OPTIMAL FUNCTIONS AND SYMMETRY BREAKING
IN FUNCTIONAL INEQUALITIES

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Introduction: Notion of symmetry breaking

* Physics: solutions with symmetry have long been studied, even
more active solutions for which symmetry is broken, and this has led
to the idea that "high" symmetric cases are more likely to
occur. The common belief has been summarized by the "asymmetry
principle of Rineau" which asserts that "effects are at least as
symmetric as their causes". This was thought in view of relations in
electromagnetism a century ago, but since then several examples
have shown that reality is not as simple: ferromagnetism for instance
can be more complex only by requiring that all local interactions
are likely to occur thus symmetry is broken after magnetization. Symmetry
breaking is even more usual in quantum field (today just that
symmetry breaking automatically leads to 100% for most physicists)
but considerations in this topic are far beyond the scope of this talk.

* Mathematics: proceeding for uniqueness and understanding of
interactions mechanisms has evolved quite recently. When one refers
to symmetry in (elliptic) PDEs, usually the procedure is to reduce
the problem to a symmetric setting (for instance, to locally symmetric
solutions) in order to have a simpler problem to solve (for instance under
a desired scheme which has preserved since the beginning of the 70s in
dealing with the symmetry of the solutions in nonlinear scalar field
theory as the following:

1. instead of working with actual points (minimizers) directly
work with the solutions of the Euler-Lagrange equations and only
keep some properties of the solutions. The notion of "ground state" for
existence has evolved from the notion of a minimizer of a functional
(which is the correct notion in physics) to the notion of a positive
solution (for instance on an Euclidean space) which has just
zero at infinity
2. use standard symmetry
3. establish uniqueness of the solutions of the corresponding PDE.
which has been done in the detection of solutions symmetric breaking
and in studying the properties of the corresponding solutions (this is more
difficult, of course).

Hence, I will not understand symmetry versus symmetry breaking
from linearization. Physicists are not in my understanding.
From Symmetry to Symmetry Breaking: Higgs-Phases Approach

In two celebrated papers 50 years ago, Gidea, N. & Niemczyk have given the first precise result of symmetry breaking. If

$$\Delta u + \lambda u = 0 \quad \text{in } B, \quad u = \lambda = 0 \quad \text{on } \partial B,$$

then for any positive solution $$\lambda > 0$$, the fixed point $$u = \lambda$$ is stable, decreasing along any line $$i, i < 0$$.

Gidea, N. & Niemczyk, 1987, A Remark (and a result in $$\mathbb{R}^d$$, given below:)

requirements on the behavior of the solution at infinity.)

A first extension: $$\Delta u + \lambda f(u) = 0, \quad u = \lambda = 0 \quad \text{in } B$$

If $$\lambda > 0$$, then the same result holds... but this extension is somewhat trivial (the monotonically increasing $$\lambda$$-dependent case is simply obtained by reducing to the $$\lambda$$-independent case).

Thm. 1: (D-P. Felmer, 1997) Consider solutions of $$\Delta u + \lambda f(u) = 0$$ and suppose that $$f \in C(\mathbb{R}, \mathbb{R})$$. We make no assumption on the sign of $$\lambda$$

There exists $$\lambda_1, \lambda_2$$ with $$0 < \lambda_1 < \lambda_2$$ such that:

(i) If $$\lambda \in (0, \lambda_1)$$ then $$\Delta u + \lambda f(u) = 0$$ on $$B$$ implies $$u \equiv 0$$.

(ii) If $$\lambda \in (\lambda_2, \infty)$$, then $$u$$ is locally symmetric.

This result explains the role of a counter-example given by N. and Gidea, which consists of a non-constant solution, whose positivity is given by the fixed point $$u = \lambda$$.

This explains the role of the local problem in the case of a constant $$\lambda$$: positivity implies symmetry, while positivity implies local uniqueness.

This can be illustrated by the following example:

Consider the case where $$\lambda = 1$$, it is locally below $$\lambda_1$$.

The proof:

(i) Assume $$\lambda < 1$$:

$$\Delta u + \lambda f(u) = 0$$

so we have a solution $$u = \lambda$$, because of $$\lambda = 1$$, $$u = \lambda$$.

