OPTIMAL FUNCTIONS AND SYMMETRY BREAKING
IN FUNCTIONAL INEQUALITIES
Madrid Autonoma - Math Cambridge - CMS Applied PDEs Days
September 29, 2009

INTRODUCTION : Notion of symmetry breaking

* physics : solutions with symmetry have for long been studied than more delicate solutions for which symmetry is broken, and this has to be related with the idea that "simple" symmetric cases are more likely to occur. The common belief has been summarized by the "symmetry principle of Pierre Curie" which asserts that "effects are at least as symmetric as their causes". This was thought in view of applications in electromagnetism a century ago, but since then, several examples have shown that reality is not as simple: ferromagnetism for instance can be made compatible only by requiring that all final states are equally likely to occur also symmetry is broken after magnetization. Symmetry breaking is even more crucial in quantum field (to the point that symmetry breaking automatically refers to QFT for most physicists) but considerations on this topic are far beyond the scope of this talk.

* mathematics: preoccupations for uniqueness and the understanding of bifurcation mechanisms have emerged quite recently. When one refers to symmetry in (elliptic) PDEs, usually the preoccupation is to reduce the problem to a symmetric setting (for instance, to radially symmetric solutions) in order to have a simpler problem to solve (for instance an ODE). The classical scheme which has emerged since the beginning of the 70's when dealing with the symmetry of the solutions in nonlinear scalar field theory is the following:

1) instead of working with critical points (minimizers) directly work with the solutions of the Euler-Lagrange equations, and only keep some properties of the solutions. The notion of "ground state" for instance has evolved from the notion of a minimizer of a functional (which is the correct notion in physics) to the notion of a positive solution (for instance on an euclidean space) which has limit zero at infinity

2) prove radial symmetry

3) establish uniqueness of the solutions of the corresponding ODE.

Much less has been done in the direction of proving symmetry breaking and in studying the properties of the corresponding solutions (this is more difficult, of course).

The goal of this talk is to understand symmetry versus symmetry breaking. Proper linearization plays a crucial role in my understanding of it.

From Symmetry to Symmetry Breaking. $\boxed{\text{Moving Planes Approach}}$

In two celebrated papers 30 years ago, Gidas, Ni and Nirenberg have given the first generic result of symmetry.

THM [Gidas, Ni & Nirenberg '79] Let $u \in C^2(B)$, $B = B(0,1)$ be a solution of $\Delta u + f(u) = 0$ in B, $u = 0$ ∂B, where f is a Lipschitz function. If u is positive then u is radial and decreasing along any radius: $u' < 0$.

[Gidas, Ni & Nirenberg '81] a similar result in $\mathbb{R}^d$ (more technical: requirements on the behaviour of the solution at infinity).

A first extension: $\Delta u + f(r,u) = 0$, $r = |x|$, $x \in B$

If $\frac{\partial f}{\partial r} \leq 0$, then the same result holds.... but this extension is somewhat trivial (the monotonicity is such that the r-dependent case is immediately reduced to the r-independent case).

THM [J. D, P. Felmer '99] Consider solutions of $\Delta u + \lambda f(r,u) = 0$ and assume that $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$. We make no assumption on the sign of $\frac{\partial f}{\partial r}$. There exists λ_1, λ_2 with $0 < \lambda_1 \leq \lambda_2$ such that:

(i) if $\lambda \in (0, \lambda_1)$ then $\frac{d}{dr}(u - \lambda u_0) < 0$ where u_0 is the solution of $-\Delta u_0 + f(r, u_0) = 0$ on B, $u_0|_{\partial B} = 0$.

(ii) if $\lambda \in (0, \lambda_2)$, then u is radially symmetric.

This results explains the role of a counter-example given by GNN and heuristically means that radial symmetry is a property associated to a second eigenvalue, while positivity or decay has to do with the first eigenvalue. This explains the very paradoxical statement in the GNN result: positivity implies symmetry, since positivity means that the parameter λ (hidden in the GNN theorem) being below λ_1, it is also below λ_2.

