Recent results of stability in functional inequalities

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Outline

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Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

A joint work with M. Bonforte, B. Nazaret and N. Simonov

Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method

arXiv:2007.03674, to appear in *Memoirs of the AMS*

Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

Strategy. Rewrite (GNS) in non-scale invariant form

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{2} + \|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{p+1} \geq \mathcal{K}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p\,\gamma}$$

Use the fast diffusion flow

$$\frac{\partial\rho}{\partial t} = \Delta\rho^m \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d$$

with initial datum $\rho(t=0,\cdot)=|f|^{2p}$ and apply entropy methods

Range of exponents

$$1<p\leq\frac{d}{d-2} \quad\Longleftrightarrow\quad \frac{d-1}{d} =: m_1 \leq m < 1$$

- Sobolev inequality: $p=\frac{d}{d-2}$, $m=m_1$
- Logarithmic Sobolev inequality: $p=1$, $m=1$

Entropy – entropy production inequality

Fast diffusion equation (written in self-similar variables)

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \left(v \left(\nabla v^{m-1} - 2\,x \right) \right) = 0 \qquad (r\,\mathrm{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1} \left(v - \mathcal{B} \right) \right) dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2 dx$$

satisfy an *entropy – entropy production inequality*

$$\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$$

[del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[v_0]\,e^{-4\,t}$$

The *entropy – entropy production inequality*

$$\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$$

is equivalent to the *Gagliardo-Nirenberg-Sobolev inequalities*

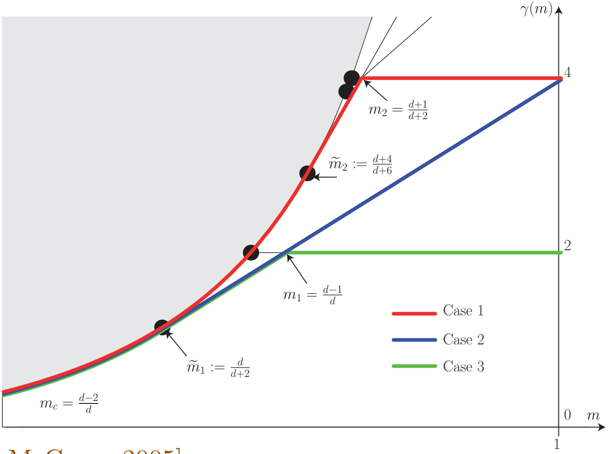
$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

with equality if and only if $|f|^{2\,p}$ is the *Barenblatt profile* such that

$$|f(x)|^{2p} = \mathcal{B}(x) = \left(1+|x|^2\right)^{\frac{1}{m-1}}$$

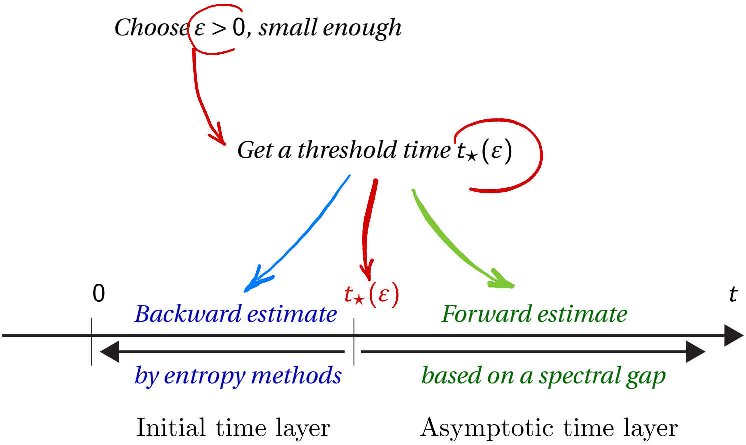
$v = f^{2\,p}$ so that $v^m = f^{p+1}$ and $v\left|\nabla v^{m-1}\right|^2 = (p-1)^2\,|\nabla f|^2$

Spectral gap and Taylor expansion around $\mathcal{B}$



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Strategy of the method



A constructive stability result (subcritical case)