We define $$\Delta u + \lambda f(u) = 0$$

and choose $$\lambda_1$$ and $$\lambda_2$$ as follows:

$$\lambda_1 = \sup \{ \lambda > 0 : \Delta u + \lambda f(u) = 0 \}$$

and

$$\lambda_2 = \inf \{ \lambda > 0 : \Delta u + \lambda f(u) = 0 \}$$

Then $$\lambda_1$$ and $$\lambda_2$$ depend on $$c$$ and are finite on $$\lambda_1$$, but are infinite otherwise with respect to $$u$$ (minimum and maximum can be given on $$\lambda$$).
We already divers the phenomenon widely occurring in separable breaking of large-field ground states: the completion of the broken symmetry, which tends to concentrate the solution into a solitary-like profile with the explicit $x$ dependence caused by its perturbation by an external potential, which tries to push the solution away from the origin. It is energetically favorable to have a solution which is concentrated, but away from the origin as that compared to a solution which is still radially symmetric.

\[ + \text{decaying profile} \]

\[ \text{This decaying solution is energetically better than} \]

Notice that we lose uniqueness: an isolated solution is still present, but it is not the "ground state" (least energy solution) and there is a whole continuum of solutions which is generated - the symmetry of the equation is not invariant.

Instead of looking at traditional polygons and their sectors, we shall use as our main tool, the least exponential functional inequalities with homogeneous polygons and involve rearrangements.

**APPARELLI-KOHN-NIRENBERG INEQUALITIES (CKW)**

\[
(\int |y|^p)^{\frac{p}{d}} \leq C_{0,p} \int |y|^{2p} \quad p = \frac{2d}{2d - 2(6 - a)}
\]

\[ c \geq 2, \quad a \leq b \leq a + 1, \quad a < \frac{d - 2}{2} \]

Some problem when restricted to radial functions (in d dimensions).

Let $\mathcal{L}$ denote by $C_{0,\mathcal{L}}$ an corresponding "best constant."

\[ C_{0,\mathcal{L}} \leq C_{0,\mathcal{L}^R} \]
Some remarks on CKN inequalities:

1. If $d = 2$, the inequalities are reduced to the case $a \leq 0$.

2. When considering radial solutions, one is dealing with

$\int_{\mathbb{R}^d} \left( \sum_{i=1}^n (\nabla u_i)^2 \right)^{\frac{p}{2}} \, dx \leq C_{a,b} \int_{\mathbb{R}^d} \left( \sum_{i=1}^n (\lambda^{1/2} a_i)^2 \right)^{\frac{p}{2}} \, dx$

where the constant has to do with Sobolev's embedding in $\mathbb{R}^2$ with $d = 2$.

3. At least for $b = a$, $p = \frac{2d}{d-2}$, and the inequality is of Sobolev type.

4. When $b = a$, $p = \frac{2d}{d-2}$, and it is of Hardy type.

For my thoughts on the problem of uncentered Sobolev and Hardy inequalities, I recall that this is not known for the exponential, except when $a = 0$.

The exponent is easily recovered by a scaling (using $u$ by $u^{ab}$) among radial functions, equality (optimally) is achieved by

$u(x) \equiv \frac{\Gamma(\frac{d-2}{2})}{\Gamma(\frac{d}{2})} \frac{1}{|x|^{d-2}}$

Do we have $C_{a,b} = C_{a,b}^* \Rightarrow u(x)$ is an optimal function for CKN?

The restriction $a < d - 2$ is not really needed since we can extend infinitely to the case $a > d - 2$ by considering a modified Kelvin transformation $[\text{below}]$.

However, the restriction $a \neq d - 2$ is a severe one (it's like trying to get a strange equality in dimension $d > 2$). If $a > d - 2$, the function $u(x)$ has to be avoided in the computation of $\int_{\mathbb{R}^d} \left( \sum_{i=1}^n (\lambda^{1/2} a_i)^2 \right)^{\frac{p}{2}} \, dx$.

Symmetry results: [Chen, 1994], [Harindranath, 1997]

\[ \begin{align*}
& d > 3 \\
& b = a \Rightarrow C_{a,b}^* \leq C_{a,b} \Rightarrow u(x) \text{ is optimal}
\end{align*} \]

A modified proof: $[\text{below}]$.

(CKN) $\Rightarrow$ $C_{a,b}^{-1} \left( \int |u(x)|^p \, dx \right)^{\frac{1}{p}} \leq a \int (\nabla u)^2 \, dx + b \int (\nabla u)^2 \, dx$

This completely fails if $d = 2$!
THEOREM 1: 
\[ d = 2 : (T, F, \text{other}) \]

\[ \forall x > 0 \exists \gamma \text{ s.t. } \text{trivial rotation symmetry} \]

\[ \text{All other  functions are exactly symmetric if } b > -\frac{\pi}{2} , c > 0 \]

\[ b = -a \]

\[ \gamma = \frac{\pi}{2} \]

\[ (\gamma, \theta, \phi) \text{ if } d = 3 \]

