Diffusions non-linéaires: entropies relatives, "best matching" et délais

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A. Fast diffusion equations: entropy, linearization, inequalities, improvements

1. entropy methods
2. linearization of the entropy
3. improved Gagliardo-Nirenberg inequalities

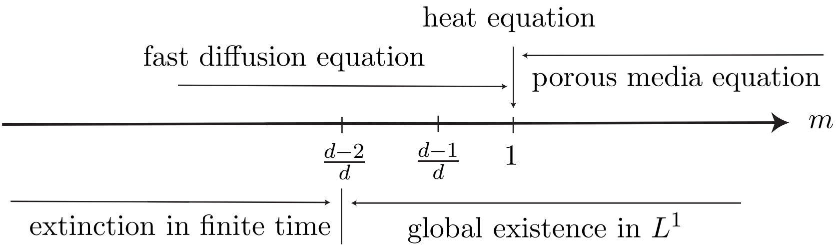
Fast diffusion equations: entropy methods

Existence, classical results

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$ as $t \to +\infty$

[Friedmann, Kamin, 1980] $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$



Existence theory, critical values of the parameter m

Time-dependent rescaling, Free energy

- *Time-dependent rescaling*: Take $u(\tau, y) = R^{-d}(\tau)\, v\,(t, y/R(\tau))$ where

$$\frac{dR}{d\tau} = R^{d(1-m)-1}\ , \quad R(0) = 1\ , \quad t = \log R$$

- The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x\, v)\ , \quad v_{|\tau=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{F}[v] := \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}\,|x|^2 v \right) dx - \mathcal{F}_0$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{F}[v] = -\mathcal{I}[v]\ , \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$

Relative entropy and entropy production

- *Stationary solution:* choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) := \left(C + \tfrac{1-m}{2\,m}\,|x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{F}_0$ so that $\mathcal{F}[v_\infty] = 0$

- *Entropy – entropy production inequality*

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[v] \geq 2\,\mathcal{F}[v]$$

Corollary

A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[u_0]\,e^{-\,2\,t}$

An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\mathcal{F}[v] = \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \mathcal{F}_0 \le \frac{1}{2} \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} \mathcal{I}[v]$$

Rewrite it with $p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \ge 0$$

- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} v\, dx = \int_{\mathbb{R}^d} w^{2p}\, dx \right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal

Theorem

[Del Pino, J.D.] *With* $1 < p \le \frac{d}{d-2}$ *(fast diffusion case) and* $d \ge 3$

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le \mathcal{C}^{\rm GN}_{p,d} \, \|\nabla w\|^\theta_{L^2(\mathbb{R}^d)} \, \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

$$\mathcal{C}^{\rm GN}_{p,d} = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}, \; \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \; y = \frac{p+1}{p-1}$$

... a proof by the Bakry-Emery method

Consider the generalized Fisher information

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v\,|z|^2\,dx \quad \text{with} \quad z := \frac{\nabla v^m}{v} + x$$

and compute

$$\frac{d}{dt}\mathcal{I}[v(t,\cdot)] + 2\,\mathcal{I}[v(t,\cdot)] = -2\,(m-1)\int_{\mathbb{R}^d} u^m\,(\mathrm{div} z)^2\,dx - 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} u^m\,(\partial_i z^j)^2\,dx$$

- the Fisher information decays exponentially:

$$\mathcal{I}[v(t,\cdot)] \le \mathcal{I}[u_0]\,e^{-2\,t}$$

- $\lim_{t\to\infty}\mathcal{I}[v(t,\cdot)] = 0$ and $\lim_{t\to\infty}\mathcal{F}[v(t,\cdot)] = 0$
- $\frac{d}{dt}\Big(\mathcal{I}[v(t,\cdot)] - 2\,\mathcal{F}[v(t,\cdot)]\Big) \le 0$ means $\mathcal{I}[v] \ge 2\,\mathcal{F}[v]$

[Carrillo, Toscani], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]

The Bakry-Emery method: details (1/2)

With $z(x,t) := \eta\,\nabla u^{m-1} - 2\,x$, the equation can be rewritten as

$$\frac{\partial u}{\partial t} + \nabla\cdot(u\,z) = 0$$

(up to a time rescaling, which introduces a factor 2) and we have

$$\frac{\partial z}{\partial t} = \eta\,(1-m)\nabla\big(u^{m-2}\,\nabla\cdot(u\,z)\big) \quad \text{and} \quad \nabla\otimes z = \eta\,\nabla\otimes\nabla u^{m-1} - 2\,\mathrm{Id}$$

$$\frac{d}{dt}\int_{\mathbb{R}^d} u\,|z|^2\,dx = \underbrace{\int_{\mathbb{R}^d}\frac{\partial u}{\partial t}\,|z|^2\,dx}_{\text{(I)}} + \underbrace{2\int_{\mathbb{R}^d} u\,z\cdot\frac{\partial z}{\partial t}\,dx}_{\text{(II)}}$$

$$\begin{aligned}
\text{(I)} &= \int_{\mathbb{R}^d}\frac{\partial u}{\partial t}\,|z|^2\,dx = \int_{\mathbb{R}^d}\nabla\cdot(u\,z)\;|z|^2 dx\\
&= 2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-2}\,(\nabla u\cdot z)^2 dx + 2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-1}\,(\nabla u\cdot z)\,(\nabla\cdot z)\;dx\\
&\quad + 2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-1}\,(z\otimes\nabla u) : (\nabla\otimes z)\;dx - 4\int_{\mathbb{R}^d} u\,|z|^2 dx
\end{aligned}$$

The Bakry-Emery method: details (2/2)

$$
\begin{aligned}
\text{(II)} &= 2\int_{\mathbb{R}^d} u\,z\cdot\frac{\partial z}{\partial t}\,dx\\
&= -2\,\eta\,(1-m)\int_{\mathbb{R}^d}\left[u^m(\nabla\cdot z)^2+2\,u^{m-1}(\nabla u\cdot z)\,(\nabla\cdot z)+u^{m-2}(\nabla u\cdot z)^2\right]dx
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\frac{\partial u}{\partial t}\,|z|^2dx+4\int_{\mathbb{R}^d} u\,|z|^2dx\\
&\qquad= -2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-2}\left[u^2(\nabla\cdot z)^2+u\,(\nabla u\cdot z)\,(\nabla\cdot z)\right.\\
&\qquad= -2\,\eta\,\frac{1-m}{m}\int_{\mathbb{R}^d} u^m\left(|\nabla z|^2-(1-m)\,(\nabla\cdot z)^2\right)\,dx
\end{aligned}
$$

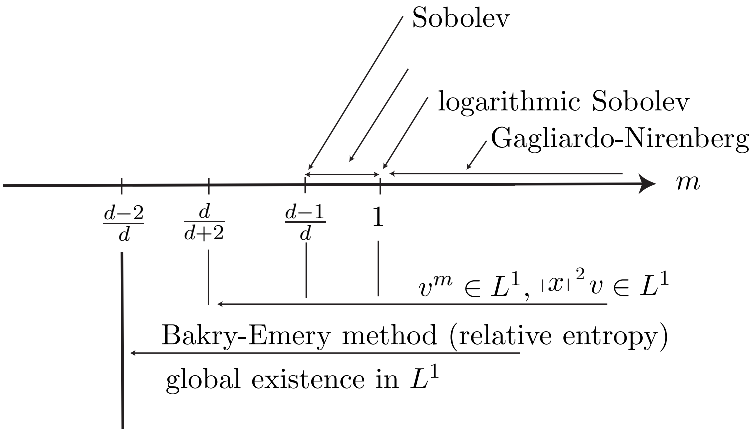
By the arithmetic geometric inequality, we know that

$$
|\nabla z|^2-(1-m)\,(\nabla\cdot z)^2\geq 0
$$

if $1-m\leq 1/d$, that is, if $m\geq m_1=1-1/d$

Fast diffusion: finite mass regime

Inequalities...



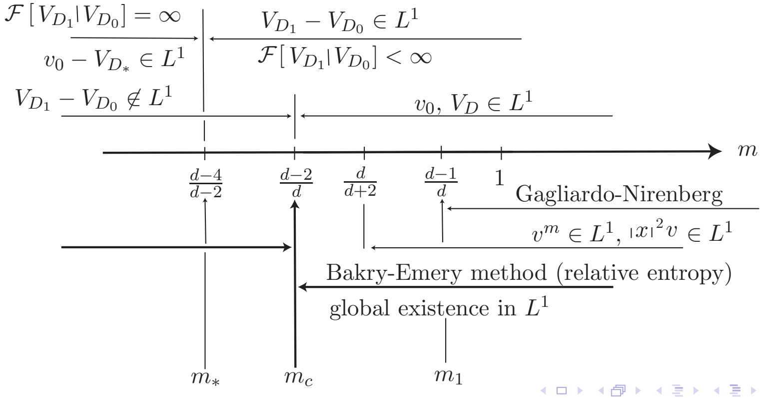
... existence of solutions of $u_t = \Delta u^m$

Fast diffusion equations: the infinite mass regime by linearization of the entropy

Extension to the infinite mass regime, finite time vanishing

- If $m > m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in $L^1(\mathbb{R}^d)$ and the Barenblatt self-similar solution has finite mass
- For $m \leq m_c$, the Barenblatt self-similar solution has infinite mass

Extension to $m \leq m_c$? Work in relative variables !



