- Nonlinear *carré du champ* techniques and Felli & Schneider (FS)
- [JD, Zhang, 2021]: weights $\left(1+\varepsilon-z^2\right)\Longrightarrow$ First parabolic proof

Details

$$\mathcal{D}'(t) = -\frac{8}{(p+2)^2}\int_{\mathbb{S}^d} v^m \left(\mathbf{K}[\mathrm{P}] - m\,n\,\alpha^2\,|\mathbf{D}\mathrm{P}|^2\right) d\sigma_n$$

with $\mathbf{K}[\mathrm{P}] := \frac{1}{2}\,\mathbf{L}\left(|\mathbf{D}\mathrm{P}|^2\right) - \mathbf{D}\mathrm{P}\cdot\mathbf{D}(\mathbf{L}\mathrm{P}) - \frac{1}{n}\,(\mathbf{L}\mathrm{P})^2$

$$\mathbf{K}[\mathrm{P}] = m\left|\alpha^2\left(1-z^2\right)\mathrm{P}'' - \frac{\Delta\mathrm{P}}{(n-1)\left(1-z^2\right)}\right|^2 + 2\,\alpha^2\left|\nabla\mathrm{P}' + \frac{z\,\nabla\mathrm{P}}{1-z^2}\right|^2$$

$$+\,\alpha^2\,(n-1)\,|\mathbf{D}\mathrm{P}|^2$$

$$+\left(1-z^2\right)^{-2}\left(\frac{1}{2}\,\Delta(|\nabla\mathrm{P}|^2) - \nabla\mathrm{P}\cdot\nabla\Delta\mathrm{P} - \frac{(\Delta\mathrm{P})^2}{n-1} - (n-2)\,\alpha^2\,|\nabla\mathrm{P}|^2\right)$$

Corollary (Bou Dagher, JD)

If $n>d\ge 3$, $m=(n-1)/n$ and $p=2\,n/(n-2)$, then

$$\int_{\mathbb{S}^{d-1}}\rho^q\left(\frac{1}{2}\,\Delta(|\nabla\mathrm{P}|^2) - \nabla\mathrm{P}\cdot\nabla\Delta\mathrm{P} - \frac{(\Delta\mathrm{P})^2}{n-1}\right)d\omega$$

$$= \mathsf{a}\int_{\mathbb{S}^{d-1}}\rho^q\left\|\mathrm{LP} - \frac{\mathsf{b}}{\mathsf{a}}\,\mathrm{MP}\right\|^2 d\omega + \left(\mathsf{c} - \frac{\mathsf{b}^2}{\mathsf{a}}\right)\int_{\mathbb{S}^{d-1}}\rho^q\,\frac{|\nabla\mathrm{P}|^4}{\mathrm{P}^2}\,d\omega$$

$$+\,(n-2)\,\frac{d-1}{n-1}\int_{\mathbb{S}^{d-1}}\rho^q\,|\nabla\mathrm{P}|^2\,d\omega$$

$(n-2)\,\frac{d-1}{n-1} = (n-2)\,\alpha_{\mathrm{FS}}^2$ Regularization as in [JD, Zhang]

Symmetry results for spinors in dimension two

Symmetry results for spinors in dimension $d = 2$

- the $d = 2$ *spinorial Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}^2} \frac{|\sigma\cdot\nabla\psi|^2}{|x|^{2\alpha}}\,dx \ge C_{\alpha,p}\left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^{\beta p}}\,dx\right)^{2/p} \qquad \text{(SCKN)}$$

for spinor valued functions $\psi : \mathbb{R}^2 \to \mathbb{C}^2$

▷ the *logarithmic Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}}\int_{\mathbb{S}^1}\left(|\partial_s\phi(s,\theta)|^2 + |(\alpha - i\sigma_3\partial_\theta)\phi(s,\theta)|^2\right) ds\,d\theta \ge C_{\alpha,p}\left(\int_{\mathbb{R}}\int_{\mathbb{S}^1}|\phi(s,\theta)|^p\,ds\,d\theta\right)^{2/p}$$

- *Interpolation inequalities for Aharonov–Bohm magnetic fields* $A(x) = (x_2, -x_1)/|x|^2$

$$\int_{\mathbb{R}^2}|(-i\nabla - \alpha A)\psi|^2\,dx \ge C^{\rm AB}_{\alpha,p}\left(\int_{\mathbb{R}^2}\frac{|\psi|^p}{|x|^2}\,dx\right)^{2/p} \qquad \text{(AB)}$$

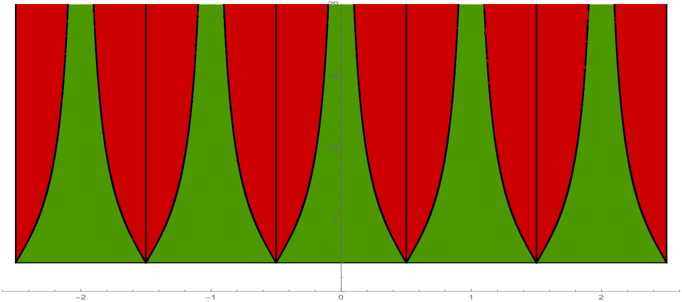
Theorem (JD, Frank, Weixler)

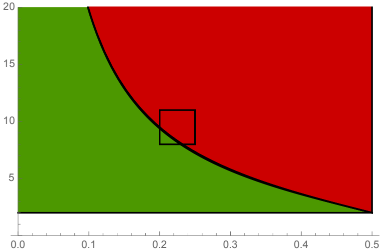
$C_{\alpha,p} = C^{\rm AB}_{\alpha,p}$ for any $(\alpha,p) \in (0,1/2)\times(2,+\infty)$

Symmetry *versus* symmetry breaking

Theorem (JD, Frank, Weixler)