*Schwarz Relativistic Symmetry* [Smets-Willem]

Partial Symmetry:

Critical functions are achieved on cycles with difference \( d \) and \( \theta \).

\[ u \in \mathbb{R}^d , \theta \in [0, \pi] \]

Symmetry breaking: [Caldeno & Weng], [Felli, Schneider], [DET]

1st result: Assume that \( \phi \) is optimal and consider the functional

\[ (Ru) = \frac{\partial}{\partial x^2} \left( \frac{u^2}{2} \right) \]

Consider the function associated to the second variation and look for a negative eigenvalue.

\[ d \geq 3 \]

\[ d = 2 \]

\[ b \leq b_{\text{fs}} \]

\[ b_{\text{fs}} = \frac{d (d - 2 a)}{2 \sqrt{(d - 2 a)^2 + 4 a \cdot 1}} - \frac{d - 2 a}{2} \]

2nd result: [Cadeno & Weng]: \( a \to -\infty \)

Minimizes both the strain function and the doublon.

Criteria: Minimizing integration inequalities with congruence transform.
RESULTS

THEOREM 2: If $A < 0 \exists \varepsilon > 0$ such that extremals are radially symmetric if $a + \varepsilon < b < a + \varepsilon$ and $a \in (A, 0)$

![Diagram]

$\frac{\kappa}{d^2} = \frac{\varepsilon}{d^2}$  if $d \geq 3$

THEOREM 3: $\exists a^* \in C^0(\Omega, \Omega) \rightarrow (\infty, 0)$

$\lim_{p \rightarrow 0^+} a_x(p) = -\infty, \lim_{p \rightarrow 2^+} a_x(p) = 0$

such that: $a > a_x(p)$ all extremals are radially symmetric

$a < a_x(p)$ no of the extremals is radially symmetric

![Diagram]

Philosophy:
Symmetry and symmetry breaking regimes are smoothly connected and separated by a continuous curve

Conjecture: $a_x(p)$ corresponds to the curve of Helffer and Schneider.
**THE ENDEN-FOHLER TRANSFORMATION**

To a point \( x \in \mathbb{R}^d \) written as \( x = (x_1, \frac{x}{x_1}) \) in radial coordinates, we may associate the point \((Ew, t)\) where

\[
(t, w) \in \mathbb{R} \times S^{d-1} = \mathcal{R}^2
\]

\[
u(x) = \frac{1}{x_1} 1 - \frac{d-2}{2} w \]

\[
u(Ew) = \left( \int_0^1 W^p dt \right)^{\frac{2}{p}} \leq C_{a,b} \left[ \left( \int_0^1 \nu_0^2 dt \right)^{\frac{1}{2}} + \frac{w_0}{2} \nu_0 \right] \]

A. & B. Theorem... existence of \( \nu \)

Recall that \( p = \frac{2d}{d-2+2(\rho-\alpha)} \)

Radially symmetric functions are transformed into \( t \) dependent solutions (no dependence of \( w \)) satisfying

\[
\left( \int_0^1 (W^p dt) \right)^{\frac{2}{p}} \leq C_{a,b} \left( \int_0^1 (W_0^2 + \nu_0^2) dt \right)
\]

**Some remarks:**

- \( W'' + \lambda W = \text{Const.} \left( W \right)^{p-1} W 

- \( W'' + \lambda W = W^{p-1} \)

**Translation invariance (with respect to \( x = e^t \))**

\( W(0) = \max_{t \in \mathbb{R}} W(t) \)

\[-\frac{\lambda W}{2} + \frac{W^2}{2} = \frac{W^p}{p} + \text{Cons.} \]

\( \lim_{t \to \pm \infty} \left( \frac{W^2}{2} + \frac{W^2}{2} \right) = 0 \Rightarrow \text{Const} = 0 \)

\( W(0) = \left( \frac{p}{2} \right)^{\frac{2}{p-2}} \)

**Writhe:**

\( W(t) = \frac{W(0)}{ \left( \sinh (pt) \right)^{\frac{3}{2}}} \) for \( \sinh (pt) > 0 \)

A family of \( L^p \) (radial) functions which extends the

**Main Results:**

- **Solutions** (the optimality in scalar inequalities).
Symmetry breaking: the method of Cattorino, Wang & Felli - Schröder

\[ \mathcal{F}(\omega) = \sum_b \left[ \omega_b \mathcal{W}_b^2 - \mathcal{C}_b \left( \mathcal{W}_b \right)^2 \right] \geq 0. \]

\( \mathcal{C}_b = \mathcal{C}_b^{\infty} \Rightarrow \mathcal{F}(\omega_{\infty}) = 0. \)