Entropy methods and linearization: intermediate asymptotics, vanishing

[A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez]

$$\frac{\partial u}{\partial \tau} = -\nabla \cdot (u \nabla u^{m-1}) = \frac{1-m}{m} \Delta u^m \tag{1}$$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$

$$R(\tau) := (T+\tau)^{\frac{1}{d\,(m-m_c)}}$$

- $0 < m < m_c$, $T < +\infty$: vanishing in finite time $\lim_{\tau \nearrow T} u(\tau, y) = 0$

$$R(\tau) := (T-\tau)^{-\frac{1}{d\,(m_c-m)}}$$

Self-similar *Barenblatt type solutions* exists for any m

$$t := \frac{1-m}{2} \log\left(\frac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\frac{1}{2\,d\,|m-m_c|}}\,\frac{y}{R(\tau)}$$

Generalized Barenblatt profiles: $V_D(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}}$

Sharp rates of convergence

Assumptions on the initial datum v_0

(H1) *$V_{D_0} \le v_0 \le V_{D_1}$ for some $D_0 > D_1 > 0$*

(H2) *if $d \ge 3$ and $m \le m_*$, $(v_0 - V_D)$ is integrable for a suitable $D \in [D_1, D_0]$*

Theorem

[Blanchet, Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if $m < 1$ and $m \ne m_* := \frac{d-4}{d-2}$, the entropy decays according to*

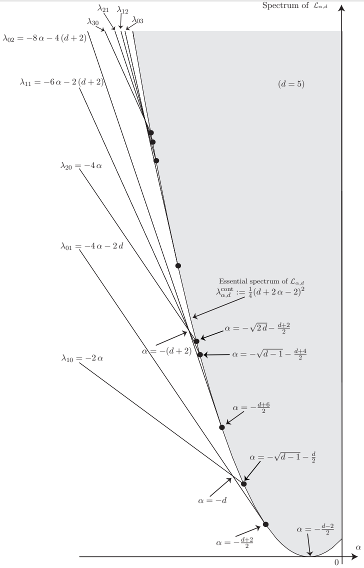
$$\mathcal{F}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\, t \ge 0$$

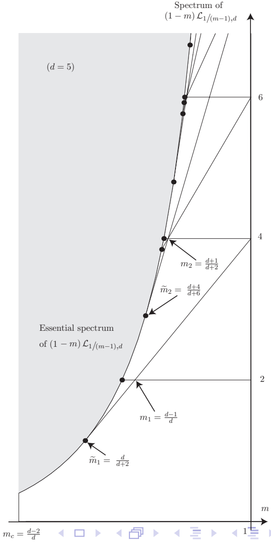
where $\Lambda_{\alpha,d} > 0$ is the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

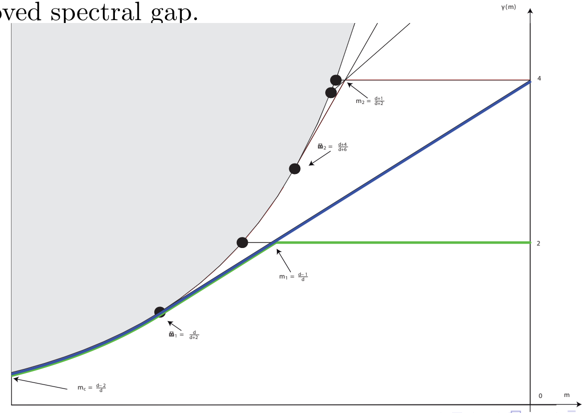
Plots ($d = 5$)


Spectrum of $\mathcal{L}_{\alpha,d}$
λ_{21}
λ_{12}
λ_{30}
λ_{03}
$\lambda_{02} = -8\,\alpha - 4\,(d+2)$
$\lambda_{11} = -6\,\alpha - 2\,(d+2)$
$(d = 5)$
$\lambda_{20} = -4\,\alpha$
$\lambda_{01} = -4\,\alpha - 2\,d$
Essential spectrum of $\mathcal{L}_{\alpha,d}$
$\lambda^{\rm cont}_{\alpha,d} := \frac{1}{4}(d+2\,\alpha-2)^2$
$\alpha = -\sqrt{2\,d} - \frac{d+2}{2}$
$\alpha = -(d+2)$
$\alpha = -\sqrt{d-1} - \frac{d+4}{2}$
$\lambda_{10} = -2\,\alpha$
$\alpha = -\frac{d+6}{2}$
$\alpha = -\sqrt{d-1} - \frac{d}{2}$
$\alpha = -d$
$\alpha = -\frac{d-2}{2}$
$\alpha = -\frac{d+2}{2}$
α
0


Spectrum of
$(1-m)\,\mathcal{L}_{1/(m-1),d}$
$(d = 5)$
6
4
$m_2 = \frac{d+1}{d+2}$
$\widetilde{m}_2 = \frac{d+4}{d+6}$
Essential spectrum
of $(1-m)\,\mathcal{L}_{1/(m-1),d}$
2
$m_1 = \frac{d-1}{d}$
$\widetilde{m}_1 = \frac{d}{d+2}$
m
$m_c = \frac{d-2}{d}$
1

Improved asymptotic rates

[Bonforte, J.D., Grillo, Vázquez] Assume that $m \in (m_1, 1)$, $d \geq 3$. Under Assumption (H1), if v is a solution of the fast diffusion equation with initial datum v_0 such that $\int_{\mathbb{R}^d} x\, v_0\, dx = 0$, then the asymptotic convergence holds with an improved rate corresponding to the improved spectral gap.



Higher order matching asymptotics

[J.D., G. Toscani] For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1} \right) = 0$$

Without choosing R, we may define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t, x) \;, \quad R = R(\tau) \;, \quad t = \tfrac{1}{2} \log R \;, \quad x = \frac{y}{R}$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \, \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2\, x \right) \right] = 0 \quad t > 0 \;, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on R)

$$2\, \sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \frac{1}{\sigma} |x|^2 \right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d \tag{2}$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution (it plays the role of a *local Gibbs state*) but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1} (v - B_\sigma) \right] dx$$

The time derivative of this relative entropy is

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt} \left(\frac{d}{d\sigma} \mathcal{F}_\sigma[v] \right)_{|\sigma=\sigma(t)}}_{\text{choose it } = 0} + \frac{m}{m-1} \int_{\mathbb{R}^d} \left(v^{m-1} - B_{\sigma(t)}^{m-1} \right) \frac{\partial v}{\partial t}\, dx$$

$\Longleftrightarrow$ Minimize $\mathcal{F}_\sigma[v]$ w.r.t. σ $\Longleftrightarrow$ $\int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx = \int_{\mathbb{R}^d} |x|^2\, v\, dx$

The entropy / entropy production estimate

Using the new change of variables, we know that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} v\left|\nabla\left[v^{m-1}-B_{\sigma(t)}^{m-1}\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[v] = \frac{m}{1-m}\int_{\mathbb{R}^d}\left[w-1-\frac{1}{m}\left(w^m-1\right)\right]B_\sigma^m\,dx$$

(Repeating) define the *relative Fisher information* by

$$\mathcal{I}_\sigma[v] := \int_{\mathbb{R}^d}\left|\frac{1}{m-1}\,\nabla\left[\left(w^{m-1}-1\right)B_\sigma^{m-1}\right]\right|^2 B_\sigma\,w\,dx$$

so that $\quad \frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\,m\,(1-m)\,\sigma(t)\,\mathcal{I}_{\sigma(t)}[v(t,\cdot)] \quad \forall\, t>0$

When linearizing, one more mode is killed and $\sigma(t)$ scales out

Improved rates of convergence

Theorem (J.D., G. Toscani)

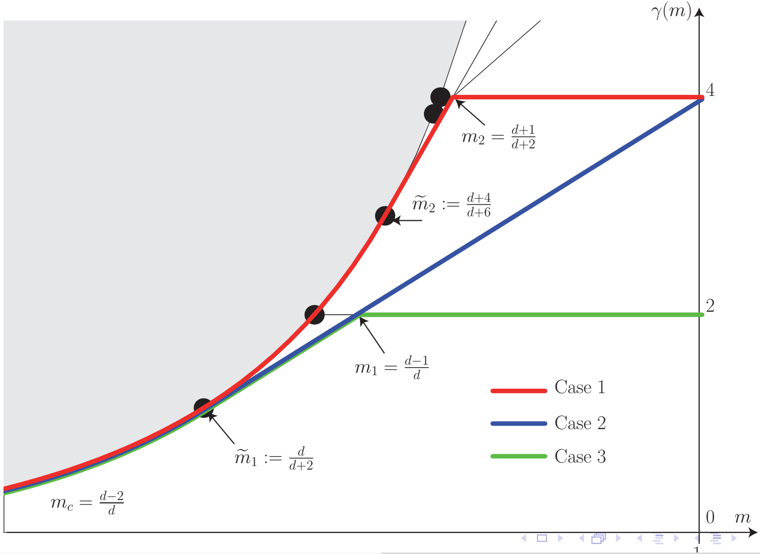
Let $m \in (\widetilde{m}_1, 1)$, $d \geq 2$, $v_0 \in L^1_+(\mathbb{R}^d)$ *such that* v_0^m, $|y|^2\, v_0 \in L^1(\mathbb{R}^d)$

$$\mathcal{F}[v(t,\cdot)] \leq C\, e^{-2\,\gamma(m)\,t} \quad \forall\; t \geq 0$$

where

$$\gamma(m) = \begin{cases} \frac{((d-2)\,m-(d-4))^2}{4\,(1-m)} & \text{if } m \in (\widetilde{m}_1, \widetilde{m}_2] \\ 4\,(d+2)\,m - 4\,d & \text{if } m \in [\widetilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$

Spectral gaps and best constants



Comments

1. A result by [Denzler, Koch, McCann] *Higher order time asymptotics of fast diffusion in Euclidean space: a dynamical systems approach*
2. The constant C in

$$\mathcal{F}[v(t,\cdot)] \leq C\, e^{-2\,\gamma(m)\,t} \quad \forall\, t \geq 0$$

can be made explicit, under additional restrictions on the initial data [Bonforte, J.D., Grillo, Vázquez]

An explicit constant C ?