- For every $\alpha \in (0, 1/2)$ and $p > 2$, there is an optimizer with $C_{\alpha,p} > 0$ and $\lim_{\alpha\to 0_+} C_{\alpha,p} = 0$. Symmetry holds if and only if $\alpha \in \big(0, \alpha(p)\big]$ for some function $p \mapsto \alpha(p) : (2, \infty) \to (0, 1/2)$
- The symmetry and symmetry breaking regions are symmetric with respect to $\alpha = 0$ and 1-periodic





(SCKN) *with* $d = 2$. *Horizontal axis:* $\alpha \in (0, 1/2)$. *Vertical axis:* $p \in (2, \infty)$

- *Symmetry range: green, by the equivalence with Aharonov-Bohm problem and entropy methods for flows associated to (CKN) inequalities*
- *Symmetry breaking range: red and blue; Undecided in the tiny white gap*

- magnetic ring: an interpolation inequality on $\mathbb{S}^1$
[JD, Esteban, Laptev, Loss]
- Aharonov-Bohm and Caffarelli-Kohn-Nirenberg inequalities
[Bonheure, JD, Esteban, Laptev, Loss]
- a Gegenbauer polynomial basis to study linear instability
[JD, Frank, Weixler]

Stability results based on entropy methods

▷ Gagliardo-Nirenberg inequalities on $\mathbb{R}^d$ and on $\mathbb{S}^d$

▷ Sobolev inequality: the Bianchi-Egnell stability estimate made constructive

▷ The Gaussian logarithmic Sobolev inequality seen as an infinite dimensional limit; further results on logarithmic Sobolev inequalities

Stability results for Gagliardo-Nirenberg inequalities on $\mathbb{R}^d$ based on entropy methods and fast diffusion equations on $\mathbb{R}^d$

Rényi entropy powers, inequalities and flow, a formal approach

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

▷ *How do we relate Gagliardo-Nirenberg-Sobolev inequalities on* $\mathbb{R}^d$

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\geq\mathcal{C}_{\mathrm{GNS}}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\qquad\text{(GNS)}$$

and the fast diffusion equation

$$\frac{\partial u}{\partial t}=\Delta u^m\qquad\text{(FDE)}$$

Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With $m \geq m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} u(t,x)\, dx = 0$$

(ii) *Second moment.* With $m > d/(d+2)$ and $u_0 \in \mathrm{L}^1_+\left(\mathbb{R}^d, (1+|x|^2)\, dx\right)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x|^2\, u(t,x)\, dx = 2\, d \int_{\mathbb{R}^d} u^m(t,x)\, dx$$

(iii) *Entropy estimate.* With $m \geq m_1 := (d-1)/d$, $u_0^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \in \mathrm{L}^1_+\left(\mathbb{R}^d, (1+|x|^2)\, dx\right)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} u^m(t,x)\, dx = \frac{m^2}{1-m} \int_{\mathbb{R}^d} u\, |\nabla u^{m-1}|^2\, dx$$

Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\, dx \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2} \int_{\mathbb{R}^d} u\, |\nabla u^{m-1}|^2\, dx$$

Entropy growth rate as a consequence of (GNS)

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^\theta\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\ge\mathcal{C}_{\mathrm{GNS}}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\qquad\text{(GNS)}$$

$$p=\frac{1}{2\,m-1}\quad\Longleftrightarrow\quad m=\frac{p+1}{2\,p}\in[m_1,1)$$

$u=f^{2\,p}$ so that $u^m=f^{p+1}$ and $u\,|\nabla u^{m-1}|^2=(p-1)^2\,|\nabla f|^2$

$$\mathcal{M}=\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{2\,p}\,,\quad\mathsf{E}[u]=\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{p+1}\,,\quad\mathsf{I}[u]=(p+1)^2\,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

If u solves (FDE) $\frac{\partial u}{\partial t}=\Delta u^m$, then $\mathsf{E}'=m\,\mathsf{I}$

$$\mathsf{E}'\ge\frac{p-1}{2\,p}\,(p+1)^2\,\mathcal{C}_{\mathrm{GNS}}^{\frac{2}{\theta}}\,\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{\frac{2}{\theta}}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{-\frac{2\,(1-\theta)}{\theta}}=C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d}u^m(t,x)\,dx\ge\left(\int_{\mathbb{R}^d}u_0^m\,dx+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\quad\forall\,t\ge0$$

Self-similar solutions

$$\int_{\mathbb{R}^d} u^m(t,x)\,dx \ge \left(\int_{\mathbb{R}^d} u_0^m\,dx + \frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}} \quad \forall\,t\ge 0$$

Equality case is achieved if and only if, up to a normalisation and a translation

$$u(t,x) = \frac{c_1}{R(t)^d}\,\mathcal{B}\left(\frac{c_2\,x}{R(t)}\right)$$

where $\mathcal{B}$ is the ***Barenblatt self-similar solution***

$$\mathcal{B}(x) := \left(1+|x|^2\right)^{\frac{1}{m-1}}$$

Notice that $\mathcal{B} = \varphi^{2p}$ means that

$$\varphi(x) = \left(1+|x|^2\right)^{-\frac{1}{p-1}}$$

is an ***Aubin-Talenti profile***

Pressure variable and decay of the Fisher information

The derivative of the *Rényi entropy power* $\mathsf{E}^{\frac{2}{d}\,\frac{1}{1-m}-1}$ is proportional to

$$\mathsf{I}^\theta\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

▷ *Pressure variable*

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

▷ *Fisher information*

$$\mathsf{I}[u] = \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx$$

If u solves (FDE), then

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla\mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta\mathsf{P} + |\nabla\mathsf{P}|^2\Big)\,dx$$
$$= -\,2\int_{\mathbb{R}^d} u^m\left(\|\mathrm{D}^2\mathsf{P}\|^2 - (1-m)\,(\Delta\mathsf{P})^2\right)dx$$