Idea of the proof. If u is a solution of $\Delta u + \lambda f(r,u) = 0$, then $\bar{u}(x) = u(\bar{x})$, $\bar{x} = (-x_1, x')$ if $x = (x_1, x')$ is also a solution: $\Delta \bar{u} + \lambda f(r, \bar{u}) = 0$ because $|\bar{x}| = |x| = r$! Let $v = \bar{u} - u$, $c_u = \frac{f(r,\bar{u}) - f(r,u)}{\bar{u} - u}$ and observe that $\Delta v + \lambda c_u v = 0$

λ_1 and λ_2 are defined as follows:

$\lambda_1 = \sup\{\lambda > 0 \text{ s.t. } \{\Delta v + \lambda c v = 0^B_{v=0\,\partial B} \Rightarrow v = 0\}$

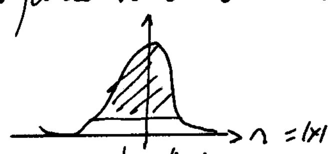
$\lambda_2 =$ idem, under the additional condition that v changes sign. If $\Delta v + \lambda c v = 0$, then $v > 0$ or $v \equiv 0$.

λ_1 and λ_2 depend on c and therefore on u, but can be evaluated uniformly with respect to u (more general conditions can be given on f).

Assume that $\lambda < \lambda_2$: $\Delta v + \lambda c_u v = 0 \Rightarrow \lambda = \lambda_1 \Rightarrow v > 0$ or $v \equiv 0$ but $\bar{u}(x) = u(\bar{x})$ so that $v(\bar{x}) = \bar{u}(\bar{x}) - u(\bar{x}) = u(x) - u(\bar{x}) = -v(x)$ so that v changes sign. This means $v \equiv 0$, i.e. $u = \bar{u}$. Since the axis x_1 can be chosen arbitrarily, u is symmetric w.r.t. to any hyperplane containing the origin, hence it is radially symmetric.

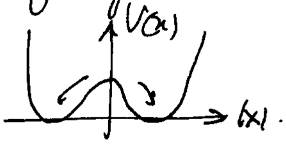
$\frac{d}{dr}(u - \lambda u_0) < 0$ can be established with similar arguments.

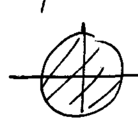
We already discover the main phenomenon which is causing the symmetry breaking at least for ground states: the competition of the nonlinearity, which tends to concentrate the solution into a soliton-like profile with the explicit x dependence caused for instance by an external potential, which tends to push the solution away from the origin. It is energetically favourable to have a solution which is concentrated, but away from the origin (at least compared to a solution which is still radially symmetric)

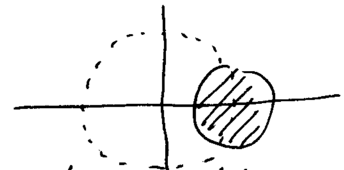


no potential

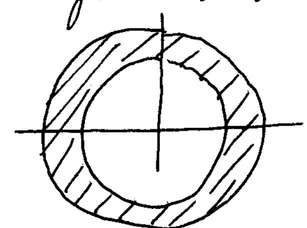
+ decentering potential







This decentered solution is energetically better than



Notice that we loose uniqueness: the radial solution is still present, but is no more the "ground state" (lowest energy solution) and there is a whole continuum of solutions which is generated by the symmetries which leave the equation invariant.

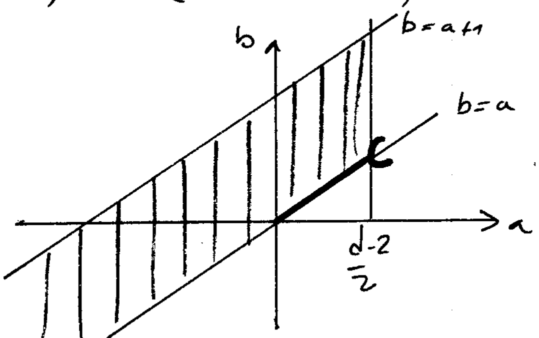
$\frac{1}{2}\int|\nabla u|^2+\int F(|x|,u)$
$\partial F/\partial u=-f(|x|,u)$

Instead of looking at variational problems and general functionals we shall take as an emblematic example functional inequalities which have homogeneity properties and involve nonlinearities and weights:

CAFFARELLI - KOHN - NIRENBERG INEQUALITIES (CKN)

$$\left(\int_{\mathbb{R}^d}\frac{|u|^p}{|x|^{bp}}\right)^{2/p}\le C_{a,b}\int\frac{|\nabla u|^2}{|x|^{2a}}\qquad p=\frac{2d}{d-2+2(b-a)}$$

$$d\geq 2\,,\quad a\le b\le a+1\,,\quad a<\frac{d-2}{2}$$



|||| equality is achieved in $\mathcal{D}_{a,b}(\mathbb{R}^d)=\{|x|^{-b}u\in L^p,\ |x|^{-a}\nabla u\in L^2(\mathbb{R}^d)\}$

Same problem when restricted to radial functions (u depends only on $|x|$ a.e.) we shall denote by $C^*_{a,b}$ the corresponding "best constant".