$$\frac{d}{dt}\mathcal{F}[w(t,\cdot)] = -\mathcal{I}[w(t,\cdot)] \quad \forall\, t>0$$

$$h^{m-2}\int_{\mathbb{R}^d}|f|^2\,V_D^{2-m}\,dx \le 2\,\mathcal{F}[w] \le h^{2-m}\int_{\mathbb{R}^d}|f|^2\,V_D^{2-m}\,dx$$

where $f := (w-1)\,V_D^{m-1}$, $h := \max\{\sup_{\mathbb{R}^d} w(t,\cdot), 1/\inf_{\mathbb{R}^d} w(t,\cdot)\}$

$$\int_{\mathbb{R}^d}|\nabla f|^2\,V_D\,dx \le h^{5-2m}\,\mathcal{I}[w] + d\,(1-m)\left[h^{4(2-m)}-1\right]\int_{\mathbb{R}^d}|f|^2\,V_D^{2-m}\,dx$$

$$0 \le h-1 \le \mathsf{C}\,\mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$$

Corollary

$\mathcal{F}[w(t,\cdot)] \le G\big(t, h(0), \mathcal{F}[w(0,\cdot)]\big)$ *for any* $t \ge 0$, *where*

$$\frac{dG}{dt} = -2\frac{\Lambda_{\alpha,d} - Y(h)}{[1+X(h)]\,h^{2-m}}\,G\,,\quad h = 1 + \mathsf{C}\,G^{\frac{1-m}{d+2-(d+1)m}}\,,\quad G(0) = \mathcal{F}[w(0,\cdot)]$$

Gagliardo-Nirenberg and Sobolev inequalities : improvements

[J.D., G. Toscani]

Best matching Barenblatt profiles

(Repeating) Consider the *fast diffusion equation*

$$\frac{\partial u}{\partial t}+\nabla\cdot\Big[u\,\Big(\sigma^{\frac{d}{2}(m-m_c)}\,\nabla u^{m-1}-2\,x\Big)\Big]=0\quad t>0\,,\quad x\in\mathbb{R}^d$$

with a nonlocal, time-dependent diffusion coefficient

$$\sigma(t)=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u(x,t)\,dx\,,\quad K_M:=\int_{\mathbb{R}^d}|x|^2\,B_1(x)\,dx$$

where

$$B_\lambda(x):=\lambda^{-\frac{d}{2}}\left(C_M+\tfrac{1}{\lambda}\,|x|^2\right)^{\frac{1}{m-1}}\quad\forall\,x\in\mathbb{R}^d$$

and define the relative entropy

$$\mathcal{F}_\lambda[u]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m-B_\lambda^m-m\,B_\lambda^{m-1}\,(u-B_\lambda)\right]\,dx$$

Three ingredients for *global improvements*

1. $\inf_{\lambda>0}\mathcal{F}_\lambda[u(x,t)]=\mathcal{F}_{\sigma(t)}[u(x,t)]$ so that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[u(x,t)]=-\mathcal{J}_{\sigma(t)}[u(\cdot,t)]$$

where the relative Fisher information is

$$\mathcal{J}_\lambda[u]:=\lambda^{\frac{d}{2}(m-m_c)}\,\frac{m}{1-m}\int_{\mathbb{R}^d}u\,\left|\nabla u^{m-1}-\nabla B_\lambda^{m-1}\right|^2\,dx$$

2. In the *Bakry-Emery method*, there is *an additional (good) term*

$$4\left[1+2\,C_{m,d}\,\frac{\mathcal{F}_{\sigma(t)}[u(\cdot,t)]}{M^\gamma\,\sigma_0^{\frac{d}{2}(1-m)}}\right]\frac{d}{dt}\left(\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\right)\geq\frac{d}{dt}\left(\mathcal{J}_{\sigma(t)}[u(\cdot,t)]\right)$$

3. The *Csiszár-Kullback inequality* is also improved

$$\mathcal{F}_\sigma[u]\geq\frac{m}{8\int_{\mathbb{R}^d}B_1^m\,dx}\,C_M^2\|u-B_\sigma\|^2_{\mathrm{L}^1(\mathbb{S}^d)}$$

improved decay for the relative entropy

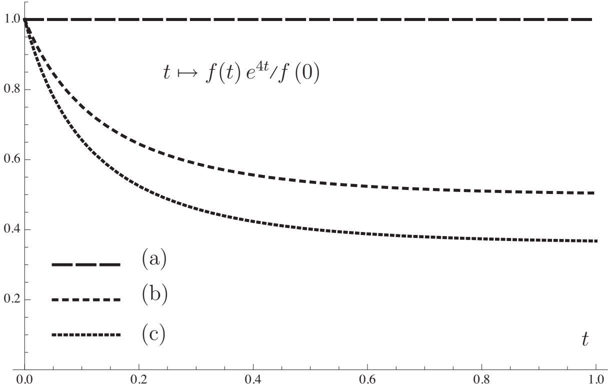


Figure: Upper bounds on the decay of the relative entropy: $t \mapsto f(t)\, e^{4t}/f(0)$
(a): estimate given by the entropy-entropy production method
(b): exact solution of a simplified equation
(c): numerical solution (found by a shooting method)

A Csiszár-Kullback(-Pinsker) inequality

Let $m \in (\widetilde{m}_1, 1)$ with $\widetilde{m}_1 = \frac{d}{d+2}$ and consider the relative entropy

$$\mathcal{F}_\sigma[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - B_\sigma^m - m\, B_\sigma^{m-1}\,(u - B_\sigma)\right]\, dx$$

Theorem

Let $d \geq 1$, $m \in (\widetilde{m}_1, 1)$ and assume that u is a nonnegative function in $\mathcal{L}^1(\mathbb{R}^d)$ such that u^m and $x \mapsto |x|^2\, u$ are both integrable on $\mathbb{R}^d$. If $\|u\|_{\mathrm{L}^1(\mathbb{S}^d)} = M$ and $\int_{\mathbb{R}^d} |x|^2\, u\, dx = \int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx$, then

$$\frac{\mathcal{F}_\sigma[u]}{\sigma^{\frac{d}{2}(1-m)}} \geq \frac{m}{8 \int_{\mathbb{R}^d} B_1^m\, dx} \left(C_M \|u - B_\sigma\|_{\mathrm{L}^1(\mathbb{S}^d)} + \frac{1}{\sigma} \int_{\mathbb{R}^d} |x|^2\, |u - B_\sigma|\, dx \right)^2$$

Csiszár-Kullback(-Pinsker): proof (1/2)

Let $v := u/B_\sigma$ and $d\mu_\sigma := B_\sigma^m\,dx$

$$\int_{\mathbb{R}^d}(v-1)\,d\mu_\sigma = \int_{\mathbb{R}^d} B_\sigma^{m-1}\,(u-B_\sigma)\,dx$$

$$= \sigma^{\frac{d}{2}(1-m)}\,C_M \int_{\mathbb{R}^d}(u-B_\sigma)\,dx + \sigma^{\frac{d}{2}(m_c-m)} \int_{\mathbb{R}^d} |x|^2\,(u-B_\sigma)\,dx = 0$$

$$\int_{\mathbb{R}^d}(v-1)\,d\mu_\sigma = \int_{v>1}(v-1)\,d\mu_\sigma - \int_{v<1}(1-v)\,d\mu_\sigma = 0$$

$$\int_{\mathbb{R}^d}|v-1|\,d\mu_\sigma = \int_{v>1}(v-1)\,d\mu_\sigma + \int_{v<1}(1-v)\,d\mu_\sigma$$

$$\int_{\mathbb{R}^d}|u-B_\sigma|\,B_\sigma^{m-1}\,dx = \int_{\mathbb{R}^d}|v-1|\,d\mu_\sigma = 2\int_{v<1}|v-1|\,d\mu_\sigma$$

Csiszár-Kullback(-Pinsker): proof (2/2)

A Taylor expansion shows that

$$\mathcal{F}_\sigma[u] = \frac{1}{m-1}\int_{\mathbb{R}^d}\left[v^m - 1 - m\,(v-1)\right]d\mu_\sigma = \frac{m}{2}\int_{\mathbb{R}^d}\xi^{m-2}\,|v-1|^2\,d\mu_\sigma$$
$$\geq \frac{m}{2}\int_{v<1}|v-1|^2\,d\mu_\sigma$$

Using the Cauchy-Schwarz inequality, we get

$$\left(\int_{v<1}|v-1|\,d\mu_\sigma\right)^2 = \left(\int_{v<1}|v-1|\,B_\sigma^{\frac{m}{2}}\,B_\sigma^{\frac{m}{2}}\,dx\right)^2 \leq \int_{v<1}|v-1|^2\,d\mu_\sigma\int_{\mathbb{R}^d}B_\sigma^m\,dx$$

and finally obtain that

$$\mathcal{F}_\sigma[u] \geq \frac{m}{2}\,\frac{\left(\int_{v<1}|v-1|\,d\mu_\sigma\right)^2}{\int_{\mathbb{R}^d}B_\sigma^m\,dx} = \frac{m}{8}\,\frac{\left(\int_{\mathbb{R}^d}|u-B_\sigma|\,B_\sigma^{m-1}\,dx\right)^2}{\int_{\mathbb{R}^d}B_\sigma^m\,dx}$$

An improved Gagliardo-Nirenberg inequality: the setting

The inequality

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{S}^d)} \le \mathcal{C}^{\mathrm{GN}}_{p,d}\,\|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{S}^d)}\,\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{S}^d)}$$

with $\theta=\theta(p):=\frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, $1<p\le\frac{d}{d-2}$ if $d\ge3$ and $1<p<\infty$ if $d=2$, can be rewritten, in a non-scale invariant form, as

$$\int_{\mathbb{R}^d}|\nabla f|^2\,dx+\int_{\mathbb{R}^d}|f|^{p+1}\,dx\ge \mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d}|f|^{2\,p}\,dx\right)^{\gamma}$$

with $\gamma=\gamma(p,d):=\frac{d+2-p\,(d-2)}{d-p\,(d-4)}$. Optimal function are given by

$$f_{M,y,\sigma}(x)=\frac{1}{\sigma^{\frac d2}}\left(C_M+\frac{|x-y|^2}{\sigma}\right)^{-\frac{1}{p-1}}\quad\forall\,x\in\mathbb{R}^d$$

where C_M is determined by $\int_{\mathbb{R}^d}f^{2\,p}_{M,y,\sigma}\,dx=M$

$$\mathfrak{M}_d:=\left\{f_{M,y,\sigma}\;:\;(M,y,\sigma)\in\mathcal{M}_d:=(0,\infty)\times\mathbb{R}^d\times(0,\infty)\right\}$$