And now\( \mathcal{W}_b = \mathcal{W}_b^{\omega_{\infty}} \).

\[ \mathcal{F}(\omega_{\infty} + \varepsilon) - \mathcal{F}(\omega_{\infty}) = \varepsilon^2 \left[ \mathcal{C}_b \mathcal{W}_b^{2 \omega_{\infty}} - \mathcal{C}_b \left( \mathcal{W}_b^{\omega_{\infty}} \right)^2 \right] \mathcal{W}_b^{2 \omega_{\infty}} \mathcal{W}_b^{\omega_{\infty}}. \]

Study the eigenvalues of

\[ -\Delta + \Lambda - \mathbf{K} \]

Decompose on spherical harmonics and Legendre's functions.

**Proof of Theorem 2:** Approaching \( \beta = \alpha + 1 \) (\( p \to 2 \))

\[ \sum_{\lambda \geq \alpha + 1} \mathcal{C}_\lambda \mathcal{W}_\lambda^2 \leq \mathcal{C}_\alpha \mathcal{W}_\alpha^2 \]

\( \mathcal{C}_\alpha \mathcal{W}_\alpha^2 = \Lambda^{-1} \)

\[ \Lambda \mathcal{W}_\lambda^2 \leq \mathcal{S} \left( \mathcal{W}_\lambda^2 + \Lambda \mathcal{W}_\lambda^2 \right) \]

Let \( (\mathcal{W}_\lambda)_{\lambda \neq \alpha} \) be a sequence of minimizers corresponding to \( \beta \) (or \( \Lambda \)) fixed

\[ \mathcal{P}_\lambda \to 2 \]

\[ \mathcal{P}_\lambda \text{ such that } \mathcal{P}_\lambda = \frac{2\Lambda}{\alpha + 2 + 2(b_n - 2\alpha)} \]

and assume that \( \mathcal{W}_\lambda^{\alpha} \) is normalized so that

\[ \sum \mathcal{W}_\lambda^2 + \Lambda \mathcal{W}_\lambda^2 = \mathcal{S} \mathcal{W}_\lambda^2 \]

Hence \( -\Delta \mathcal{W}_\lambda + \Lambda \mathcal{W}_\lambda = \mathcal{P}_\lambda \mathcal{W}_\lambda^{\lambda-1} \) and \( \mathcal{C}_\alpha \mathcal{W}_\alpha = \mathcal{S} \mathcal{W}_\alpha^{\alpha-1} \)

\( \mathcal{W}_\alpha^{\alpha} \) is bounded in \( H^1 \)

Let \( \mathcal{W}_\alpha = \mathcal{W}_\alpha \mathcal{W}_\alpha^{\alpha} \) be such that

\[ \mathcal{W}_\alpha^{\alpha} \mathcal{W}_\alpha^{\alpha-1} = 1. \]

\( \mathcal{W}_\alpha \to \mathcal{W}_\alpha \) in \( H^1 \) to some \( \mathcal{W}_\alpha \).

\[ \mathcal{C}_\alpha \mathcal{W}_\alpha = \mathcal{P}_\alpha \mathcal{W}_\alpha^{\alpha-1} \]

\( \mathcal{W}_\alpha \to \mathcal{W}_\alpha \) in \( L^p \) for \( p < \infty \)

\[ \mathcal{W}_\alpha \to \mathcal{W}_\alpha \text{ in } L^p \]

\( \mathcal{W}_\alpha \to \mathcal{W}_\alpha \text{ in } L^p \text{ for } p < \infty \)}
\[ W \in H^1 \text{ and } -\Delta W + \lambda W \leq \lambda W \]

implying unless \( W \equiv 0 \).

\[ d = 2 \quad X_n = \Theta \frac{\Theta}{\Theta} \chi_2 \quad \omega \in S^2.\]

\[ d = 3 \quad X_n = (\sin \Theta)^{-d/2} \quad \Theta (d \Theta/\Theta)^{d/2} \quad \tilde{W}_n \]

\[ -\Delta X_n + \lambda X_n = (\rho_{n-1}) C_{n-2}^{2p/n} \quad \tilde{W}_n \]

\[ \int (dx)^n (\tilde{W}_n)^2 \leq (\rho_{n-1}) C_{n-2}^{2p/n} \quad \tilde{W}_n (\omega)^{2p/n} \quad X_n \]

as a function of \( \Theta \), \( X_n \) has zero average on \( S^d \).