The ***stability in the entropy - entropy production estimate*** $\mathcal{I}[v] - 4\,\mathcal{F}[v] \ge \zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \ge \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

$$\mathrm{A}[\rho] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} \rho\,dx \le A < \infty$$

Theorem

Let $d \ge 1$ and $p \in (1, p^*)$. There is an explicit $\mathcal{C} = \mathcal{C}[f] > 0$ such that, for any $f \in \mathrm{L}^{2p}\big(\mathbb{R}^d, (1+|x|^2)\,dx\big)$ s.t. $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathrm{A}\big[f^{2p}\big] < \infty$

$$(p-1)^2\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + 4\,\frac{d - p\,(d-2)}{p+1}\,\|f\|^{p+1}_{\mathrm{L}^{p+1}(\mathbb{R}^d)} - \mathcal{K}_{\mathrm{GNS}}\,\|f\|^{2\,p\,\gamma}_{\mathrm{L}^{2\,p}(\mathbb{R}^d)} \ge \mathcal{C}[f] \inf_{\varphi \in \mathfrak{M}} \int_{\mathbb{R}^d} \left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 dx$$

A constructive stability result (critical case)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \geq 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d)\,:\,\nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\;\;|x|\,f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d)\right\}$$

Theorem

Let $d \geq 3$ and $A > 0$. For any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d}(1,x,|x|^2)\,f^{2^*}\,dx = \int_{\mathbb{R}^d}(1,x,|x|^2)\,\mathrm{g}\,dx \text{ and } \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\,dx \leq A$$

we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\mathcal{C}_\star(A)}{4+\mathcal{C}_\star(A)}\int_{\mathbb{R}^d}\left|\nabla f + \tfrac{d-2}{2}\,f^{\frac{d}{d-2}}\,\nabla \mathrm{g}^{-\frac{2}{d-2}}\right|^2 dx$$

$\mathcal{C}_\star(A) = \mathcal{C}_\star(0)\left(1+A^{1/(2\,d)}\right)^{-1}$ *and* $\mathcal{C}_\star(0) > 0$ *depends only on* d

Logarithmic Sobolev and Gagliardo-Nirenberg inequalities on the sphere

A joint work with G. Brigati and N. Simonov

Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results

arXiv:2211.13180

Stability for subcritical GNS inequalities on $\mathbb{S}^d$

For any $p \in [1,2) \cup (2,2^*)$, with $d\mu$: uniform probability measure
$2^* := 2\,d/(d-2)$ if $d \geq 3$ and $2^* = +\infty$ otherwise

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu \geq \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\,F \in \mathrm{H}^1(\mathbb{S}^d,d\mu)$$

Optimal constant: test functions $F_\varepsilon(x) = 1 + \varepsilon\,x\cdot\nu$, $\nu \in \mathbb{S}^d$, $\varepsilon \to 0$
logarithmic Sobolev inequality: obtained by taking the limit as $p \to 2$

Theorem

Let $d \geq 1$ and $p \in (1,2) \cup (2,2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d,d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu - \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$
$$\geq \mathscr{S}_{d,p}\left(\frac{\|\nabla\Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla(\mathrm{Id}-\Pi_1)\,F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

A joint work with JD, M.J. Esteban, A. Figalli, R. Frank, M. Loss
Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence
arXiv: 2209.08651

Stability results for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq S_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g(x) = c\,\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

Theorem

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

[JD, Esteban, Figalli, Frank, Loss]

Some important features of this result:

- The (estimate of the) constant β is explicit
- No compactness argument is involved

Some history

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq S_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$$

▷ $2^* = 2\,d/(d-2)$ is the *critical Sobolev exponent*
▷ $S_d = \frac{1}{4}\,d\,(d-2)\,|\mathbb{S}^d|^{2/d}$ is the sharp Sobolev constant
[Rodemich, 1966], [Rosen, 1971], [Aubin, 1976] and [Talenti, 1976] ▷ *Sobolev deficit*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