An improved Gagliardo-Nirenberg inequality (1/2)

Relative entropy functional

$$\mathcal{R}^{(p)}[f] := \inf_{g \in \mathfrak{M}_d^{(p)}} \int_{\mathbb{R}^d} \left[g^{1-p} \left(|f|^{2\,p} - g^{2\,p} \right) - \tfrac{2\,p}{p+1} \left(|f|^{p+1} - g^{p+1} \right) \right] dx$$

Theorem

Let $d \geq 2$, $p > 1$ and assume that $p < d/(d-2)$ if $d \geq 3$. If

$$\frac{\int_{\mathbb{R}^d} |x|^2\,|f|^{2\,p}\,dx}{\left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx\right)^\gamma} = \frac{d\,(p-1)\,\sigma_*\,M_*^{\gamma-1}}{d+2-p\,(d-2)}\,,\quad \sigma_*(p) := \left(4\,\frac{d+2-p\,(d-2)}{(p-1)^2\,(p+1)}\right)^{\frac{4\,p}{d-p\,(d-4)}}$$

for any $f \in \mathcal{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$, then we have

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx - \mathsf{K}_{p,d} \left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx \right)^\gamma \geq \mathsf{C}_{p,d}\,\frac{\left(\mathcal{R}^{(p)}[f]\right)^2}{\left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx\right)^\gamma}$$

An improved Gagliardo-Nirenberg inequality (2/2)

A Csiszár-Kullback inequality

$$\mathcal{R}^{(p)}[f] \geq \mathsf{C}_{\mathrm{CK}} \, \|f\|_{\mathrm{L}^{2p}(\mathbb{S}^d)}^{2p(\gamma-2)} \inf_{g \in \mathfrak{M}_d^{(p)}} \left\| |f|^{2p} - g^{2p} \right\|_{\mathrm{L}^1(\mathbb{S}^d)}^2$$

with $\mathsf{C}_{\mathrm{CK}} = \frac{p-1}{p+1} \, \frac{d+2-p\,(d-2)}{32\,p} \, \sigma_*^{d\,\frac{p-1}{4p}} \, M_*^{1-\gamma}$. Let

$$\mathfrak{C}_{p,d} := \mathsf{C}_{d,p} \, {\mathsf{C}_{\mathrm{CK}}}^2$$

Corollary

Under previous assumptions, we have

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, dx + \int_{\mathbb{R}^d} |f|^{p+1} \, dx - \mathsf{K}_{p,d} \left(\int_{\mathbb{R}^d} |f|^{2p} \, dx \right)^\gamma \geq \mathfrak{C}_{p,d} \, \|f\|_{\mathrm{L}^{2p}(\mathbb{S}^d)}^{2p(\gamma-4)} \inf_{g \in \mathfrak{M}_d(p)} \left\| |f|^{2p} - g^{2p} \right\|_{\mathrm{L}^1(\mathbb{S}^d)}^4$$

Conclusion 1: improved inequalities

- We have found an improvement of an optimal Gagliardo-Nirenberg inequality, which provides an explicit measure of the distance to the manifold of optimal functions.
- The method is based on the nonlinear flow
- The explicit improvement gives (is equivalent to) an improved entropy – entropy production inequality

Conclusion 2: improved rates

If $m \in (m_1, 1)$, with

$$f(t) := \mathcal{F}_{\sigma(t)}[u(\cdot, t)]$$
$$\sigma(t) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 \, u(x,t) \, dx$$
$$j(t) := \mathcal{J}_{\sigma(t)}[u(\cdot, t)]$$

$$\mathcal{J}_\sigma[u] := \frac{m \, \sigma^{\frac{d}{2}(m-m_c)}}{1-m} \int_{\mathbb{R}^d} u \left| \nabla u^{m-1} - \nabla \mathfrak{B}_\sigma^{m-1} \right|^2 dx$$

we can write a system of coupled ODEs

$$\begin{cases} f' = -j \le 0 \\ \sigma' = -2\,d\,\frac{(1-m)^2}{m\,K_M}\,\sigma^{\frac{d}{2}(m-m_c)}\,f \le 0 \\ j' + 4\,j = \frac{d}{2}\,(m-m_c)\left[\frac{j}{\sigma} + 4\,d\,(1-m)\,\frac{f}{\sigma}\right]\sigma' - \mathrm{r} \end{cases} \tag{3}$$

In the rescaled variables, we have found an *improved decay* (algebraic rate) of the relative entropy. This is a new nonlinear effect, which matters for the initial time layer

Conclusion 3: Best matching Barenblatt profiles are delayed

Let u be such that

$$v(\tau,x)=\frac{\mu^d}{R(D\,\tau)^d}\,u\left(\frac{1}{2}\log R(D\,\tau),\frac{\mu\,x}{R(D\,\tau)}\right)$$

with $\tau\mapsto R(\tau)$ given as the solution to

$$\frac{1}{R}\,\frac{dR}{d\tau}=\left(\frac{\mu^2}{K_M}\int_{\mathbb{R}^d}|x|^2\,v(\tau,x)\,dx\right)^{-\frac{d}{2}(m-m_c)}\,,\quad R(0)=1$$

Then

$$\frac{1}{R}\,\frac{dR}{d\tau}=\left[R^2(\tau)\,\sigma\left(\frac{1}{2}\log R(D\,\tau)\right)\right]^{-\frac{d}{2}(m-m_c)}$$

that is $R(\tau)=R_0(\tau)\leq R_0(\tau)$ where $\frac{1}{R}\,\frac{dR_0}{d\tau}=\left(R_0^2(\tau)\,\sigma\,(0)\right)^{-\frac{d}{2}(m-m_c)}$
and asymptotically as $\tau\to\infty$, $R(\tau)=R_0(\tau-\delta)$ for some delay $\delta>0$

B. Fast diffusion equations: New points of view

1. improved inequalities and scalings
2. scalings and a concavity property
3. improved rates and best matching

Improved inequalities and scalings

The logarithmic Sobolev inequality

$d\mu = \mu\,dx$, $\mu(x) = (2\,\pi)^{-d/2}\,e^{-|x|^2/2}$, on $\mathbb{R}^d$ with $d \ge 1$

Gaussian logarithmic Sobolev inequality

$$\int_{\mathbb{R}^2} |\nabla u|^2\,d\mu \ge \frac{1}{2}\int_{\mathbb{R}^2} |u|^2\,\log|u|^2\,d\mu$$

for any function $u \in \mathrm{H}^1(\mathbb{R}^d, d\mu)$ such that $\int_{\mathbb{R}^2} |u|^2\,d\mu = 1$

$$\varphi(t) := \frac{d}{4}\left[\exp\left(\frac{2\,t}{d}\right) - 1 - \frac{2\,t}{d}\right] \quad \forall\, t \in \mathbb{R}$$

[Bakry, Ledoux (2006)], [Fathi et al. (2014)], [Dolbeault, Toscani (2014)]

Proposition

$$\int_{\mathbb{R}^2} |\nabla u|^2\,d\mu - \frac{1}{2}\int_{\mathbb{R}^2} |u|^2\,\log|u|^2\,d\mu \ge \varphi\left(\int_{\mathbb{R}^2} |u|^2\,\log|u|^2\,d\mu\right)$$

$$\forall\, u \in \mathrm{H}^1(\mathbb{R}^d, d\mu) \quad s.t. \quad \int |u|^2\,d\mu = 1 \quad and \quad \int |x|^2\,|u|^2\,d\mu = d$$

Consequences for the heat equation

Ornstein-Uhlenbeck equation (or backward Kolmogorov equation)

$$\frac{\partial f}{\partial t} = \Delta f - x \cdot \nabla f$$

with initial datum $f_0 \in \mathrm{L}^1_+(\mathbb{R}^d, (1+|x|^2)\,d\mu$ and define the *entropy* as

$$\mathcal{E}[f] := \int_{\mathbb{R}^2} f\,\log f\,d\mu\,, \quad \frac{d}{dt}\mathcal{E}[f] = -\,4\int_{\mathbb{R}^2} |\nabla\sqrt{f}|^2\,d\mu \le -\,2\,\mathcal{E}[f]$$

thus proving that $\mathcal{E}[f(t,\cdot)] \le \mathcal{E}[f_0]\,e^{-2t}$. Moreover,

$$\frac{d}{dt}\int_{\mathbb{R}^2} f\,|x|^2\,d\mu = 2\int_{\mathbb{R}^2} f\,(d-|x|^2)\,d\mu$$

Theorem

Assume that $\mathcal{E}[f_0]$ is finite and $\int_{\mathbb{R}^2} f_0\,|x|^2\,d\mu = d\,\int_{\mathbb{R}^2} f_0\,d\mu$. Then

$$\mathcal{E}[f(t,\cdot)] \le -\frac{d}{2}\,\log\left[1-\left(1-e^{-\frac{2}{d}\,\mathcal{E}[f_0]}\right)\,e^{-2t}\right] \quad \forall\,t\ge 0$$

Gagliardo-Nirenberg inequalities and the FDE

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{S}^d)}^{\vartheta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{S}^d)}^{1-\vartheta}\geq \mathsf{C}_{\rm GN}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{S}^d)}$$

With the right choice of the constants, the functional

$$\mathsf{J}[w]:=\frac{1}{4}\,(q^2-1)\int_{\mathbb{R}^d}|\nabla w|^2\,dx+\beta\int_{\mathbb{R}^d}|w|^{q+1}\,dx-\mathcal{K}\,\mathsf{C}_{\rm GN}^{\alpha}\left(\int_{\mathbb{R}^d}|w|^{2q}\,dx\right)^{\frac{\alpha}{2q}}$$

is nonnegative and $\mathsf{J}[w]\geq\mathsf{J}[w_*]=0$

Theorem

[Dolbeault-Toscani] *For some nonnegative, convex, increasing φ*

$$\mathsf{J}[w]\geq\varphi\left[\beta\left(\int_{\mathbb{R}^d}|w_*|^{q+1}\,dx-\int_{\mathbb{R}^d}|w|^{q+1}\,dx\right)\right]$$

for any $w\in\mathrm{L}^{q+1}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}|\nabla w|^2\,dx<\infty$ and $\int_{\mathbb{R}^d}|w|^{2q}\,|x|^2\,dx=\int_{\mathbb{R}^d}w_*^{2q}\,|x|^2\,dx$