$C^*_{a,b}\le C_{a,b}$.

Some remarks on CKN inequalities:

* if $d=2$, the inequalities are restricted to the case $a<0$.
* when considering radial solutions, one is dealing with

$$|S^{d-1}|^{\frac{2}{p}-1}\left(\int r^{d-1-bp}\, u^p\, dr\right)^{2/p} \le C^*_{a,b}\int r^{d-1-2a}\, u'^2\, dr$$

hence the equation has a lot to do with Sobolev's embedding in $\mathbb{R}^{\tilde d}$ with $\tilde d = d-2a$, at least for $b=a$: $p=\frac{2\tilde d}{\tilde d-2}$ (--)

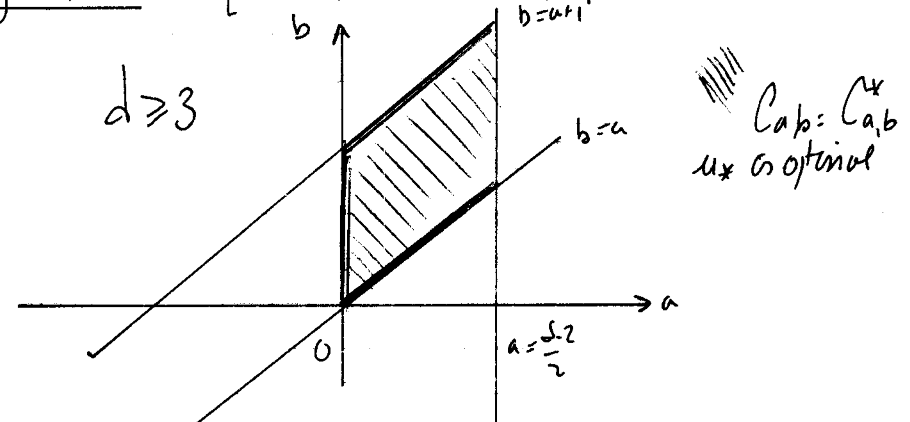
* when $b=a$, $p=\frac{2d}{\tilde d-2}$ and the inequality is of Sobolev type.
* when $b=a+1$, $p=2$, and it is of Hardy type
* one may think to the problem as an interpolation between Sobolev and Hardy in dimension $d-2a$ (but this is not correct for the exponent p, except when $a=0$).
* the exponent is easily recovered by a scaling (replace u by $u(\lambda x)$)
* among radial functions, equality (optimality) is achieved by

$$u_*(x) := \left(1+|x|^{-2a\frac{1+a-b}{b-a}}\right)^{-\frac{b-a}{1+a-b}}$$

Do we have $C_{a,b}=C^*_{a,b}$? is u_* an optimal function for CKN?

* the restriction $a<\frac{d-2}{2}$ is not really serious since we can extend the inequality to the case $a>\frac{d-2}{2}$ by considering a modified Kelvin transform [JD, M. Esteban, G. Tarantello]. However, the restriction $a\neq\frac{d-2}{2}$ is a severe one (it is like trying to get a Hardy inequality in dimension $d=2$). If $a>\frac{d-2}{2}$, the functional space to be considered is the completion of $\mathcal{D}(\mathbb{R}^d\setminus\{0\})$ with respect to the norm $\|u\|:=\||x|^{-b}u\|^2_{L^p(\mathbb{R}^d)}+\||x|^{-a}\nabla u\|^2_{L^2(\mathbb{R}^d)}$.

Symmetry results [Chou, Chu '93], [Horiuchi '97]



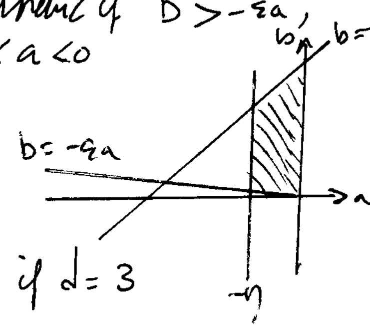
A simplified proof [JD, M. Esteban, M. Loss, G. Tarantello] $u=|x|^a v$

$$(CKN) \iff C_{a,b}^{-1}\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{(b-a)p}}\right)^{2/p} + a(d-2-a)\int_{\mathbb{R}^d}\frac{|v|^2}{|x|^2}\le\int_{\mathbb{R}^d}|\nabla v|^2$$

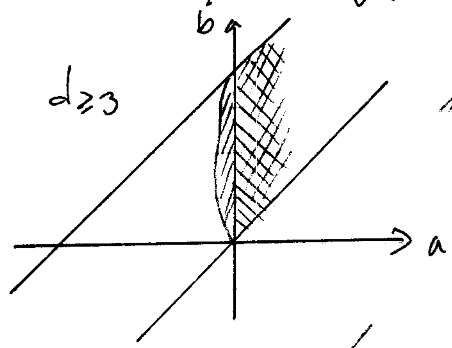
+ Schwarz symmetrization. NB. all optimal functions are radially symmetric

This completely fails if $d=2$!