▷ [Brezis, Lieb, 1985]: is it possible to bound the deficit on $\dot{\mathrm{H}}^1(\mathbb{R}^d)$ from below by some distance to $\mathcal{M}$?
▷ [Lions, 1985] if the deficit is small for some function f, then f has to be close to $\mathcal{M}$
▷ [Bianchi, Egnell, 1991] for any $d \geq 3$ there is a constant $c_{\mathrm{BE}} > 0$ s.t.

$$\mathcal{E}(f) := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} \geq c_{\mathrm{BE}} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$$

▷ [König, 2022] c_{BE} is achieved and $c_{\mathrm{BE}} < 4/(d+4)$
▷ [Figalli, 2013] for more historical details

Comments

▷ The power two of the distance to $\mathcal{M}$ is optimal.

▷ The strategy of Bianchi-Egnell is based

- on a local analysis in a neighborhood of $\mathcal{M}$ (spectral analysis)
- on a reduction of the global estimate to a local estimate by the concentration-compactness method based on Lions's analysis

▷ The method is widely applicable to many problems in the Calculus of Variations

▷ Because of either compactness estimates or arguments by contradiction, no estimate of c_{BE} was known so far

Our strategy is to make both steps of the strategy of Bianchi-Egnell constructive and based on

▷ The "far away" regime and the "neighborhood" of $\mathcal{M}$

▷ Competing symmetries and a notion of a continuous flow (based on Steiner's symmetrization) to reduce the global estimate to a local estimate

Stability for Sobolev: main steps of the proof

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d}\,\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

The proof is divided into several steps

▷ We prove the inequality for nonnegative functions close to $\mathcal{M}$ with an explicit remainder term (without dimensional dependence)

▷ We prove the inequality for *nonnegative* functions far from $\mathcal{M}$ using the method of *competing symmetries* and a continuous symmetrization

▷ The inequality for *sign changing* functions is deduced from the inequality for *nonnegative* functions by convexity arguments

▷ To get the asymptotic dependence in the dimension requires a refined analysis of the local step: a cutting at various scales, uniform bounds on spherical harmonics, and some concavity properties

The sphere and the stereographic projection

We denote by $\omega=(\omega_1,\omega_2,\ldots,\omega_{d+1})$ the coordinates in $\mathbb{R}^{d+1}$
Stereographic coordinates on the unit sphere $\mathbb{S}^d\subset\mathbb{R}^{d+1}$

$$\omega_j=\frac{2\,x_j}{1+|x|^2}\,,\quad j=1,\ldots,d\,,\quad \omega_{d+1}=\frac{1-|x|^2}{1+|x|^2}$$

To a function f on $\mathbb{R}^d$ we associate a function F on $\mathbb{S}^d$ via

$$F(\omega)=\frac{f(x)}{g_\star(x)}\,,\quad g_\star(x):=\left(\frac{1+|x|^2}{2}\right)^{-\frac{d-2}{2}}\quad\forall\,x\in\mathbb{R}^d$$

If $d\mu$ is the uniform probability measure on $\mathbb{S}^d$, then the sharp Sobolev inequality on $\mathbb{S}^d$ for any $F\in\mathrm{H}^1(\mathbb{S}^d,d\mu)$ is

$$\int_{\mathbb{S}^d}\left(|\nabla F|^2+A\,|F|^2\right)d\mu\ge A\left(\int_{\mathbb{S}^d}|F|^{2^*}\,d\mu\right)^{2/2^*}$$

with $A=\frac14\,d\,(d-2)$. Equality holds exactly for the functions

$$G(\omega)=c\,\left(a+b\cdot\omega\right)^{-\frac{d-2}{2}}$$

and $a>0$, $b\in\mathbb{R}^d$ and $c\in\mathbb{R}$ are constants

A preliminary result (without optimal dependence in d)

$$\mathcal{E}[f] := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\,,\quad \nu(\delta) := \sqrt{\frac{\delta}{1-\delta}}$$