To conclude that

\[ \Gamma_{n-1} \quad \int S_n (\tilde{W}_n)^2 \leq (\rho_{n-1}) C_{n-2}^{2p/n} \quad (\tilde{W}_n (\omega)^{2p/n} \quad X_n \]

for \( n \in N \) large enough : \( X_n \equiv 0 \), which contradicts \( W \) does not depend on \( \Theta \).

**Proof of Theorem 3:** \( t \in H^1(\mathbb{C})\), \( \sigma > 0 \), \( \omega \in (\sigma, 0) \)

\[ F(w) = F_{\lambda, p}(w) = \left( \frac{\sigma}{\sigma} \right)^{1/2} \quad \left( \frac{\sigma}{\sigma} \right)^{1/2} \]

\[ F_{\lambda, p}(w) = \sigma^{-1/2} \quad F_{\lambda, p}(w) \quad \sigma^{-1/2} \quad \left( \frac{\sigma}{\sigma} \right)^{1/2} \quad \left( \frac{\sigma}{\sigma} \right)^{1/2} \]

In \( \lambda, p \) fixed

\[ \text{symmetry here...} \]

\[ \text{...implies symmetry here} \]

\[ \text{Lemma:} \quad C_{\lambda, p} = C_{\lambda, p} \quad \forall \lambda \leq \Lambda \]

\[ C_{\lambda, p} > C_{\lambda, p} \quad \Rightarrow \quad C_{\lambda, p} > C_{\lambda, p} \quad \forall \lambda > \Lambda \]

+ upper \& lower bounds + continuity + detailed.
Theorem 4: \[
\frac{d}{dx} \varphi = \frac{\varphi}{x^2} \quad \frac{1}{\pi} \frac{1}{\left(1 + x^2 + \epsilon^2 \right)^{\frac{1}{2}}} \, dx, \quad x > 0
\]

\( \epsilon \in (0,1) \)

\[
\log \left( \int \frac{e^{-x}}{x} \, dx \right) \leq \frac{1}{16\pi} \int_0^\infty \frac{e^{-x}}{x} \, dx
\]

Proof.
\( \epsilon \in (0,1) \), \( a < 0 \)
\( a = \frac{-\epsilon}{1-\epsilon} \), \( b = a + \frac{\epsilon}{1-\epsilon} \)
\( u \leq (1 + \epsilon)^{\frac{1}{2}(\epsilon+1)} \), \( \omega_0 = (1 + \epsilon)^{\frac{1}{2}(\epsilon+1)} \)
\( \omega_0 = (1 + \epsilon)^{\frac{1}{2}(\epsilon+1)} \omega_0 \)

\[
\frac{1 + \epsilon^2}{2} \int \omega_0 \, dx \leq C_{\epsilon, b} \omega_0 \cdot \int \left[ \epsilon + 8(1+\omega_0^2) \frac{\omega_0^2}{\omega_0} \right] \frac{d\omega}{\omega_0} + O(\omega_0^3)
\]

Getting symmetry in Theorem 1?

Arise by contradiction: \( u \) a sequence of positive smooth functions. \( \partial \) first order continuous to a solution of the Liouville equation. 

Extraneous energy from a Bôcher type identity on \( S^1 \). The limit is actually \( \omega_0 \) non-regular.

Differentiate with respect to the angle. \( \varphi \) a contradiction!

Remark: Theorem 4 on \( \omega \).
\( d\varphi = \frac{\varphi}{x^2} \frac{dx}{x} \)
\( \log \left( \int e^{-x} \, dx \right) \leq \frac{1}{16\pi} \int_0^\infty \frac{e^{-x}}{x} \, dx \)

Logarithmic Hardy inequalities
\[
\int \omega^2 \, dx = \int \omega^2 \, dx + k(\omega) \int \omega^2 \, dx \leq 2\gamma \omega^2 \log \left( \frac{\gamma \omega^2 + A}{\omega^2} \right)
\]

\( \forall \omega \in H^1(\Omega) \).

How to do with generalized CKN inequalities.
Some observations: results obtained in collaboration with
Esteban (E)
less (L)
Terentello (T)
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Ideas points:
- Nature of symmetry breaking in physics and mathematics
- Noether's theorems (vector field theory)
- Supersymmetric - Kohn - Nirenberg inequalities. Thm 1
- Results Thm 2 & Thm 3
- Emden - Fowler
- Supercritical - Log Hardy. Thm 4

Concluding remarks
- How symmetry breaks is quite well understood.
- It is considered that known obstructions could be refined.
- And, in general, only limited cases of symmetry are known, and
  some continuation problems. There is a lack of understanding
  of the function at a global level (energy landscape)
  and the finite flows (exhaustive equations) might be a subject
  for further research.