Rényi entropy powers and interpolation inequalities

▷ Integrations by parts and completion of squares: with $m_1 = \frac{d-1}{d}$

$$-\frac{\mathsf{I}}{2\,\theta}\,\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^\theta\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)$$
$$=\int_{\mathbb{R}^d} u^m\left\|\mathrm{D}^2\mathsf{P}-\frac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right\|^2 dx+(m-m_1)\int_{\mathbb{R}^d} u^m\left|\Delta\mathsf{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2 dx$$

▷ Analysis of the asymptotic regime as $t\to+\infty$

$$\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^\theta\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{\mathcal{M}^{\frac{2\,\theta}{p}}}=\frac{\mathsf{I}[\mathcal{B}]^\theta\,\mathsf{E}[\mathcal{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathcal{B}\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\frac{2\,\theta}{p}}}=(p+1)^{2\,\theta}\,\mathcal{C}_{\mathrm{GNS}}^{2\,\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$\mathsf{I}[u]^\theta\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\ge(p+1)^{2\,\theta}\left(\mathcal{C}_{\mathrm{GNS}}\right)^{2\,\theta}\mathcal{M}^{\frac{2\,\theta}{p}}$$

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$

Joint papers with M. Bonforte, B. Nazaret and N. Simonov

Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method

arXiv:2007.03674, *Memoirs of the AMS 308 (2025)*

Constructive stability results in interpolation inequalities and explicit improvements of decay rates of fast diffusion equations

DCDS, 43 (3&4): 10701089, 2023

Entropy – entropy production inequality

The fast diffusion equation on $\mathbb{R}^d$ in self-similar variables

$$\frac{\partial v}{\partial t} + \nabla \cdot \Big[v \left(\nabla v^{m-1} - 2\,x \right) \Big] = 0 \qquad \text{(FDE)}$$

admits a stationary Barenblatt solution $\mathcal{B}(x) := \left(1 + |x|^2\right)^{\frac{1}{m-1}}$

$$\frac{d}{dt} \mathcal{F}[v(t,\cdot)] = -\,\mathcal{I}[v(t,\cdot)]$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1}\,(v - \mathcal{B}) \right) dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} - \nabla \mathcal{B}^{m-1} \right|^2 dx$$

are such that $\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$ [del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[v_0]\, e^{-4\,t}$$

Entropy growth rate

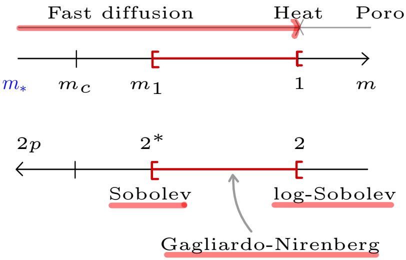
$\mathcal{I}[v] \geq 4\,\mathcal{F}[v] \Longleftrightarrow$ *Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

with optimal constant. Under appropriate mass normalization

$v = f^{2p}$ so that $v^m = f^{p+1}$ and $v\,|\nabla v^{m-1}|^2 = (p-1)^2\,|\nabla f|^2$

$$p = \frac{1}{2m-1} \quad \Longleftrightarrow \quad m = \frac{p+1}{2p} \in [m_1, 1)$$



$m_* = \frac{d-2}{d+2}$

$m_1 = \frac{d-1}{d}$

$m_c = \frac{d-2}{d}$

Asymptotic regime as $t \to +\infty$

Take $f_\varepsilon := \mathcal{B}\,(1+\varepsilon\,\mathcal{B}^{1-m}\,w)$ and expand $\mathcal{F}[f_\varepsilon]$ and $\mathcal{I}[f_\varepsilon]$ at order $O(\varepsilon^2)$

linearized free energy and *linearized Fisher information*

$$\mathsf{F}[w] := \frac{m}{2}\int_{\mathbb{R}^d} w^2\,\mathcal{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[w] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla w|^2\,\mathcal{B}\,dx$$

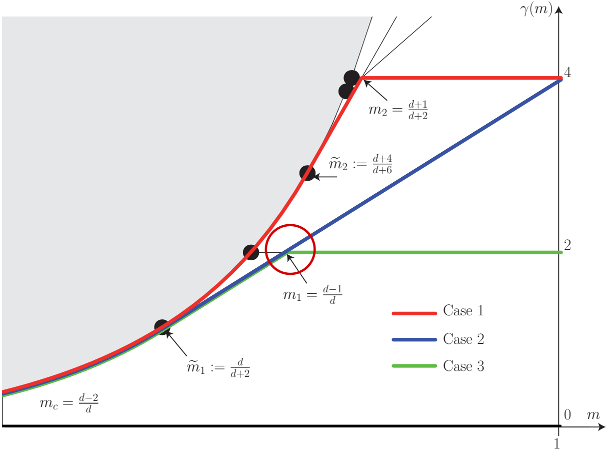
Proposition (Hardy-Poincaré inequality)

[BBDGV, BDNS] Let $m \in [m_1, 1)$ if $d \ge 3$, $m \in (1/2, 1)$ if $d = 2$, and $m \in (1/3, 1)$ if $d = 1$. If $w \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^{2-m}\,dx)$ is such that $\nabla w \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}\,dx)$, $\int_{\mathbb{R}^d} w\,\mathcal{B}^{2-m}\,dx = 0$, then

$$\mathsf{I}[w] \ge 4\,\alpha\,\mathsf{F}[w]$$

with $\alpha = 1$, or $\alpha = 2 - d\,(1-m)$ if $\int_{\mathbb{R}^d} x\,w\,\mathcal{B}^{2-m}\,dx = 0$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