Theorem

[JD, Esteban, Figalli, Frank, Loss] Let $d \ge 3$, $q = 2\,d/(d-2)$. If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a *non-negative* function, then

$$\mathcal{E}[f] \ge \sup_{0<\delta<1} \delta\,\mu(\delta)$$

where $\mu(\delta) \ge \mathsf{m}\big(\nu(\delta)\big)$ and

$$\mathsf{m}(\nu) := \frac{4}{d+4} - \frac{2}{q}\,\nu^{q-2} \qquad \text{if} \quad d \ge 6$$

$$\mathsf{m}(\nu) := \frac{4}{d+4} - \frac{1}{3}\,(q-1)\,(q-2)\,\nu - \frac{2}{q}\,\nu^{q-2} \qquad \text{if} \quad d = 4, 5$$

$$\mathsf{m}(\nu) := \frac{4}{7} - \frac{20}{3}\,\nu - 5\,\nu^2 - 2\,\nu^3 - \frac{1}{3}\,\nu^4 \qquad \text{if} \quad d = 3$$

Strategy: two regions

- *Taylor expansion, spectral estimates:* in the region $\inf_{g\in\mathcal{M}} \|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, prove that

$$\mathcal{E}[f] \ge \mu(\delta)$$

- *Continuous flow argument:* [Christ, 2017] if $\inf_{g\in\mathcal{M}} \|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, build a *flow* $(f_\tau)_{0\le\tau<\infty}$ s.t.

$$f_0 = f\,,\quad \|f_\tau\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = \|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\,,\quad \tau\mapsto\|\nabla f_\tau\|_{\mathrm{L}^2(\mathbb{R}^d)}\ \text{is}\ \searrow$$

$$\lim_{\tau\to\infty}\inf_{g\in\mathcal{M}}\|\nabla(f_\tau-g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = 0$$

$$\mathcal{E}[f] \ge \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} = 1 - S_d\,\frac{\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} \ge \frac{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f_{\tau_0}\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}$$

for some τ_0 (it exists ?) s.t. $\inf_{g\in\mathcal{M}}\|\nabla(f_{\tau_0}-g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = \delta\,\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

... then $\mathcal{E}[f] \ge \mathcal{E}(f_{\tau_0}) \ge \delta\,\mu(\delta)$

Step 1: Taylor expansion in the neighborhood of $\mathcal{M}$

Proposition

Let $\nu>0$, $r\in\mathrm{H}^1(\mathbb{S}^d)$ *such that* $1+r\ge0$, $\|r\|_{\mathrm{L}^q(\mathbb{S}^d)}\le\nu$ *and*

$$\int_{\mathbb{S}^d} r\,d\mu=0=\int_{\mathbb{S}^d}\omega_j\,r\,d\mu\,,\quad j=1,\ldots,d+1$$

$$\int_{\mathbb{S}^d}\left(|\nabla r|^2+A\,(1+r)^2\right)d\mu-A\left(\int_{\mathbb{S}^d}(1+r)^q\,d\mu\right)^{2/q}\ge \mathrm{m}(\nu)\int_{\mathbb{S}^d}\left(|\nabla r|^2+A\,r^2\right)d\mu$$

$$\mathrm{m}(\nu):=\tfrac{4}{d+4}-\tfrac{2}{q}\,\nu^{q-2}\qquad \textit{if}\quad d\ge6$$

$$\mathrm{m}(\nu):=\tfrac{4}{d+4}-\tfrac{1}{3}\,(q-1)\,(q-2)\,\nu-\tfrac{2}{q}\,\nu^{q-2}\qquad \textit{if}\quad d=4\,,\,5$$

$$\mathrm{m}(\nu):=\tfrac{4}{7}-\tfrac{20}{3}\,\nu-5\,\nu^2-2\,\nu^3-\tfrac{1}{3}\,\nu^4\qquad \textit{if}\quad d=3$$

Analysis close to the manifold of optimizers

Proposition

Let X be a measure space and $u, r \in \mathrm{L}^q(X)$ for some $q \geq 2$ with $u \geq 0$ and $u + r \geq 0$. Assume also that $\int_X u^{q-1}\, r\, dx = 0$. If $2 \leq q \leq 3$, then

$$\|u + r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \left((q-1) \int_X u^{q-2}\, r^2\, dx + \frac{2}{q} \int_X r_+^q\, dx \right)$$