Consequence for decay rates of relative Rényi entropies: see [Carrillo-Toscani]

Scalings and a concavity property

The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial u}{\partial t} = \Delta u^p$$

with initial datum $u(x, t=0) = u_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} u_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, u_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathcal{B}_\star\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(p-1)\,, \quad \kappa := \left|\frac{2\,\mu\,p}{p-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(p-1)} & \text{if } p > 1 \\ \left(C_\star + |x|^2\right)^{1/(p-1)} & \text{if } p < 1 \end{cases}$$

The entropy

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} u^p \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} u \, |\nabla v|^2 \, dx \quad \text{with} \quad v = \frac{p}{p-1} \, u^{p-1}$$

If u solves the fast diffusion equation, then

$$\mathsf{E}' = (1-p) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial v}{\partial t} = (p-1) \, v \, \Delta v + |\nabla v|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-p)} = 1 + \frac{2}{1-p} \left(\frac{1}{d} + p - 1 \right) = \frac{2}{d} \, \frac{1}{1-p} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The concavity property

Theorem

[Toscani-Savaré] *Assume that* $p \geq 1 - \frac{1}{d}$ *if* $d > 1$ *and* $p > 0$ *if* $d = 1$. *Then* $F(t)$ *is increasing,* $(1-p)\,\mathsf{F}''(t) \leq 0$ *and*

$$\lim_{t\to+\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-p)\,\sigma \lim_{t\to+\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-p)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

[Dolbeault-Toscani] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \geq \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{S}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{S}^d)}^{1-\theta} \geq \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{S}^d)}$$

if $1 - \frac{1}{d} \leq p < 1$. Hint: $u^{p-1/2} = \frac{w}{\|w\|_{\mathrm{L}^{2q}(\mathbb{S}^d)}}$, $q = \frac{1}{2\,p-1}$

The proof

Lemma

$$\mathsf{I}' = \frac{d}{dt}\int_{\mathbb{R}^d} u\,|\nabla v|^2\,dx = -\,2\int_{\mathbb{R}^d} u^p\left(\|\mathrm{D}^2 v\|^2 + (p-1)\,(\Delta v)^2\right)dx$$

$$\|\mathrm{D}^2 v\|^2 = \frac{1}{d}\,(\Delta v)^2 + \left\|\mathrm{D}^2 v - \frac{1}{d}\,\Delta v\,\mathrm{Id}\right\|^2$$

$$\begin{aligned}\frac{1}{\sigma\,(1-p)}\,\mathsf{E}^{2-\sigma}\,\left(\mathsf{E}^\sigma\right)'' = {} & (1-p)\,(\sigma-1)\left(\int_{\mathbb{R}^d} u\,|\nabla v|^2\,dx\right)^2 \\ & -\,2\left(\frac{1}{d}+p-1\right)\int_{\mathbb{R}^d} u^p\,dx\int_{\mathbb{R}^d} u^p\,(\Delta v)^2\,dx \\ & -\,2\int_{\mathbb{R}^d} u^p\,dx\int_{\mathbb{R}^d} u^p\left\|\mathrm{D}^2 v - \frac{1}{d}\,\Delta v\,\mathrm{Id}\right\|^2 dx\end{aligned}$$

Improved rates and best matching

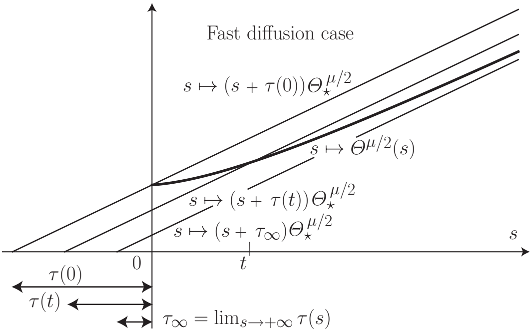
Temperature (fast diffusion case)

The *second moment functional* (temperature) is defined by

$$\Theta(t) := \frac{1}{d} \int_{\mathbb{R}^d} |x|^2 \, u(t,x) \, dx$$

and such that

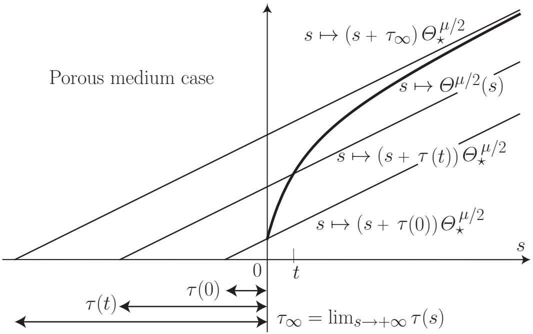
$$\Theta' = 2\,\mathsf{E}$$



Temperature (porous medium case) and delay

Let $\mathcal{U}_\star^s$ be the *best matching Barenblatt* function, in the sense of relative entropy $\mathcal{F}[u\,|\,\mathcal{U}_\star^s]$, among all Barenblatt functions $(\mathcal{U}_\star^s)_{s>0}$. We define s as a function of t and consider the *delay* given by

$$\tau(t) := \left(\frac{\Theta(t)}{\Theta_\star}\right)^{\frac{\mu}{2}} - t$$



A result on delays

Theorem

Assume that $p \geq 1 - \frac{1}{d}$ and $p \neq 1$. The best matching Barenblatt function of a solution u is $(t,x) \mapsto \mathcal{U}_\star(t+\tau(t),x)$ and the function $t \mapsto \tau(t)$ is nondecreasing if $p > 1$ and nonincreasing if $1 - \frac{1}{d} \leq p < 1$

With $\mathsf{G} := \Theta^{1-\frac{\eta}{2}}$, $\eta = d\,(1-p) = 2-\mu$, the *Rényi entropy power* functional $\mathsf{H} := \Theta^{-\frac{\eta}{2}}\,\mathsf{E}$ is such that

$$\mathsf{G}' = \mu\,\mathsf{H} \quad \text{with} \quad \mathsf{H} := \Theta^{-\frac{\eta}{2}}\,\mathsf{E}$$

$$\frac{\mathsf{H}'}{1-p} = \Theta^{-1-\frac{\eta}{2}}\left(\Theta\,\mathsf{I} - d\,\mathsf{E}^2\right) = \frac{d\,\mathsf{E}^2}{\Theta^{\frac{\eta}{2}+1}}\,(\mathsf{q}-1) \quad \text{with} \quad \mathsf{q} := \frac{\Theta\,\mathsf{I}}{d\,\mathsf{E}^2} \geq 1$$

$$d\,\mathsf{E}^2 = \frac{1}{d}\left(-\int_{\mathbb{R}^d} x\cdot\nabla(u^p)\,dx\right)^2 = \frac{1}{d}\left(\int_{\mathbb{R}^d} x\cdot u\,\nabla v\,dx\right)^2$$
$$\leq \frac{1}{d}\int_{\mathbb{R}^d} u\,|x|^2\,dx \int_{\mathbb{R}^d} u\,|\nabla v|^2\,dx = \Theta\,\mathsf{I}$$

An estimate of the delay

Theorem

If $p > 1 - \frac{1}{d}$ and $p \neq 1$, then the delay satisfies

$$\lim_{t \to +\infty} |\tau(t) - \tau(0)| \geq |1 - p| \, \frac{\Theta(0)^{1 - \frac{d}{2}(1-p)}}{2\,\mathsf{H}_\star} \, \frac{\big(\mathsf{H}_\star - \mathsf{H}(0)\big)^2}{\Theta(0)\,\mathsf{I}(0) - d\,\mathsf{E}(0)^2}$$

C. Fast diffusion equations on manifolds and sharp functional inequalities

Interpolation inequalities on the sphere

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss

A family of interpolation inequalities on the sphere

The following interpolation inequality holds on the sphere

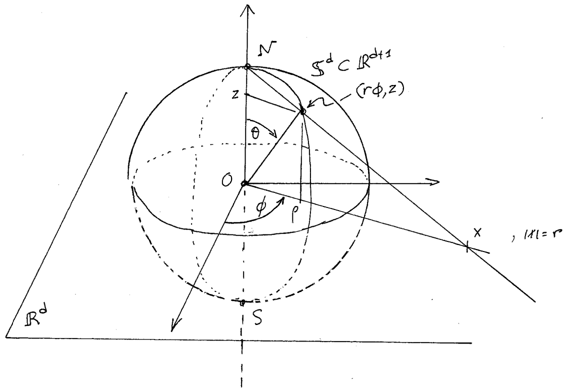
$$\frac{p-2}{d}\int_{\mathbb{S}^d}|\nabla u|^2\,d\,v_g+\int_{\mathbb{S}^d}|u|^2\,d\,v_g\ge\left(\int_{\mathbb{S}^d}|u|^p\,d\,v_g\right)^{2/p}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d,dv_g)$$

- for any $p\in(2,2^*]$ with $2^*=\frac{2\,d}{d-2}$ if $d\ge3$
- for any $p\in(2,\infty)$ if $d=2$

Here dv_g is the uniform probability measure: $v_g(\mathbb{S}^d)=1$

- 1 is the optimal constant, equality achieved by constants
- $p=2^*$ corresponds to Sobolev's inequality...