The asymptotic time layer improvement

Proposition

Let $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$, $\eta = 2\,(d\,m - d + 1)$ *and* $\chi = m/(266 + 56\,m)$. *If* $\int_{\mathbb{R}^d} v\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ *and*

$$(1 - \varepsilon)\,\mathcal{B} \leq v \leq (1 + \varepsilon)\,\mathcal{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, *then*

$$\mathcal{I}[v] \geq (4 + \eta)\,\mathcal{F}[v]$$

Uniform convergence in relative error: threshold time

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $t_\star \geq 0$ such that, if u is a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \Big[v \left(\nabla v^{m-1} - 2\,x \right) \Big] = 0 \qquad \text{(FDE)}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\mathsf{A}[u_0] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \qquad (\mathsf{H}_A)$$

$\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} B\,dx = \mathcal{M}$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq t_\star$$

The initial time layer improvement: backward estimate

By the *carré du champ* method, we have
Away from the Barenblatt solutions, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt} \le \mathcal{Q}\,(\mathcal{Q} - 4)$$

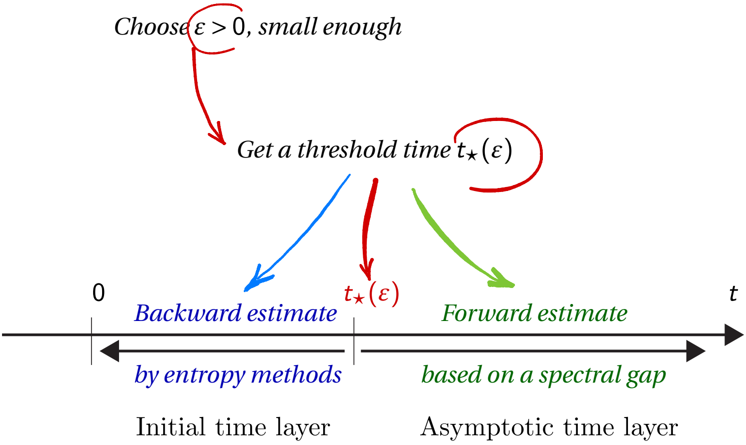
Lemma

Assume that $m > m_1$ and v is a solution to (FDE) with nonnegative initial datum v_0. If for some $\eta > 0$ and $t_\star > 0$, we have $\mathcal{Q}[v(t_\star, \cdot)] \ge 4 + \eta$, then

$$\mathcal{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\,e^{-4\,t_\star}}{4 + \eta - \eta\,e^{-4\,t_\star}} \quad \forall\, t \in [0, t_\star]$$

Stability in Gagliardo-Nirenberg-Sobolev inequalities

Our strategy



Two consequences (subcritical case)

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (FDE) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ and v_0 satisfies (H_A), then

$$\mathcal{F}[v(t,.)] \leq \mathcal{F}[v_0]\,e^{-\,(4+\zeta)\,t} \quad \forall\,t \geq 0$$

▷ The ***stability of the entropy - entropy production inequality*** $\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

A constructive stability result (critical case)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \geq 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\;\; |x|\,f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Deficit of the Sobolev inequality: $\delta[f] := \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$

Theorem

Let $d \geq 3$ and $A > 0$. Then for any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, dx \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \leq A$$

we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\mathcal{C}_\star(A)}{4+\mathcal{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathcal{C}_\star(A) = \mathcal{C}_\star(0)\,\big(1 + A^{1/(2\,d)}\big)^{-1}$ *and* $\mathcal{C}_\star(0) > 0$ *depends only on* d

Constructive stability results in subcritical Gagliardo-Nirenberg inequalities on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2,2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* = +\infty$ if $d = 1$ or 2

Joint paper with G. Brigati and N. Simonov
Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results
Annales IHP, Analyse non linéaire, 362, 2023
arXiv: 2211.13180

Gagliardo-Nirenberg inequalities: stability

An improved inequality under orthogonality constraint (Π_1 is a projection on some positive spherical harmonic functions) and the stability inequality arising from the *carré du champ* method can be combined *in the subcritical case* as follows

Theorem (Brigati, JD, Simonov)

Let $d \ge 1$ and $p \in (1, 2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu - d\, \mathcal{E}_p[F] \ge \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

▷ The result holds true for the logarithmic Sobolev inequality ($p = 2$), again with an explicit constant $\mathscr{S}_{d,2}$, for any finite dimension d

▷ A ***spectral*** estimate based on harmonic analysis (Funk-Hecke)

▷ The ***far away*** regime: use an improved interpolation inequality

If $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} / \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \ge \vartheta_0 > 0$, by the convexity of ψ

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right) \ge \frac{d}{\vartheta_0}\,\psi\left(\frac{\vartheta_0}{d}\right)\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

▷ The ***local*** case: $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta_0\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$

Take $\|F\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$, assume that $\frac{d\,\vartheta_0}{d-(p-2)\,\vartheta_0} > 0$ and deduce from the Poincaré inequality that

$$1 - \frac{\vartheta}{d} < \left(\int_{\mathbb{S}^d} F\,d\mu\right)^2 \le 1$$

+ a Taylor expansion using a partial decomposition on spherical harmonics

Constructive stability results for the Sobolev inequality

... the Bianchi-Egnell stability estimate made constructive

Stability for the Sobolev inequality: the history

▷ [Rodemich, 1969], [Aubin, 1976], [Talenti, 1976]
In the inequality $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mathsf{S}_d\, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, the optimal constant is

$$\mathsf{S}_d = \tfrac{1}{4}\, d\,(d-2)\,|\mathbb{S}^d|^{1-2/d}$$

with equality on the manifold $\mathcal{M} = \{g_{a,b,c}\}$ of the *Aubin-Talenti functions*

▷ [Lions] a qualitative stability result

$$\textit{if } \lim_{n\to\infty} \|\nabla f_n\|_2^2/\|f_n\|_{2^*}^2 = \mathsf{S}_d\,,\ \textit{then } \lim_{n\to\infty}\inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2/\|\nabla f_n\|_2^2 = 0$$