$2 \leq q = \frac{2\,d}{d-2} \leq 3$ means $d \geq 6$ and is the most difficult case for Taylor

Corollary

Let $q = 2^*$, $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ which realizes

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2$$

Set $r := f - u$ and $\sigma := \|r\|_q / \|u\|_q$. If $d \geq 6$, we have

$$\|\nabla f\|_2^2 - S_d\, \|f\|_q^2 \geq \int_{\mathbb{R}^d} \left(|\nabla r|^2 - S_d\, (q-1)\, \|u\|_q^{2-q}\, u^{q-2}\, r^2 \right) dx - \tfrac{2}{q}\, \|\nabla r\|_2^2\, \sigma^{q-2}$$

Spectral gap estimate

Cf. [Rey, 1990] and [Bianchi, Egnell, 1991]

Lemma

Let $d \geq 3$, $q = 2^*$, $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ be such that $\|\nabla f - \nabla u\| = \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|$. Then $r := f - u$ satisfies

$$\int_{\mathbb{R}^d} \left(|\nabla r|^2 - S_d\,(q-1)\,\|u\|_q^{2-q}\,|u|^{q-2}\,r^2 \right) dx \geq \frac{4}{d+4} \int_{\mathbb{R}^d} |\nabla r|^2\,dx$$

Corollary

Let $q = 2^*$ and $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$. Set $\mathcal{D}[f] := \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2$ and $\tau := \mathcal{D}[f]/(\|\nabla f\|_2^2 - \mathcal{D}[f]^2)^{1/2}$. If $d \geq 6$, we have

$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \geq \left(\tfrac{4}{d+4} - \tfrac{2}{q}\,\tau^{q-2}\right) \mathcal{D}[f]^2$$

Step 2: The "far away" regime for nonnegative solutions

▷ We prove the inequality for *nonnegative* functions far from $\mathcal{M}$ using the method of *competing symmetries* and a continuous symmetrization

Competing symmetries

[Carlen, Loss, 1990]

- *Conformal rotation*

$$(UF)(s) = F(s_1, s_2, \ldots, s_{d+1}, -s_d)$$

On $\mathbb{R}^d$, the function that corresponds to UF on $\mathbb{R}^d$ is given by

$$(Uf)(x) = \left(\frac{2}{|x-e_d|^2}\right)^{\frac{d-2}{2}} f\left(\frac{x_1}{|x-e_d|^2}, \ldots, \frac{x_{d-1}}{|x-e_d|^2}, \frac{|x|^2-1}{|x-e_d|^2}\right)$$

where $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$ and $\mathcal{E}(Uf) = \mathcal{E}[f]$

- *Symmetric decreasing rearrangement*: if $f \geq 0$, let

$$\mathcal{R}f(x) = f^*(x)$$

f and f^* are equimeasurable and $\|\nabla f^*\|_2 \leq \|\nabla f\|_2$

... continuous Steiner symmetrization

On $\mathbb{R}^d$, let

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}$$

Theorem

[Carlen, Loss] Let $f \in \mathrm{L}^{2^*}(\mathbb{R}^d)$ be a non-negative function. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ of functions

$$f_n = (\mathcal{R}U)^n f$$

Then $h_f = \|f\|_{2^*}\, g_* \in \mathcal{M}$ and

$$\lim_{n\to\infty} \|f_n - h_f\|_{2^*} = 0$$

If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$, then $(\|\nabla f_n\|_2)_{n\in\mathbb{N}}$ is a non-increasing sequence

Define $\mathcal{M}_1$ to be the set of the elements in $\mathcal{M}$ with 2^*-norm equal to 1

Lemma

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(f, g^{2^*-1}\right)^2$$

Lemma

For the sequence $(f_n)_{n\in\mathbb{N}}$ of the Theorem of [Carlen, Loss] we have that

$$n \mapsto \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_{2^*}^2 \quad \textit{is strictly decreasing}$$

$$\lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d \|f\|_{2^*}^2$$

Two alternatives

Lemma

Let $0 \le f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$ s.t. $\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 \ge \delta\, \|\nabla f\|_2^2$
One of the following alternatives holds:

(a) for all $n = 0, 1, 2 \ldots$ $\inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 \ge \delta\, \|\nabla f_n\|_2^2$

(b) $\exists n_0 \in \mathbb{N}$ such that

$$\inf_{g\in\mathcal{M}} \|\nabla f_{n_0} - \nabla g\|_2^2 \ge \delta\, \|\nabla f_{n_0}\|_2^2 \quad \text{and} \quad \inf_{g\in\mathcal{M}} \|\nabla f_{n_0+1} - \nabla g\|_2^2 < \delta\, \|\nabla f_{n_0+1}\|_2^2$$

In case (a) we have

$$\mathcal{E}[f] = \frac{\|\nabla f\|_2^2 - S_d\, \|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2} \ge \frac{\|\nabla f\|_2^2 - S_d\, \|f\|_{2^*}^2}{\|\nabla f\|_2^2} \ge \frac{\|\nabla f_n\|_2^2 - S_d\, \|f\|_{2^*}^2}{\|\nabla f_n\|_2^2} \ge \delta$$

because by the Theorem of [Carlen, Loss]

$$\lim_{n\to\infty} \|\nabla f_n\|_2^2 \le \frac{1}{\delta} \lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \frac{1}{\delta} \left(\lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\, \|f\|_{2^*}^2 \right)$$

Continuous rearrangement

Let $\mathsf{f}_0 = U f_{n_0}$ and denote by $(\mathsf{f}_\tau)_{0\le\tau\le\infty}$ the continuous rearrangement starting at f_0 and ending at $\mathsf{f}_\infty = f_{n_0+1}$
We find $\tau_0 \in [0,\infty)$ such that

$$\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2 = \delta\,\|\nabla \mathsf{f}_{\tau_0}\|_2^2$$

and conclude using

$$\mathcal{E}(\mathsf{f}_0) \ge 1 - S_d\,\frac{\|\mathsf{f}_0\|_{2^*}^2}{\|\nabla \mathsf{f}_0\|_2^2} \ge 1 - S_d\,\frac{\|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\|\nabla \mathsf{f}_{\tau_0}\|_2^2} = \delta\,\frac{\|\nabla \mathsf{f}_{\tau_0}\|_2^2 - S_d\,\|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\inf_{g\in\mathcal{M}}\|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2} \ge \delta\,\mu(\delta)$$

Existence of τ_0 not granted: a discussion is needed !
(use a semi-continuity property)

Step 3: removing the positivity assumption

The *Bianchi-Egnell stability estimate*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge c_{\mathrm{BE}}\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

Nonnegative functions: $c_{\mathrm{BE}}^{\mathrm{pos}} \ge \delta\,\mu(\delta)$ and $c_{\mathrm{BE}} \le c_{\mathrm{BE}}^{\mathrm{pos}} \le \frac{4}{d+4}$

Sign-changing solutions. Take $m := \|u_-\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ and assume that $1-m = \|u_+\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$. We argue that $2\,h(1/2)\,m \le h(m)$ if

$$h(m) := m^{1-\frac{2}{d}} + (1-m)^{1-\frac{2}{d}} - 1$$

With $\mathrm{D}(v) := \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|v\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ and (...), we obtain

$$\mathrm{D}(u) \ge c_{\mathrm{BE}}^{\mathrm{pos}}\,\|\nabla u_+ - \nabla g_+\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{2\,h(1/2)}{2\,h(1/2)+\xi_d}\,\|\nabla u_-\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

$$\boxed{c_{\mathrm{BE}} \ge \tfrac{1}{2}\,\delta\,\mu(\delta)}$$

Dimensional dependence and stability results for the log-Sobolev inequality

An equivalent form of the stability inequality

Bianchi-Egnell stability estimate

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \frac{\beta(d)}{d}\,\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