Stereographic projection



Sobolev inequality

The stereographic projection of $\mathbb{S}^d \subset \mathbb{R}^d \times \mathbb{R} \ni (\rho\,\phi, z)$ onto $\mathbb{R}^d$: to $\rho^2 + z^2 = 1$, $z \in [-1,1]$, $\rho \ge 0$, $\phi \in \mathbb{S}^{d-1}$ we associate $x \in \mathbb{R}^d$ such that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}\,, \quad \rho = \frac{2\,r}{r^2+1}$$

and transform any function u on $\mathbb{S}^d$ into a function v on $\mathbb{R}^d$ using

$$u(y) = \left(\tfrac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\tfrac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1-z)^{-\frac{d-2}{2}}\, v(x)$$

- $p = 2^*$, $\mathsf{S}_d = \frac{1}{4}\,d\,(d-2)\,|\mathbb{S}^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2\,dx \ge \mathsf{S}_d \left[\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\,dx\right]^{\frac{d-2}{d}} \quad \forall\, v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

Extended inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \geq \frac{d}{p-2} \left[\left(\int_{\mathbb{S}^d} |u|^p \, d\,v_g \right)^{2/p} - \int_{\mathbb{S}^d} |u|^2 \, d\,v_g \right] \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

is valid

- for any $p \in (1,2) \cup (2,\infty)$ if $d = 1,\, 2$
- for any $p \in (1,2) \cup (2,2^*]$ if $d \geq 3$

- Case $p = 2$: Logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \geq \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \log \left(\frac{|u|^2}{\int_{\mathbb{S}^d} |u|^2 \, d\,v_g} \right) d\,v_g \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

- Case $p = 1$: Poincaré inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \geq d \int_{\mathbb{S}^d} |u - \bar{u}|^2 \, d\,v_g \quad \text{with} \quad \bar{u} := \int_{\mathbb{S}^d} u \, d\,v_g \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

Optimality: a perturbation argument

- For any $p \in (1, 2^*]$ if $d \geq 3$, any $p > 1$ if $d = 1$ or 2, it is remarkable that

$$\mathcal{Q}[u] := \frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \geq \inf_{u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)} \mathcal{Q}[u] = \frac{1}{d}$$

is achieved in the limiting case

$$\mathcal{Q}[1 + \varepsilon\, v] \sim \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|v\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \quad \text{as} \quad \varepsilon \to 0$$

when v is an eigenfunction associated with the first nonzero eigenvalue of Δ_g, thus proving the optimality

- $p < 2$: a proof by semi-groups using Nelson's hypercontractivity lemma. $p > 2$: no simple proof based on spectral analysis is available: [Beckner], an approach based on Lieb's duality, the Funk-Hecke formula and some (non-trivial) computations

- elliptic methods / Γ_2 formalism of Bakry-Emery / nonlinear flows

Schwarz symmetrization and the ultraspherical setting

$(\xi_0,\, \xi_1, \dots \xi_d) \in \mathbb{S}^d$, $\xi_d = z$, $\sum_{i=0}^d |\xi_i|^2 = 1$ [Smets-Willem]

Lemma

Up to a rotation, any minimizer of $\mathcal{Q}$ depends only on $\xi_d = z$

- Let $d\sigma(\theta) := \frac{(\sin\theta)^{d-1}}{Z_d}\, d\theta$, $Z_d := \sqrt{\pi}\, \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}$: $\forall\, v \in \mathrm{H}^1([0,\pi], d\sigma)$

$$\frac{p-2}{d} \int_0^\pi |v'(\theta)|^2 \, d\sigma + \int_0^\pi |v(\theta)|^2 \, d\sigma \ge \left(\int_0^\pi |v(\theta)|^p \, d\sigma \right)^{\frac{2}{p}}$$

- Change of variables $z = \cos\theta$, $v(\theta) = f(z)$

$$\frac{p-2}{d} \int_{-1}^1 |f'|^2 \, \nu \, d\nu_d + \int_{-1}^1 |f|^2 \, d\nu_d \ge \left(\int_{-1}^1 |f|^p \, d\nu_d \right)^{\frac{2}{p}}$$

where $\nu_d(z)\, dz = d\nu_d(z) := Z_d^{-1}\, \nu^{\frac{d}{2}-1}\, dz$, $\nu(z) := 1 - z^2$

The ultraspherical operator

With $d\nu_d = Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1 - z^2$, consider the space $\mathrm{L}^2((-1,1), d\nu_d)$ with scalar product

$$\langle f_1, f_2\rangle = \int_{-1}^{1} f_1\, f_2\, d\nu_d\,, \quad \|f\|_p = \left(\int_{-1}^{1} f^p\, d\nu_d\right)^{\frac{1}{p}}$$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\, f := (1-z^2)\, f'' - d\, z\, f' = \nu\, f'' + \frac{d}{2}\,\nu'\, f'$$

which satisfies $\langle f_1, \mathcal{L}\, f_2\rangle = -\int_{-1}^{1} f_1'\, f_2'\, \nu\, d\nu_d$

Proposition

Let $p \in [1,2) \cup (2, 2^*]$, $d \geq 1$

$$-\langle f, \mathcal{L}\, f\rangle = \int_{-1}^{1} |f'|^2\, \nu\, d\nu_d \geq d\, \frac{\|f\|_p^2 - \|f\|_2^2}{p-2} \quad \forall\, f \in \mathrm{H}^1([-1,1], d\nu_d)$$

Flows on the sphere

- Heat flow and the Bakry-Emery method
- Fast diffusion (porous media) flow and the choice of the exponents

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss

Heat flow and the Bakry-Emery method

With $g = f^p$, *i.e.* $f = g^\alpha$ with $\alpha = 1/p$

$$\text{(Ineq.)} \qquad -\langle f, \mathcal{L} f \rangle = -\langle g^\alpha, \mathcal{L}\, g^\alpha \rangle =: \mathcal{I}[g] \ge d\, \frac{\|g\|_1^{2\,\alpha} - \|g^{2\,\alpha}\|_1}{p-2} =: \mathcal{F}[g]$$

Heat flow

$$\frac{\partial g}{\partial t} = \mathcal{L}\, g$$

$$\frac{d}{dt}\, \|g\|_1 = 0\,, \quad \frac{d}{dt}\, \|g^{2\,\alpha}\|_1 = -\,2\,(p-2)\,\langle f, \mathcal{L}\, f \rangle = 2\,(p-2) \int_{-1}^{1} |f'|^2\, \nu\; d\nu_d$$

which finally gives

$$\frac{d}{dt}\mathcal{F}[g(t,\cdot)] = -\frac{d}{p-2}\,\frac{d}{dt}\,\|g^{2\,\alpha}\|_1 = -\,2\,d\,\mathcal{I}[g(t,\cdot)]$$

$$\text{Ineq.} \iff \frac{d}{dt}\mathcal{F}[g(t,\cdot)] \le -\,2\,d\,\mathcal{F}[g(t,\cdot)] \Longleftarrow \frac{d}{dt}\mathcal{I}[g(t,\cdot)] \le -\,2\,d\,\mathcal{I}[g(t,\cdot)]$$

The equation for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L} f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1} |f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f, \mathcal{L} f\rangle = \langle \mathcal{L} f, \mathcal{L} f\rangle + (p-1)\,\langle \frac{|f'|^2}{f}\,\nu, \mathcal{L} f\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\,d\,\mathcal{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1} |f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1} |f'|^2\,\nu\,d\nu_d$$

$$= -\,2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} = 2^\# < \frac{2\,d}{d-2} = 2^*$$

... up to the critical exponent: a proof in two slides

$$\left[\frac{d}{dz},\mathcal{L}\right] u = (\mathcal{L}\,u)' - \mathcal{L}\,u' = -2\,z\,u'' - d\,u'$$

$$\int_{-1}^{1} (\mathcal{L}\,u)^2\,d\nu_d = \int_{-1}^{1} |u''|^2\,\nu^2\,d\nu_d + d\int_{-1}^{1} |u'|^2\,\nu\,d\nu_d$$

$$\int_{-1}^{1} (\mathcal{L}\,u)\,\frac{|u'|^2}{u}\,\nu\,d\nu_d = \frac{d}{d+2}\int_{-1}^{1} \frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d - 2\,\frac{d-1}{d+2}\int_{-1}^{1} \frac{|u'|^2\,u''}{u}\,\nu^2\,d\nu_d$$

On $(-1,1)$, let us consider the *porous medium (fast diffusion)* flow

$$u_t = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$$

If $\kappa = \beta\,(p-2)+1$, the L^p norm is conserved

$$\frac{d}{dt}\int_{-1}^{1} u^{\beta p}\,d\nu_d = \beta\,p\,(\kappa - \beta\,(p-2) - 1)\int_{-1}^{1} u^{\beta(p-2)}\,|u'|^2\,\nu\,d\nu_d = 0$$

$f = u^\beta$, $\|f'\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right) \geq 0$?

$$\mathcal{A} := \int_{-1}^{1} |u''|^2\,\nu^2\,d\nu_d - 2\,\frac{d-1}{d+2}\,(\kappa+\beta-1)\int_{-1}^{1} u''\,\frac{|u'|^2}{u}\,\nu^2\,d\nu_d + \left[\kappa\,(\beta-1) + \frac{d}{d+2}\,(\kappa+\beta-1)\right]\int_{-1}^{1}\frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d$$

$\mathcal{A}$ is nonnegative for some β if

$$\frac{8\,d^2}{(d+2)^2}\,(p-1)\,(2^*-p) \geq 0$$

$\mathcal{A}$ is a sum of squares if $p \in (2, 2^*)$ for an arbitrary choice of β in a certain interval (depending on p and

$$\mathcal{A} = \int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2 \nu^2\,d\nu_d \geq 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

The rigidity point of view

Which computation have we done ? $u_t = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$

$$-\mathcal{L}\,u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\,d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\,d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\,d\nu_d$$

The two terms cancel and we are left only with the two-homogenous terms

Improvements of the inequalities (subcritical range)

- as long as the exponent is either in the range $(1, 2)$ or in the range $(2, 2^*)$, on can establish *improved inequalities*
- An improvement automatically gives an explicit stability result of the optimal functions in the (non-improved) inequality
- By duality, this provides a stability result for Keller-Lieb-Tirring inequalities

What does “improvement” mean ?