▷ [Brezis, Lieb, 1985] a quantitative stability result ?
▷ [Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|_2^2 \ge \mathsf{S}_d\,\|f\|_{2^*}^2 + c_{\mathrm{BE}} \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

- The strategy of Bianchi & Egnell involves two steps:
– a local (spectral) analysis: the *neighbourhood* of $\mathcal{M}$
– a local-to-global extension based on concentration-compactness : the *far away regime*
- The constant c_{BE} is not explicit

An explicit stability result for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

Theorem (JD, Esteban, Figalli, Frank, Loss)

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \to +\infty$

The global and the local problem

$$\mathsf{d}(u,v)^2 := \mathsf{q}[u-v] \quad \text{where} \quad \mathsf{q}[w] := \|\nabla w\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \tfrac{d}{p-2}\,\|w\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

- *deficit* : $\delta[u] := \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right)$, $\quad p = \frac{2\,d}{d-2}$
- *distance* to the set $\mathscr{M}$ of the Aubin-Talenti (optimal) functions

$$\mathsf{d}(u,\mathscr{M}) := \inf_{v\in\mathscr{M}} \mathsf{d}(u,v)$$

$\lim_{t\to+\infty} \mathsf{d}\big(u(t,\cdot),\mathscr{M}\big) = 0$ and $\delta[u(t,\cdot)]$ is monotone non-increasing if

$$\frac{\partial u}{\partial t} = m\,u^{(m-1)\,p}\left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u}\right)$$

For a given $\varepsilon\in(0,1)$, u is in the ***far away*** regime if

$$\mathsf{d}(u,\mathscr{M})^2 > \varepsilon\,\mathsf{q}[u]$$

and in the neighbourhood of $\mathscr{M}$ if $\mathsf{d}(u,\mathscr{M})^2 \le \varepsilon\,\mathsf{q}[u]$

local stability **:** $\mathcal{I}(\varepsilon) := \inf\left\{\frac{\delta[u]}{\mathsf{d}(u,\mathscr{M})^2}\;:\; u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)\,,\;\mathsf{d}(u,\mathscr{M})^2\le\varepsilon\,\mathsf{q}[u]\right\}$

A new proof for the global to local reduction

[Bonforte, JD], [JD, Esteban, Figalli, Frank, Loss], based on an idea by Christ. If we start in the ***far away*** regime, which means

$$\mathsf{d}(u_{|t=0},\mathscr{M})^2 > \varepsilon\,\mathsf{q}[u_{|t=0}]$$

using $\mathsf{d}(u_{|t=0},\mathscr{M}) \le \mathsf{d}(u_{|t=0},0) = \mathsf{q}[u_{|t=0}]$, $\|u(t,\cdot)\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$ we obtain

$$\frac{\delta[u_{|t=0}]}{\mathsf{d}(u_{|t=0},\mathscr{M})^2} \ge \frac{\mathsf{q}[u_{|t=0}] - \frac{d}{p-2}}{\mathsf{q}[u_{|t=0}]} \ge 1 - \frac{\frac{d}{p-2}}{\mathsf{q}[u(t,\cdot)]} = \frac{\delta[u(t,\cdot)]}{\mathsf{q}[u(t,\cdot)]}$$

We know that

$$\lim_{t\to+\infty}\mathsf{q}[u(t,\cdot)] = \frac{d}{p-2} \quad \text{and} \quad \lim_{t\to+\infty}\mathsf{d}\big(u(t,\cdot),\mathscr{M}\big)^2 = 0$$

so that for some $t_* > 0$ we have

$$\mathsf{q}[u(t_*,\cdot)] = \tfrac{1}{\varepsilon}\,\mathsf{d}\big(u(t_*,\cdot),\mathscr{M}\big)^2$$

$$\frac{\delta[u_{|t=0}]}{\mathsf{d}(u_{|t=0},\mathscr{M})^2} \ge \frac{\delta[u(t_*,\cdot)]}{\mathsf{q}[u(t_*,\cdot)]} = \varepsilon\,\frac{\delta[u(t_*,\cdot)]}{\mathsf{d}\big(u(t_*,\cdot),\mathscr{M}\big)^2} \ge \varepsilon\,\mathcal{I}(\varepsilon)$$

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From interpolation inequalities on the sphere to Gaussian interpolation inequalities

Joint papers with G. Brigati and N. Simonov

Gaussian interpolation inequalities
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C. R. Math. Acad. Sci. Paris 41, 2024

Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results
Annales IHP, Analyse non linéaire, 362, 2023
arXiv: 2211.13180

Subcritical interpolation inequalities on the sphere

- ***Gagliardo-Nirenberg-Sobolev inequalities on*** $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2,2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* = +\infty$ if $d = 1$ or 2

- Limit $p \to 2$: the ***logarithmic Sobolev inequality***

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu \ge \frac{d}{2}\,\mathcal{E}_2[F] := \frac{d}{2}\int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\,F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

[Bakry, Emery, 1984], [Bidaut-Véron, Véron, 1991], [Beckner, 1993]

Large dimensional limit

... based on the Maxwell-Poincaré lemma [McKean, 1973]
Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)$$

Theorem (Brigati, JD, Simonov)

Let $v \in \mathrm{H}^1(\mathbb{R}^n, dx)$ with compact support, $d \geq n$ and

$$u_d(\omega) = v\left(\omega_1/\sqrt{d}, \omega_2/\sqrt{d}, \ldots, \omega_n/\sqrt{d}\right)$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. With $d\gamma(y) := (2\,\pi)^{-n/2}\, e^{-\frac{1}{2}\,|y|^2}\, dy$,