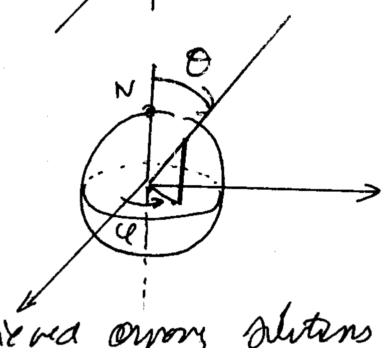
THEOREM 1: $d=2$: [JD, Esteban, Tarantello]
$\forall \varepsilon>0\ \exists \eta>0$ such that there is radial symmetry (all optimal functions are radially symmetric) if $b > -\varepsilon a$, $-\eta < a < 0$



Extensions: [Smets-Willem] [Lin-Wang]



Partial symmetry:



(r,θ,φ) if $d=3$

"Schwarz foliated symmetry" [Smets-Willem]

Optimal functions are achieved among solutions which depend only on r and θ (θ is the azimuthal angle). We may replace dx on $\mathbb{R}^d$ by

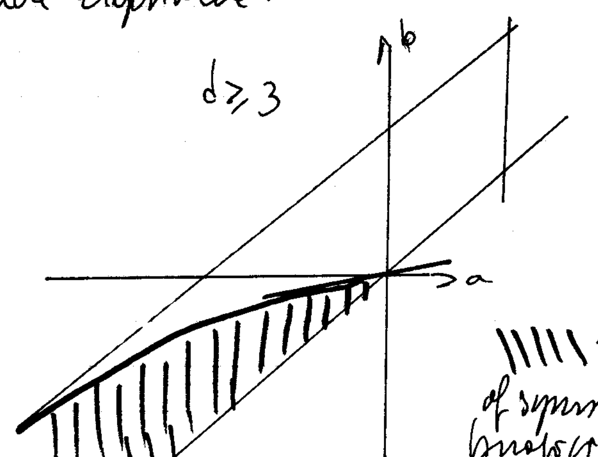
$$\sin\theta^{d-2}\,d\theta\ \ r^{d-1}\,dr\ ,\quad \theta\in(0,\pi],\ r\in[0,+\infty)$$

Symmetry breaking: [Catrina & Wang], [Felli-Schneider], [DET]

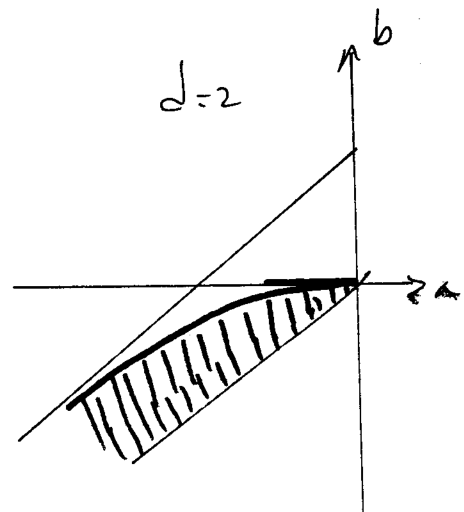
1st result: assume that u_* is optimal and consider the functional

$$F(u) = C^*_{a,b}\int_{\mathbb{R}^d}|\nabla u|^2 - \Big(\int_{\mathbb{R}^d} u^p\Big)^{2/p}$$

Consider the quadratic form associated to the second variation at $u=u_*$ and look for a negative eigenvalue.



zone of symmetry breaking



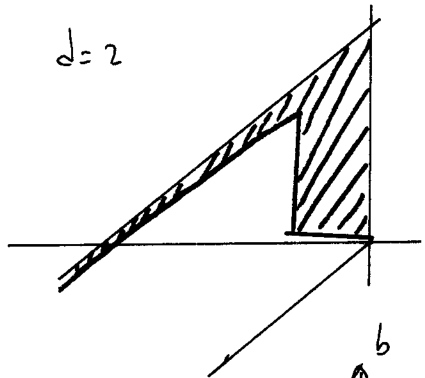
$$b \leq b^{FS}(a) := \frac{d\,(d-2-2a)}{2\sqrt{(d-2-2a)^2+4(d-1)}} - \frac{d-2-2a}{2}$$

2nd result: Catrina and Wang: $a \to -\infty$
Minimizers look like optimal functions for some (decentered) Gagliardo-Nirenberg interpolation inequalities, after an appropriate transform.