An *improved* inequality is

$$d\,\Phi\,(\mathrm{e}) \le \mathrm{i} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$$

for some function Φ such that $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi' > 0$ and $\Phi(s) > s$ for any s. With $\Psi(s) := s - \Phi^{-1}(s)$

$$\mathrm{i} - d\,\mathrm{e} \ge d\,(\Psi \circ \Phi)(\mathrm{e}) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$$

Lemma (Generalized Csiszár-Kullback inequalities)

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{d}{p-2}\left[\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right]$$

$$\ge d\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,(\Psi \circ \Phi)\left(\mathrm{C}\,\frac{\|u\|^{2\,(1-r)}_{\mathrm{L}^s(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\,\|u^r - \bar{u}^r\|^2_{\mathrm{L}^q(\mathbb{S}^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

$s(p) := \max\{2, p\}$ and $p \in (1,2)$: $q(p) := 2/p$, $r(p) := p$; $p \in (2,4)$: $q = p/2$, $r = 2$; $p \ge 4$: $q = p/(p-2)$, $r = p - 2$

Linear flow: improved Bakry-Emery method

Cf. [Arnold, JD]

$$w_t = \mathcal{L}\, w + \kappa\, \frac{|w'|^2}{w}\, \nu$$

With $2^\sharp := \frac{2\,d^2+1}{(d-1)^2}$

$$\gamma_1 := \left(\frac{d-1}{d+2}\right)^2 (p-1)\,(2^\# - p) \quad \text{if} \quad d>1\,, \quad \gamma_1 := \frac{p-1}{3} \quad \text{if} \quad d=1$$

If $p \in [1,2) \cup (2, 2^\sharp]$ and w is a solution, then

$$\frac{d}{dt}\,(\mathsf{i} - \, d\,\mathsf{e}) \le -\,\gamma_1 \int_{-1}^{1} \frac{|w'|^4}{w^2}\, d\nu_d \le -\,\gamma_1\, \frac{|\mathsf{e}'|^2}{1 - \,(p-2)\,\mathsf{e}}$$

Recalling that $\mathsf{e}' = -\,\mathsf{i}$, we get a differential inequality

$$\mathsf{e}'' + \, d\,\mathsf{e}' \ge \gamma_1\, \frac{|\mathsf{e}'|^2}{1 - \,(p-2)\,\mathsf{e}}$$

After integration: $d\,\Phi(\mathsf{e}(0)) \le \mathsf{i}(0)$

Nonlinear flow: the Hölder estimate of J. Demange

$$w_t = w^{2-2\beta}\left(\mathcal{L}\,w + \kappa\,\frac{|w'|^2}{w}\right)$$

For all $p \in [1, 2^*]$, $\kappa = \beta\,(p-2)+1$, $\frac{d}{dt}\int_{-1}^{1} w^{\beta p}\,d\nu_d = 0$

$$-\frac{1}{2\beta^2}\,\frac{d}{dt}\int_{-1}^{1}\left(|(w^\beta)'|^2\,\nu + \frac{d}{p-2}\,\left(w^{2\beta} - \overline{w}^{2\beta}\right)\right)d\nu_d \ge \gamma\int_{-1}^{1}\frac{|w'|^4}{w^2}\,\nu^2\,d\nu_d$$

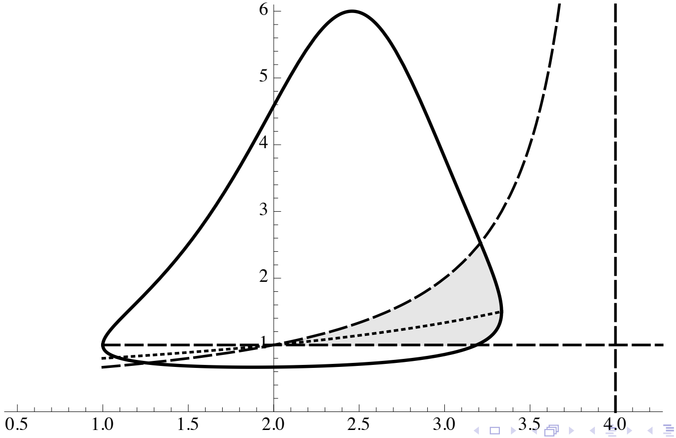
Lemma

For all $w \in \mathrm{H}^1((-1,1), d\nu_d)$, such that $\int_{-1}^{1} w^{\beta p}\,d\nu_d = 1$

$$\int_{-1}^{1}\frac{|w'|^4}{w^2}\,\nu^2\,d\nu_d \ge \frac{1}{\beta^2}\,\frac{\int_{-1}^{1}|(w^\beta)'|^2\,\nu\,d\nu_d\int_{-1}^{1}|w'|^2\,\nu\,d\nu_d}{\left(\int_{-1}^{1} w^{2\beta}\,d\nu_d\right)^{\delta}}$$

.... but there are conditions on β

Admissible (p, β) for $d = 5$



The line

- A first example of a non-compact manifold

Joint work with M.J. Esteban, A. Laptev and M. Loss

One-dimensional Gagliardo-Nirenberg-Sobolev inequalities

$$\|f\|_{\mathrm{L}^p(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|f'\|_{\mathrm{L}^2(\mathbb{R})}^{\theta}\,\|f\|_{\mathrm{L}^2(\mathbb{R})}^{1-\theta} \quad \text{if} \quad p\in(2,\infty)$$

$$\|f\|_{\mathrm{L}^2(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|f'\|_{\mathrm{L}^2(\mathbb{R})}^{\eta}\,\|f\|_{\mathrm{L}^p(\mathbb{R})}^{1-\eta} \quad \text{if} \quad p\in(1,2)$$

with $\theta=\frac{p-2}{2\,p}$ and $\eta=\frac{2-p}{2+p}$

The threshold case corresponding to the limit as $p\to 2$ is the logarithmic Sobolev inequality

$$\int_{\mathbb{R}} u^2\,\log\left(\frac{u^2}{\|u\|_{\mathrm{L}^2(\mathbb{R})}^2}\right)\,dx \le \frac{1}{2}\,\|u\|_{\mathrm{L}^2(\mathbb{R})}^2\,\log\left(\frac{2}{\pi\,e}\,\frac{\|u'\|_{\mathrm{L}^2(\mathbb{R})}^2}{\|u\|_{\mathrm{L}^2(\mathbb{R})}^2}\right)$$

If $p>2$, $u_\star(x)=(\cosh x)^{-\frac{2}{p-2}}$ solves

$$-(p-2)^2\,u''+4\,u-\,2\,p\,|u|^{p-2}\,u=0$$

If $p\in(1,2)$ consider $u_*(x)=(\cos x)^{\frac{2}{2-p}}$, $x\in(-\pi/2,\pi/2)$

Flow

Let us define on $\mathrm{H}^1(\mathbb{R})$ the functional

$$\mathcal{F}[v] := \|v'\|^2_{\mathrm{L}^2(\mathbb{R})} + \frac{4}{(p-2)^2}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R})} - \mathsf{C}\,\|v\|^2_{\mathrm{L}^p(\mathbb{R})} \quad \text{s.t. } \mathcal{F}[u_\star] = 0$$

With $z(x) := \tanh x$, consider the *flow*

$$v_t = \frac{v^{1-\frac{p}{2}}}{\sqrt{1-z^2}}\left[v'' + \frac{2\,p}{p-2}\,z\,v' + \frac{p}{2}\,\frac{|v'|^2}{v} + \frac{2}{p-2}\,v\right]$$

Theorem (Dolbeault-Esteban-Laptev-Loss)

Let $p \in (2,\infty)$. Then

$$\frac{d}{dt}\mathcal{F}[v(t)] \le 0 \quad \textit{and} \quad \lim_{t\to\infty} \mathcal{F}[v(t)] = 0$$

$$\tfrac{d}{dt}\mathcal{F}[v(t)] = 0 \quad \Longleftrightarrow \quad v_0(x) = u_\star(x - x_0)$$

Similar results for $p \in (1,2)$

The inequality ($p > 2$) and the ultraspherical operator

- *The problem on the line is equivalent to the critical problem for the ultraspherical operator*

$$\int_{\mathbb{R}} |v'|^2 \, dx + \frac{4}{(p-2)^2} \int_{\mathbb{R}} |v|^2 \, dx \ge \mathsf{C} \left(\int_{\mathbb{R}} |v|^p \, dx \right)^{\frac{2}{p}}$$

With

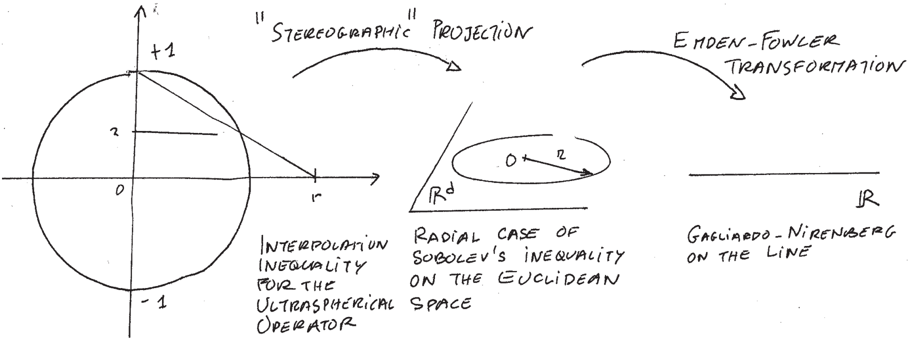
$$z(x) = \tanh x \,, \quad v_\star = (1-z^2)^{\frac{1}{p-2}} \quad \text{and} \quad v(x) = v_\star(x)\, f(z(x))$$

equality is achieved for $f = 1$ and, if we let $\nu(z) := 1 - z^2$, then

$$\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{2\,p}{(p-2)^2} \int_{-1}^{1} |f|^2 \, d\nu_d \ge \frac{2\,p}{(p-2)^2} \left(\int_{-1}^{1} |f|^p \, d\nu_d \right)^{\frac{2}{p}}$$

where $d\nu_p$ denotes the probability measure $d\nu_p(z) := \frac{1}{\zeta_p}\, \nu^{\frac{2}{p-2}}\, dz$

$$d = \frac{2\,p}{p-2} \quad \Longleftrightarrow \quad p = \frac{2\,d}{d-2}$$



Change of variables = stereographic projection + Emden-Fowler

Compact Riemannian manifolds

- no sign is required on the Ricci tensor and an improved integral criterion is established
- the flow explores the energy landscape... and shows the non-optimality of the improved criterion