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

Gaussian interpolation inequalities on $\mathbb{R}^n$

- ***Beckner interpolation inequalities***

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

▷ $1 \leq p < 2$ [Beckner, 1989], [Bakry, Emery, 1984]
▷ Poincaré inequality corresponding: $p = 1$
▷ *Gaussian logarithmic Sobolev inequality* $p \to 2$

- ***Gaussian logarithmic Sobolev inequality*** $p \to 2$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \frac{1}{2}\int_{\mathbb{R}^n} |v|^2 \log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$

$$d\gamma(y) := (2\,\pi)^{-n/2}\, e^{-\frac{1}{2}\,|y|^2}\, dy$$

Gaussian carré du champ and nonlinear diffusion

$$\frac{\partial v}{\partial t} = v^{-p\,(1-m)} \left(\mathcal{L} v + (m\,p-1)\,\frac{|\nabla v|^2}{v}\right) \quad \text{on} \quad \mathbb{R}^n$$

Ornstein-Uhlenbeck operator: $\mathcal{L} = \Delta - x \cdot \nabla$

$$m_\pm(p) := \lim_{d\to+\infty} m_\pm(d,p) = 1 \pm \frac{1}{p}\sqrt{(p-1)\,(2-p)}$$

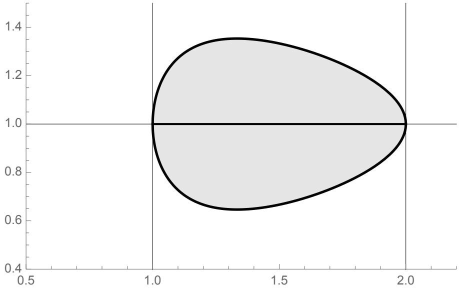


Figure: The admissible parameters $1 \le p \le 2$ and m are independent of n

A stability result for Gaussian interpolation inequalities

Theorem

For all $n \ge 1$, and all $p \in (1,2)$, there is an explicit constant $c_{n,p} > 0$ such that, for all $v \in \mathrm{H}^1(d\gamma)$,

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{p-2}\left(\|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\right)$$
$$\ge c_{n,p}\left(\|\nabla(\mathrm{Id}-\Pi_1)v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \frac{\|\nabla \Pi_1 v\|^4_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)$$

Stability for the logarithmic Sobolev inequality on $\mathbb{R}^d$

▷ [Gross, 1975] *Gaussian logarithmic Sobolev inequality* for $n \geq 1$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$

▷ [Weissler, 1979] scale invariant (but dimension-dependent) version of the Euclidean form of the inequality

▷ [Stam, 1959], [Federbush, 1969], [Costa, 1985] *Cf.* [Villani, 2008]

▷ [Bakry, Emery, 1984], [Carlen, 1991] equality iff

$$u \in \mathscr{M} := \left\{w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R}\right\} \quad \text{where} \quad w_{a,c}(x) = c\, e^{a\cdot x} \quad \forall\, x \in \mathbb{R}^n$$

[Carlen, 1991] reinforcement of the inequality (Wiener transform)

▷ [McKean, 1973], [Beckner, 92] (LSI) as a large d limit of Sobolev

▷ [Bobkov, Gozlan, Roberto, Samson, 2014], [Indrei et al., 2014-23] stability in Wasserstein distance, in $\mathrm{W}^{1,1}$, *etc.*

▷ [JD, Toscani, 2016] Comparison with Weissler's form, a (dimension dependent) improved inequality

▷ [Fathi, Indrei, Ledoux, 2016] improved inequality assuming a Poincaré inequality (Mehler formula)

A stability result for the logarithmic Sobolev inequality

- Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{1}{4}\,d\,(d-2)\left(\|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$
$$\geq \frac{\beta}{d} \inf_{G\in\mathcal{M}(\mathbb{S}^d)} \left(\|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\,d\,(d-2)\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

- Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d\to+\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi\,|x|^2}\,dx$

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta>0$ as in the result for the Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi\int_{\mathbb{R}^n} u^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)d\gamma$$
$$\geq \frac{\beta\,\pi}{2}\inf_{a\in\mathbb{R}^n,\,c\in\mathbb{R}}\int_{\mathbb{R}^n}|u-c\,e^{a\cdot x}|^2\,d\gamma$$

L^2 stability of LSI: comments

[JD, Esteban, Figalli, Frank, Loss]

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma \geq \frac{\beta\,\pi}{2} \inf_{a\in\mathbb{R}^d,\,c\in\mathbb{R}} \int_{\mathbb{R}^n} |u - c\,e^{a\cdot x}|^2\,d\gamma$$

- One dimension is lost (for the manifold of invariant functions) in the limiting process
- Euclidean forms of the stability
- The $\dot{\mathrm{H}}^1(\mathbb{R}^n)$ does not appear, it gets lost in the limit $d \to +\infty$
- $\int_{\mathbb{R}^n} |\nabla(u - c\,e^{a\cdot x})|^2\,d\gamma$? False, but makes sense under additional assumptions. Some results based on the Ornstein-Uhlenbeck flow and entropy methods: [Fathi, Indrei, Ledoux, 2016], [JD, Brigati, Simonov]
- Taking the limit is difficult because of the lack of compactness

Further results on logarithmic Sobolev inequalities

Joint papers with G. Brigati and N. Simonov

Stability for the logarithmic Sobolev inequality

Journal of Functional Analysis, 287, oct. 2024

arXiv: 2411.13271

Logarithmic Sobolev inequalities: a review on stability and instability results

La Matematica 5, 2026

arXiv: 2504.08658

▷ *Entropy methods, with constraints*

Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon\, x$ in the limit as $\varepsilon \to 0$

$$d(u_\varepsilon, 1)^2 = \|u_\varepsilon'\|^2_{\mathrm{L}^2(\mathbb{R}, d\gamma)} = \varepsilon^2 \quad \text{and} \quad \inf_{w \in \mathscr{M}} \mathrm{d}(u_\varepsilon, w)^\alpha \le \frac{1}{2}\, \varepsilon^4 + O\big(\varepsilon^6\big)$$