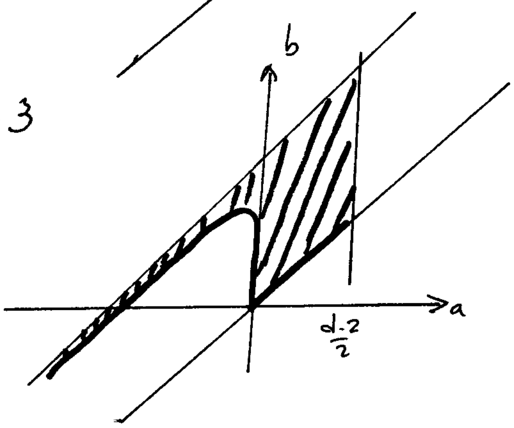
Results

Theorem 2: $\forall A<0\ \exists \varepsilon>0$ such that extremals are radially symmetric if $a+1-\varepsilon < b < a+1$ and $a \in (A,0)$

$d=2$



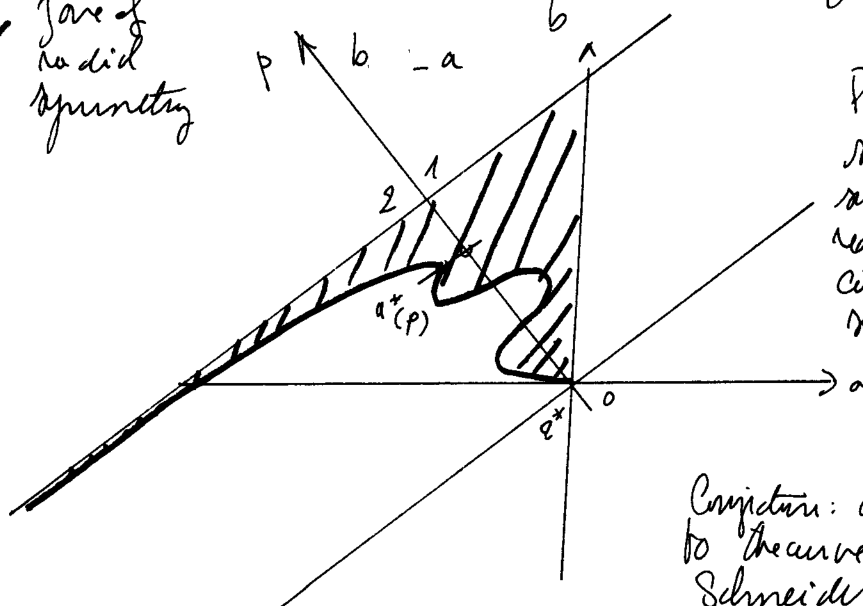
$d \geq 3$



$q^* = \infty$ if $d=2$
$q^* = \frac{2d}{d-2}$ if $d \geq 3$

Theorem 3: $\exists\, a^* \in C^0(2,q^*) \longrightarrow (-\infty,0)$
$\lim_{p\to 2} a_*(p) = -\infty$, $\lim_{p\to 2^*} a_*(p) = 0$.
such that: $a > a_*(p)$ all extremals are radially symmetric
$a < a_*(p)$ none of the extremals is radially symmetric

//// zone of radial symmetry



Philosophy: symmetry and symmetry breaking regions are simply connected and separated by a continuous curve

Conjecture: $a_*(p)$ corresponds to the curve of Felli and Schneider.

THE EMDEN-FOWLER TRANSFORMATION

To a point $x \in \mathbb{R}^d$, of written as $x = \left(|x|, \frac{x}{|x|}\right)$ in radial coordinates, we may associate the point (t,ω) where $t = \log r$, $r = |x|$ and $\omega = \frac{x}{|x|}$