We know that $\beta_\star = \liminf_{d\to+\infty}\beta(d) > 0$

With the Aubin-Talenti function $g_\star(x) := \left(1+|x|^2\right)^{1-\frac{d}{2}}$ and $u = f/g_\star$,

$$\int_{\mathbb{R}^d}|\nabla u|^2\,g_\star^2\,dx + d\,(d-2)\int_{\mathbb{R}^d}|u|^2\,g_\star^{2^*}\,dx$$
$$- d\,(d-2)\,\|g_\star\|^{2^*-2}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}|u|^{2^*}\,g_\star^{2^*}\,dx\right)^{2/2^*}$$
$$\ge \frac{\beta(d)}{d}\left(\int_{\mathbb{R}^d}|\nabla u|^2\,g_\star^2\,dx + d\,(d-2)\int_{\mathbb{R}^d}|u-1|^2\,F_\star^{2^*}\,dx\right)$$

A rescaling

$$u(x) = v\,(r_d\,x) \quad \forall\,x \in \mathbb{R}^d\,, \quad r_d = \sqrt{\frac{d}{2\,\pi}}$$

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla v|^2 \left(1 + \tfrac{1}{r_d^2}\,|x|^2\right)^2 d\mu_d \\
&\geq \pi\,(d-2)\left[\left(\int_{\mathbb{R}^d} |v|^{2^*}\,d\mu_d\right)^{2/2^*} - \int_{\mathbb{R}^d} |v|^2\,d\mu_d\right] \\
&\qquad + \frac{\beta(d)}{d}\left(\int_{\mathbb{R}^d} |\nabla v|^2\,d\mu_d + (d-2)\int_{\mathbb{R}^d} |v-1|^2\,d\mu_d\right)
\end{aligned}$$

where $d\mu_d = Z_d^{-1}\,g_\star^{2^*}\,dx$ is the probability measure given by

$$d\mu_d(x) := Z_d^{-1}\left(1 + \tfrac{1}{r_d^2}\,|x|^2\right)^{-d} dx \quad \text{with} \quad Z_d = \frac{2^{1-d}\,\sqrt{\pi}}{\Gamma\left(\frac{d+1}{2}\right)}\left(\frac{d}{2}\right)^{\frac{d}{2}}$$

The large dimensions limit in the Sobolev inequality

Let us consider a function $v(x)$ which actually depends only on $y \in \mathbb{R}^N$, where we write that $x = (y,z) \in \mathbb{R}^N \times \mathbb{R}^{d-N} \approx \mathbb{R}^N$, for some integer N such that $1 \le N < d$. With $|x|^2 = |y|^2 + |z|^2$ and

$$1 + \frac{1}{r_d^2}\,|x|^2 = 1 + \frac{1}{r_d^2}\left(|y|^2 + |z|^2\right) = \left(1 + \frac{1}{r_d^2}\,|y|^2\right)\left(1 + \frac{|z|^2}{r_d^2 + |y|^2}\right)$$

we can integrate over the z variable and notice that

$$\lim_{d\to+\infty}\left(1 + \frac{1}{r_d^2}\,|y|^2\right)^{-\frac{N+d}{2}} = e^{-\pi\,|y|^2}$$

$$\lim_{d\to+\infty}\int_{\mathbb{R}^d}|v(y)|^2\,d\mu_d = \int_{\mathbb{R}^N}|v|^2\,d\gamma$$

$$\lim_{d\to+\infty}\int_{\mathbb{R}^d}|\nabla v|^2\left(1 + \frac{1}{r_d^2}\,|x|^2\right)^2 d\mu_d = \int_{\mathbb{R}^N}|\nabla v|^2\,d\gamma$$

where $d\gamma(y) := e^{-\pi\,|y|^2}\,dy$ is a standard Gaussian probability measure
Gaussian logarithmic Sobolev inequality

$$\int_{\mathbb{R}^N}|\nabla v|^2\,d\gamma \ge \frac{1}{2}\int_{\mathbb{R}^N}|v|^2\,\log\left(\frac{|v|^2}{\int_{\mathbb{R}^N}|v|^2\,d\gamma}\right)d\gamma$$