$\mathscr{M} := \left\{ w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R} \right\}$ where $w_{a,c}(x) = c\, e^{-a \cdot x}$

Proposition

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ and $\|x\, u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le d$, we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge \frac{1}{2\, d} \left(\int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \right)^2$$

and, with $\psi(s) := s - \frac{d}{4} \log\left(1 + \frac{4}{d}\, s\right)$,

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge \psi\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \right)$$

[Bakry, Ledoux '06], [Toscani '14], [JD, Toscani '16]

Lemma

Let $d \ge 1$*. With* $\varphi(t) := \frac{d}{4}\,\left[\exp\left(\frac{2\,t}{d}\right) - 1 - \frac{2\,t}{d}\right]$

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma - \frac{1}{2}\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma \\ \ge \varphi\left(\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma + \frac{d}{2} - \frac{1}{2}\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma\right)$$

for any $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ *such that* $\|v\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = 1$

Counter-examples to the H^1 stability if $\|x\,u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > d$
[Indrei, Marcon '14], [Kim '18], [Kim, Indrei '21], [Indrei '21-'23], [Brigati, JD, Simonov '25]

Stability under log-concavity

Theorem

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $u^2\,\gamma$ is log-concave and such that

$$\int_{\mathbb{R}^d} (1, x)\;|u|^2\,d\gamma = (1, 0) \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2\,|u|^2\,d\gamma \le \mathsf{K}$$

we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{\mathscr{C}_\star}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\,d\gamma \ge 0$$

$$\mathscr{C}_\star = 1 + \frac{1}{432\,\mathsf{K}} \approx 1 + \frac{0.00231481}{\mathsf{K}}$$

Self-improving Poincaré inequality and stability for LSI

[Fathi, Indrei, Ledoux, '16]

Theorem

Let $d \geq 1$. For any $\varepsilon > 0$, there is some explicit $\mathscr{C} > 1$ depending only on ε such that, for any $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ with

$$\int_{\mathbb{R}^d} (1, x)\, |u|^2\, d\gamma = (1, 0)\,, \quad \int_{\mathbb{R}^d} |u|^2\, e^{\varepsilon\, |x|^2}\, d\gamma < \infty$$

for some $\varepsilon > 0$, then we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \geq \frac{\mathscr{C}}{2} \int_{\mathbb{R}^d} |u|^2\, \log |u|^2\, d\gamma$$

with $\mathscr{C} = 1 + \frac{\mathscr{C}_\star(\mathsf{K}_\star) - 1}{1 + R^2\, \mathscr{C}_\star(\mathsf{K}_\star)}$, $\mathsf{K}_\star := \max\left(d, \frac{(d+1)\, R^2}{1 + R^2}\right)$ *if* $\operatorname{supp}(u) \subset B(0, R)$

Compact support: [Lee, Vázquez, '03]; [Chen, Chewi, Niles-Weed, '21]

Stability in Caffarelli-Kohn-Nirenberg inequalities ?

in collaboration with M. Bonforte, B. Nazaret and N. Simonov

Constructive stability results in interpolation inequalities and explicit improvements of decay rates of fast diffusion eq.
DCDS, 43 (3 & 4): 1070-1089, 2023

Subcritical Caffarelli-Kohn-Nirenberg inequalities

On $\mathbb{R}^d$ with $d \geq 1$, let us consider the *Caffarelli-Kohn-Nirenberg interpolation inequalities*

$$\|f\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \leq \mathcal{C}_{\beta,\gamma,p}\,\|\nabla f\|_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}^{1-\theta}$$

$$\gamma-2<\beta<\frac{d-2}{d}\,\gamma\,,\quad \gamma\in(-\infty,d)\,,\quad p\in(1,p_\star]\quad\text{with}\quad p_\star:=\frac{d-\gamma}{d-\beta-2}\,,$$

with $\theta=\frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$

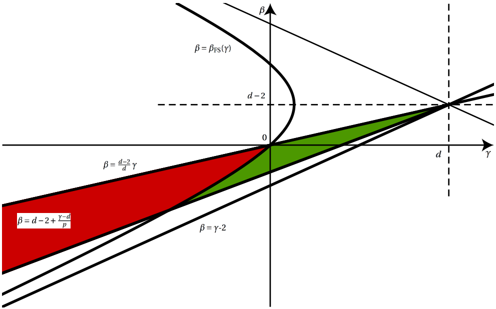
and $\|f\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d}|f|^q\,|x|^{-\gamma}\,dx\right)^{1/q}$

Symmetry: equality is achieved by the *Aubin-Talenti type functions*

$$\mathsf{g}(x)=\left(1+|x|^{2+\beta-\gamma}\right)^{-\frac{1}{p-1}}$$

[JD, Esteban, Loss, Muratori, 2017] Symmetry holds if and only if

$$\gamma<d\,,\quad\text{and}\quad \gamma-2<\beta<\frac{d-2}{d}\,\gamma\quad\text{and}\quad \beta\leq\beta_{\mathrm{FS}}(\gamma)$$



$d = 4$ and $p = 6/5$: (γ, β) admissible region

V

An improved decay rate along the flow

In self-similar variables, with $m=(p+1)/(2\,p)$

$$|x|^{-\gamma}\,\frac{\partial v}{\partial t}+\nabla\cdot\left(|x|^{-\beta}\,v\,\nabla v^{m-1}\right)=\sigma\,\nabla\cdot\left(x\,|x|^{-\gamma}\,v\right)$$

$$\mathcal{F}[v]=\frac{2\,p}{1-p}\int_{\mathbb{R}^d}\left(v^{\frac{p+1}{2\,p}}-\mathrm{g}^{p+1}-\frac{p+1}{2\,p}\mathrm{g}^{1-p}\left(v-\mathrm{g}^{2p}\right)\right)|x|^{-\gamma}\,dx$$