$(t,\omega) \in \mathbb{R} \times S^{d-1} = \mathcal{C}$

$$u(x) = |x|^{-\frac{d-2-2a}{2}}\, w(t,\omega)$$

$$(CKN) \Longleftrightarrow \left(\int_{\mathcal{C}} w^p\, dy\right)^{2/p} \leq C_{a,b}\left[\int_{\mathcal{C}} |\nabla w|^2 dy + \Lambda \int_{\mathcal{C}} w^2 dy\right]$$

$$\Lambda = \frac{1}{4}(d-2-2a)^2$$

a Gagliardo - Nirenberg inequality on $\mathcal{C}$

Recall that $p = \dfrac{2d}{d-2+2(b-a)}$

Radially symmetric functions are transformed into t dependent solution (no dependence in ω) satisfying

$$|S^{d-1}|^{\frac{2}{p}-1}\left(\int_{\mathbb{R}} w^p\, dt\right)^{2/p} \leq C_{a,b}\left(\int_{\mathbb{R}} w'^2 dt + \Lambda \int w^2 dt\right)$$

Euler-Lagrange equations:

$$-w'' + \Lambda w = \text{Const}\,.\,|w|^{p-2} w$$

Scaling in t, multiplication by constants:

$$-w'' + w = w^{p-1} \qquad \text{(extremals are positive)}$$

Translation invariance (wrt t, i.e. scaling invariance in $r = e^t$)

$$w(0) = \max_{t \in \mathbb{R}} w(t)$$

$$-\frac{w'^2}{2} + \frac{w^2}{2} = \frac{w^p}{p} + \text{Const}.$$

$$\lim_{t \to \pm\infty}\left(\frac{w'^2}{2} + \frac{w^2}{2}\right) = 0 \Longrightarrow \text{Const} = 0$$

$$w(0) = (p/2)^{\frac{1}{2-p}}$$

$$w_*(t) = \frac{w(0)}{(\cosh \alpha t)^{\frac{2}{p-2}}} \qquad \text{for some } \alpha > 0$$

A family of optimal (radial) functions which extends the Aubin-Talenti solutions (for optimality in Sobolev inequalities).

Symmetry breaking : the method of Catrina - Wang & Felli - Schneider

$$\mathcal{F}(w) = \int_{\mathcal{C}} |\nabla w|^2 + \Lambda \int_{\mathcal{C}} w^2 - C_{a,b}^{-1} \left(\int w^p\right)^{2/p} \geq 0 .$$

$$C_{a,b} = C_{a,b}^* \Rightarrow \mathcal{F}(w_*) = 0 .$$

Constraint: $\int_{\mathcal{C}} w^p dy = \int_{\mathcal{C}} w_*^p dy$.

$$\mathcal{F}(w_* + \varepsilon\phi) - \mathcal{F}(w_*) = \varepsilon^2 \Big[\int |\nabla\phi|^2 + \Lambda \int \phi^2 - \underbrace{C_{a,b}^{-1} \left(\int w_*^p\right)^{\frac{2}{p}-1}}_{K} \int |w_*|^{p-2} \phi^2\Big] + o(\varepsilon^2)$$

$$w_*^{p-2} \sim (\cosh t)^{-2}$$

Study the eigenvalues of $-\Delta + \Lambda - \frac{K}{(\cosh(\alpha t))^2}$

Decompose on spherical harmonics and Legendre's functions.

Proof of theorem 2: approaching $b = a+1$ $(p \to 2)$

$$\int \frac{|w|^2}{|x|^{2(a+1)}} \leq C_{a,a+1} \int \frac{|\nabla u|^2}{|x|^2}$$

$$C_{a,a+1} = \Lambda^{-1}$$

$$\Lambda \int w^2 \leq \int |\nabla w|^2 + \Lambda \int w^2$$

Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of minimizers corresponding to
a (or Λ) fixed
$p_n \to 2$
b_n such that $p_n = \dfrac{2d}{d-2+2(b_n-a_n)}$: $b_n \to a+1$

and assume that $w_n \geq 0$ is normalized so that

$$\int |\nabla w_n|^2 + \Lambda \int w_n^2 = \int w_n^{p_n}$$

Hence $-\Delta w_n + \Lambda w_n = p_n w_n^{p_n-1}$ and $C_{a,b_n} = \|w_n\|_{p_n}^{p_n-2}$

$$C_{a,b_n}^{-1} \leq (C_{a,b_n}^*)^{-1} \longrightarrow \frac{1}{\Lambda}$$

$(w_n)_{n\in\mathbb{N}}$ is bounded in $H1$
Let $\tilde{w}_n = c_n w_n$ be such that $\int |\nabla \tilde{w}_n|^2 + \Lambda |\tilde{w}_n|^2 = 1$.

$$-\Delta \tilde{w}_n + \Lambda \tilde{w}_n = p_n c_n^{2-p_n} \tilde{w}_n^{p_n-1}$$

$\tilde{w}_n$ converges in $H1$ to some w, $p_n c_n^{2-p_n} \to \Lambda$.

$$\tilde{w}_n^{p_n-1} = \tilde{w}_n^{p_n-2} \cdot \tilde{w}_n \leq \tilde{w}_n(0)^{p_n-2} w_n$$

$$\to \lim_{n\to\infty} \tilde{w}_n(0)^{p_n-2} \leq 1 .$$

$w \in H^1$ and $-\Delta w + \Lambda w \leq \Lambda w$
impossible unless $w \equiv 0$.