Gaussian logarithmic Sobolev inequality

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, d\gamma \geq \pi \int_{\mathbb{R}^N} |v|^2 \ln\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\gamma)}}\right) d\gamma$$

The constant π is optimal
[Carlen, 1991] equality holds if and only if

$$v(x) = c\, e^{a\cdot x}$$

for some $a \in \mathbb{R}^N$ and $c \in \mathbb{R}$

Theorem

There is an explicit constant $\kappa > 0$ such that $\forall\, N \in \mathbb{N}$ and $\forall\, v \in \mathrm{H}^1(\gamma)$

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, d\gamma - \pi \int_{\mathbb{R}^N} v^2 \ln\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\gamma)}}\right) d\gamma \geq \kappa \inf_{a\in\mathbb{R}^N,\, c\in\mathbb{R}} \int_{\mathbb{R}^N} (v - c\, e^{a\cdot x})^2 \, d\gamma$$

Step 4 for Sobolev stability: optimal dependence

Refinement of Step 1: cutting r into pieces

$$(1+r)^q - 1 - q\,r$$

for real numbers r in terms of three numbers

$$r_1 := \min\{r,\gamma\}\,,\quad r_2 := \min\{(r-\gamma)_+, M-\gamma\}\quad\text{and}\quad r_3 := (r-M)_+$$

where γ and M are parameters such that $0<\gamma<M$

$$\theta := q-2 = 2^*-2 = \tfrac{4}{d-2} \to 0 \quad\text{as}\quad d\to+\infty$$

Lemma

Given $q\in[2,3]$, $r\in[-1,\infty)$ and $\overline{M}\in[\sqrt{e},+\infty)$, we have

$$\begin{aligned}(1+r)^q - 1 - q\,r \\ &\le \tfrac12\,q\,(q-1)\,(r_1+r_2)^2 + 2\,(r_1+r_2)\,r_3 + \Big(1 + C_M\,\theta\,\overline{M}^{-1}\ln\overline{M}\Big)\,r_3^q \\ &\quad + \Big(\tfrac32\,\gamma\,\theta\,r_1^2 + C_{M,\overline{M}}\,\theta\,r_2^2\Big)\,\mathbb{1}_{\{r\le M\}} + C_{M,\overline{M}}\,\theta\,M^2\,\mathbb{1}_{\{r>M\}}\end{aligned}$$

$$\int_{\mathbb{S}^d} \left(|\nabla r|^2 + A\,(1+r)^2\right) d\mu - A\left(\int_{\mathbb{S}^d} (1+r)^q\, d\mu\right)^{2/q} \geq \theta\,\epsilon_0 \int_{\mathbb{S}^d} \left(|\nabla r|^2 + A\,r^2\right) d\mu + \sum_{k=1}^{3} I_k$$

$$I_1 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d} \left(|\nabla r_1|^2 + A\,r_1^2\right) d\mu - A\,(q-1+\epsilon_1\,\theta)\int_{\mathbb{S}^d} r_1^2\, d\mu + A\,\sigma_0\,\theta \int_{\mathbb{S}^d} (r_2^2 + r_3^2)$$

$$I_2 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d} \left(|\nabla r_2|^2 + A\,r_2^2\right) d\mu - A\left(q-1+(\sigma_0 + C_{\epsilon_1,\epsilon_2})\,\theta\right)\int_{\mathbb{S}^d} r_2^2\, d\mu$$

$$I_3 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d} \left(|\nabla r_3|^2 + A\,r_3^2\right) d\mu - \frac{2}{q}\,A\,(1+\epsilon_2\,\theta)\int_{\mathbb{S}^d} r_3^q\, d\mu - A\,\sigma_0\,\theta \int_{\mathbb{S}^d} r_3^2\, d\mu$$

for some parameter $\sigma_0 > 0$
I_1: spectral gap estimates
I_3: use the Sobolev inequality. The extra coefficient $\frac{2}{q} < 1$ gives us enough room to accomodate all error terms I_2: an improved spectral gap inequality using that $\mu\big(\{r_2 > 0\}\big)$ is small

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Thank you for your attention !