Theorem

In the symmetry region, if $v\ge0$ is a solution with a initial datum v_0 s.t.

$$A[v_0]:=\sup_{R>0}R^{\frac{2+\beta-\gamma}{1-m}-(d-\gamma)}\int_{|x|>R}v_0(x)\,|x|^{-\gamma}\,dx<\infty$$

then there are some $\zeta>0$ and some $T>0$ such that

$$\mathcal{F}[v(t,.)]\le\mathcal{F}[v_0]\,e^{-\,(4\,\alpha^2+\zeta)\,t}\quad\forall\,t\ge2\,T$$

[Bonforte, JD, Nazaret, Simonov, 2022]

Critical Caffarelli-Kohn-Nirenberg inequality

- The extension of the result for Sobolev to the critical CKN inequality

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

using the monotonicity of the associated parabolic flow after reduction of the problem to the sphere...

Joint paper with E. Bou Dagher

Caffarelli-Kohn-Nirenberg inequalities, parabolic carré du champ estimates and symmetry results for weighted interpolation inequalities on the sphere

(work in progress)

On a nonlinear kinetic Fokker-Planck equation

Joint paper with G. Brigati and G. Carlier
The fundamental solution of a nonlinear kinetic Fokker-Planck equation
arXiv: 2603.26650

Fundamental solution of a nonlinear kinetic equation

We consider the nonlinear kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f^m \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \qquad (*)$$

The *pressure* variable

$$\mathsf{P} := \frac{m}{1-m}\, f^{m-1}$$

solves

$$\partial_t \mathsf{P} = (1-m)\,\mathsf{P}\,\Delta_v \mathsf{P} - |\nabla_v \mathsf{P}|^2 - v \cdot \nabla_x \mathsf{P}$$

Special (fundamental) solution: with $A = \frac{1+d-d\,m}{3-d+d\,m}$,

$$\mathsf{P}_\star(t,x,v) = \beta(t) + \frac{(1+A)}{2\,(1-A)\,t}\left(\left|v - \frac{x}{(1-A)\,t}\right|^2 + A\left|\frac{x}{(1-A)\,t}\right|^2\right)$$

$$f_\star(t,x,v) = \left(\frac{1-m}{m}\,\mathsf{P}_\star(t,x,v)\right)_+^{\frac{1}{m-1}}$$

$$m_1 := 1 - \frac{1}{d} < m < 1 \quad \text{or} \quad 1 < m < m_2 := 1 + \frac{1}{d}$$

Intermediate asymptotics

The *fundamental solution* $f_\star$ governs the large time behaviour of all solutions of (*) with initial datum f_0 such that

$$f_0 \in \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)\,, \quad \|f_0\|_{\mathrm{L}^{\mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)}(\mathbb{R}^d)} = 1$$

$$g_{\gamma_1}(x, v-x) \le f_0(x,v) \le g_{\gamma_2}(x, v-x)$$

for any $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, with $(1-m)\,\gamma_2 > (1-m)\,\gamma_1 > 0$ and

$$g_\gamma(x,v) := \left(\tfrac{1-m}{m}\,\left(\gamma + \tfrac{1+A}{2}\,|v|^2 + \tfrac{1+A}{2}\,A\,|x|^2\right)\right)_+^{\frac{1}{m-1}}$$

Theorem

With $d \ge 1$, $m \in (m_1,1) \cup (1,m_2)$ and $1/2 < m < 3/2$ if $d=1$,

$$\lim_{t\to+\infty} \|f(t,\cdot,\cdot) - f_\star(t,\cdot,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)} = 0$$

Level lines on the phase space

The level lines of the *fundamental solution* $f_\star$ are ellipses, which rotate and expand in the phase space $\mathbb{R}^d \times \mathbb{R}^d \ni (x, v)$ as t increases

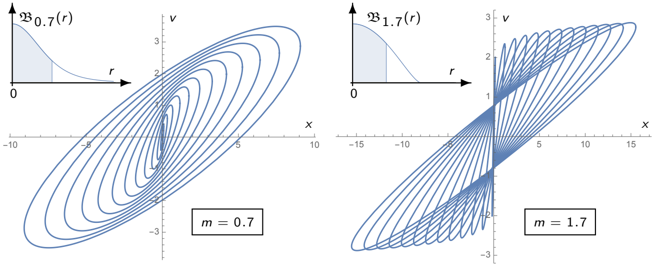


Figure: In dimension $d = 1$, the ellipse $\mathfrak{E}_m(t)$ is represented in the phase space $\mathbb{R}^d \times \mathbb{R}^d \ni (x; v)$ at times $t = 0.1$, $t = 0.6$, $1.1 \ldots 6.1$ for $m = 0.7$ (left) and $m = 1.7$ (right). In the upper left corners, the Barenblatt profiles $\mathfrak{B}_m(r) := \left(1 \pm r^2\right)_+^{1/(m-1)}$ are shown, with shaded areas corresponding to half of the mass. Here $\pm$ denotes the sign of $(1 - m)$ and r is such that $r^2 = A\,|x|^2 + |v|^2$: the function $f_\star(t, \cdot, \cdot)$ has compact support if $m > 1$

Current directions of research

- ▷ Gradient flow characterisation
  with G. Brigati, G. Carlier and F. Quattrocchi
- ▷ Diffusion limits with
  E. Bouin and A. Mellet
- ▷ Diffusion limits H^1-hypocoercivity
  with E. Bouin and C. Schmeiser

These slides can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Lectures/
▷ Lectures

The papers can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Preprints/list/
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For final versions, use `Dolbeault` as login and `Jean` as password

Thank you for your attention !