$d = 2$ $\quad \chi_n = \partial \tilde{w}_n / \partial \omega$, $\omega \in S^1$.

$d \geq 3$ $\quad \chi_n = (\sin\theta)^{2-d} \frac{\partial}{\partial\theta}\left((\sin\theta)^{d-2}\, \tilde{w}_n\right)$

$$-\Delta \chi_n + \Lambda\, \chi_n = (p_n - 1)\, C_n^{2-p_n}\, \tilde{w}_n^{\,p_n-2}\, \chi_n$$

$$\int_{\mathcal{C}} |\nabla \chi_n|^2 dy + \Lambda \int_{\mathcal{C}} |\chi_n|^2 dy \leq \underbrace{(p_n - 1)\, C_n^{2-p_n} \int_{\mathcal{C}} |\tilde{w}_n(\theta)|^{p_n-2}}_{\to \Lambda} \chi_n^2 dy$$

as a function of θ, χ_n has zero average on S^{d-1}: use the Poincaré inequality to conclude that

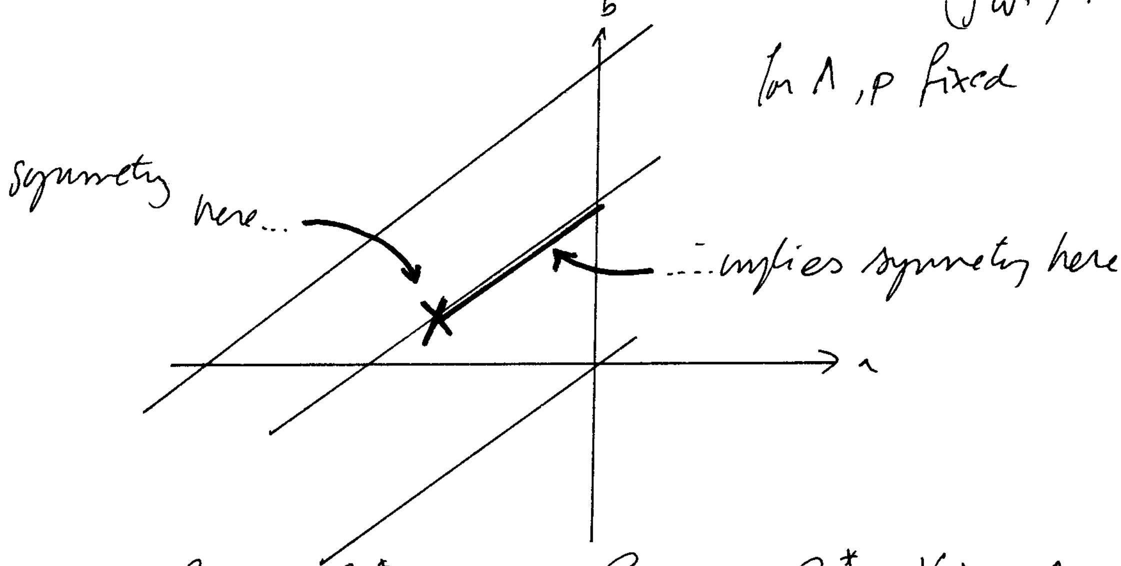
$$(N-1+\Lambda) \int_{\mathcal{C}} |\chi_n|^2 dy \leq (p_n-1)\, C_n^{2-p_n} \int |\chi_n|^2 \, |\tilde{w}_n(\theta)|^{p_n-2}$$
$$\to \Lambda \int |\chi_n|^2 .$$

For $n \in \mathbb{N}$ large enough: $\chi_n \equiv 0$, which means that w_n does not depend on θ!

<u>Proof of Theorem 3</u>: let $w \in H^1(\mathcal{C})$, $\sigma > 0$, $w_\sigma(t,\theta) = w(\sigma t, \theta)$

$$\mathcal{F}(w) = \mathcal{F}_{\Lambda,p}(w) = \left(\int_{\mathcal{C}} |\nabla w|^2 + \Lambda |w|^2\right) \left(\int_{\mathcal{C}} w^p\right)^{-2/p}$$

$$\mathcal{F}_{\sigma^2\Lambda,p}(w_\sigma) = \sigma^{-1+\frac{2}{p}}\, \mathcal{F}_{\Lambda,p}(w) - \sigma^{-1+\frac{2}{p}}(\sigma^2 - 1)\, \frac{\int |\nabla_\theta w|^2}{\left(\int w^p\right)^{2/p}}$$



for Λ, p fixed

<u>Lemma</u>: $\quad C_{\Lambda,p} = C^*_{\Lambda,p} \implies C_{\lambda,p} = C^*_{\lambda,p} \quad \forall \lambda < \Lambda$

$\quad C_{\Lambda,p} > C^*_{\Lambda,p} \implies C_{\lambda,p} > C^*_{\lambda,p} \quad \forall \lambda > \Lambda$.

+ upper semicontinuity of $a^*(p)$: easy + continuity: delicate.

ONOFRI TYPE INEQUALITIES

THEOREM 4: $d\mu_\alpha = \frac{\alpha+1}{\pi} \frac{|x|^{2\alpha}}{(1+|x|^{2(\alpha+1)})^2} dx$, $x\in\mathbb{R}^2$

$\alpha \in (-1,0]$

$$\log\left(\int_{\mathbb{R}^2} e^{v-\bar v} d\mu_\alpha\right) \le \frac{1}{16\pi(1+\alpha)} \int_{\mathbb{R}^2} |\nabla v|^2 dx$$

Proof $\varepsilon \in (0,1)$, $a<0$ $\quad a = -\frac{\varepsilon}{1-\varepsilon}(\alpha+1)$, $b = a+\varepsilon$, $p = \frac{2}{\varepsilon}$

$u_\varepsilon = (1+|x|^{2(\alpha+1)})^{-\frac{\varepsilon}{1-\varepsilon}}$ radial extremal for (CKN)

$w_\varepsilon = (1+\varepsilon v)u_\varepsilon$

$$\int (1+\varepsilon v)^{\frac{2}{\varepsilon}} \frac{f_\varepsilon dx}{\int f_\varepsilon dx} \le C_{a,b}\, \mathrm{Const}(\varepsilon) \cdot \left[\lambda_\varepsilon + \frac{8(1+\alpha)^2}{(1-\varepsilon)^2}\int \frac{u_\varepsilon^{2/\varepsilon} v}{|x|^{2(\varepsilon-\alpha)}} + \int |\nabla v|^2 \frac{u_\varepsilon^2}{|x|^{2a}} + O(a^2\varepsilon)\right]$$

$\square$

Getting symmetry in Theorem 1 ?
argue by contradiction with a sequence of non-radial minimizers. At first order converges to a solution of the Liouville equation. Estimates coming from a Pohozaev type identity on S^1. The limit is radially symmetric. Differentiate with respect to the angle. Linearizing regime: get a contradiction!

Remark: Theorem 4 on $\mathcal{C}$. $d\nu_\alpha = \frac{\alpha+1}{2} \frac{dt}{[\cosh((\alpha+1)t)]^2}$

$$\log\left(\int_{\mathcal{C}} e^{w-\bar w} d\nu_\alpha\right) \le \frac{1}{16\pi(\alpha+1)} \int_{\mathcal{C}} |\nabla w|^2 dy$$

LOGARITHMIC HARDY INEQUALITIES

$$\int w^2 \log \frac{w^2}{\int w^2 dy} dy + K(\gamma,\Lambda)\int w^2 \le 2\gamma \int w^2 \log\left(\frac{\int|\nabla w|^2}{\int w^2} + \Lambda\right)$$
$$\forall w \in H^1(\mathcal{C}).$$

has to do with weighted CKN inequalities

Some observations: results obtained in collaboration with
Esteban (= E)
Loss (= L)
Tarantello (= T)
(myself) (= D)

Key points: Notion of symmetry breaking in physics and in maths
Moving planes (scalar field theory).
Caffarelli-Kohn-Nirenberg inequalities. Thm 1
Results Thm 2 & Thm 3
Emden-Fowler
Proofs
Onofri-log Hardy. Thm 4

Concluding remarks

How symmetry breaks is quite well understood. It is conjectured that known conditions could be optimal. On the opposite, only trivial cases of symmetry are known, and some perturbation regimes. There is a lack of understanding of the functional at a global level (energy landscape) and of gradient flows (evolution equations) might be a direction for future research.