Riemannian manifolds with positive curvature

$(\mathfrak{M}, g)$ is a smooth closed compact connected Riemannian manifold dimension d, no boundary, Δ_g is the Laplace-Beltrami operator $\mathrm{vol}(\mathfrak{M}) = 1$, $\mathfrak{R}$ is the Ricci tensor, $\lambda_1 = \lambda_1(-\Delta_g)$

$$\rho := \inf_{\mathfrak{M}} \inf_{\xi \in \mathbb{S}^{d-1}} \mathfrak{R}(\xi\,,\xi)$$

Theorem (Licois-Véron, Bakry-Ledoux)

Assume $d \geq 2$ and $\rho > 0$. If

$$\lambda \leq (1-\theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} \quad \textit{where} \quad \theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} > 0$$

then for any $p \in (2, 2^*)$, the equation

$$-\,\Delta_g v + \frac{\lambda}{p-2}\,\left(v - v^{p-1}\right) = 0$$

has a unique positive solution $v \in C^2(\mathfrak{M})$: $v \equiv 1$

Riemannian manifolds: first improvement

Theorem (Dolbeault-Esteban-Loss)

For any $p \in (1,2) \cup (2,2^*)$

$$0 < \lambda < \lambda_\star = \inf_{u \in \mathrm{H}^2(\mathfrak{M})} \frac{\int_{\mathfrak{M}} \left[(1-\theta)\,(\Delta_g u)^2 + \frac{\theta\,d}{d-1}\,\mathfrak{R}(\nabla u, \nabla u) \right] d\,v_g}{\int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g}$$

there is a unique positive solution in $C^2(\mathfrak{M})$*:* $u \equiv 1$

$\lim_{p \to 1_+} \theta(p) = 0 \Longrightarrow \lim_{p \to 1_+} \lambda_\star(p) = \lambda_1$ if ρ is bounded
$\lambda_\star = \lambda_1 = d\,\rho/(d-1) = d$ if $\mathfrak{M} = \mathbb{S}^d$ since $\rho = d-1$

$$(1-\theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} \le \lambda_\star \le \lambda_1$$

Riemannian manifolds: second improvement

$\mathrm{H}_g u$ denotes Hessian of u and $\theta = \dfrac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1}$

$$\mathrm{Q}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u - \frac{(d-1)\,(p-1)}{\theta\,(d+3-p)}\left[\frac{\nabla u\otimes\nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right]$$

$$\Lambda_\star := \inf_{u\in\mathrm{H}^2(\mathfrak{M})\setminus\{0\}} \frac{(1-\theta)\displaystyle\int_{\mathfrak{M}}(\Delta_g u)^2\,d\,v_g + \frac{\theta\,d}{d-1}\int_{\mathfrak{M}}\Big[\,\|\mathrm{Q}_g u\|^2 + \mathfrak{R}(\nabla u,\nabla u)\Big]}{\displaystyle\int_{\mathfrak{M}}|\nabla u|^2\,d\,v_g}$$

Theorem (Dolbeault-Esteban-Loss)

Assume that $\Lambda_\star > 0$. For any $p \in (1,2)\cup(2,2^*)$, the equation has a unique positive solution in $C^2(\mathfrak{M})$ if $\lambda \in (0,\Lambda_\star)$: $u \equiv 1$

Optimal interpolation inequality

For any $p \in (1,2) \cup (2,2^*)$ or $p = 2^*$ if $d \geq 3$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \frac{\lambda}{p-2}\left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})}\right] \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M})$$

Theorem (Dolbeault-Esteban-Loss)

Assume $\Lambda_\star > 0$. The above inequality holds for some $\lambda = \Lambda \in [\Lambda_\star, \lambda_1]$
If $\Lambda_\star < \lambda_1$, then the optimal constant Λ is such that

$$\Lambda_\star < \Lambda \leq \lambda_1$$

If $p = 1$, then $\Lambda = \lambda_1$

Using $u = 1 + \varepsilon\,\varphi$ as a test function where φ we get $\lambda \leq \lambda_1$
A minimum of

$$v \mapsto \|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} - \tfrac{\lambda}{p-2}\left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})}\right]$$

under the constraint $\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 1$ is negative if $\lambda > \lambda_1$

The flow

The key tools the flow

$$u_t = u^{2-2\,\beta}\left(\Delta_g u + \kappa\,\frac{|\nabla u|^2}{u}\right)\,,\quad \kappa = 1+\beta\,(p-2)$$

If $v = u^\beta$, then $\frac{d}{dt}\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 0$ and the functional

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2\,d\,v_g + \frac{\lambda}{p-2}\left[\int_{\mathfrak{M}} u^{2\,\beta}\,d\,v_g - \left(\int_{\mathfrak{M}} u^{\beta\,p}\,d\,v_g\right)^{2/p}\right]$$

is monotone decaying

- J. Demange, *Improved Gagliardo-Nirenberg-Sobolev inequalities on manifolds with positive curvature*, J. Funct. Anal., 254 (2008), pp. 593–611. Also see C. Villani, *Optimal Transport, Old and New*

Elementary observations (1/2)

Let $d \geq 2$, $u \in C^2(\mathfrak{M})$, and consider the trace free Hessian

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2\,d\,v_g = \frac{d}{d-1}\int_{\mathfrak{M}} \|\,\mathrm{L}_g u\,\|^2\,d\,v_g + \frac{d}{d-1}\int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u)\,d\,v_g$$

Based on the Bochner-Lichnerovicz-Weitzenböck formula

$$\frac{1}{2}\,\Delta\,|\nabla u|^2 = \|\mathrm{H}_g u\|^2 + \nabla(\Delta_g u)\cdot\nabla u + \mathfrak{R}(\nabla u, \nabla u)$$

Elementary observations (2/2)

Lemma

$$\int_{\mathfrak{M}} \Delta_g u \, \frac{|\nabla u|^2}{u} \, d\, v_g$$

$$= \frac{d}{d+2} \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, d\, v_g - \frac{2\, d}{d+2} \int_{\mathfrak{M}} [\mathrm{L}_g u] : \left[\frac{\nabla u \otimes \nabla u}{u} \right] d\, v_g$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, d\, v_g \geq \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 \, d\, v_g \quad \forall\, u \in \mathrm{H}^2(\mathfrak{M})$$

and λ_1 is the optimal constant in the above inequality

The key estimates

$$\mathcal{G}[u] := \int_{\mathfrak{M}} \left[\theta\,(\Delta_g u)^2 + (\kappa+\beta-1)\,\Delta_g u\,\frac{|\nabla u|^2}{u} + \kappa\,(\beta-1)\,\frac{|\nabla u|^4}{u^2}\right] d\,v_g$$

Lemma

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] = -\,(1-\theta)\int_{\mathfrak{M}} (\Delta_g u)^2\,d\,v_g - \mathcal{G}[u] + \lambda\int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g$$

$$\mathrm{Q}_g^\theta u := \mathrm{L}_g u - \tfrac{1}{\theta}\,\tfrac{d-1}{d+2}\,(\kappa+\beta-1)\left[\frac{\nabla u\otimes\nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right]$$

Lemma

$$\mathcal{G}[u] = \frac{\theta\,d}{d-1}\left[\int_{\mathfrak{M}} \|\mathrm{Q}_g^\theta u\|^2\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u,\nabla u)\,d\,v_g\right] - \mu\int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2}\,d\,v_g$$

with $\mu := \frac{1}{\theta}\left(\frac{d-1}{d+2}\right)^2(\kappa+\beta-1)^2 - \kappa\,(\beta-1) - (\kappa+\beta-1)\,\frac{d}{d+2}$

The end of the proof

Assume that $d \geq 2$. If $\theta = 1$, then μ is nonpositive if

$$\beta_-(p) \leq \beta \leq \beta_+(p) \quad \forall\, p \in (1, 2^*)$$

where $\beta_\pm := \frac{\mathsf{b} \pm \sqrt{\mathsf{b}^2 - \mathsf{a}}}{2\,\mathsf{a}}$ with $\mathsf{a} = 2 - p + \left[\frac{(d-1)\,(p-1)}{d+2}\right]^2$ and $\mathsf{b} = \frac{d+3-p}{d+2}$

Notice that $\beta_-(p) < \beta_+(p)$ if $p \in (1, 2^*)$ and $\beta_-(2^*) = \beta_+(2^*)$

$$\theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2) + p - 1} \quad \text{and} \quad \beta = \frac{d+2}{d+3-p}$$

Proposition

Let $d \geq 2$, $p \in (1,2) \cup (2, 2^*)$ ($p \neq 5$ or $d \neq 2$)

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] \leq (\lambda - \Lambda_\star) \int_{\mathfrak{M}} |\nabla u|^2\, d\,v_g$$

The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban

- Extension to compact Riemannian manifolds of dimension 2...

We shall also denote by $\mathfrak{R}$ the Ricci tensor, by $\mathrm{H}_g u$ the Hessian of u and by

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

the trace free Hessian. Let us denote by $\mathrm{M}_g u$ the trace free tensor

$$\mathrm{M}_g u := \nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2$$

We define

$$\lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M})\setminus\{0\}} \frac{\int_{\mathfrak{M}} \left[\|\, \mathrm{L}_g u - \frac{1}{2}\,\mathrm{M}_g u \,\|^2 + \mathfrak{R}(\nabla u, \nabla u) \right] e^{-u/2}\, d v_g}{\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, d v_g}$$

Theorem

Assume that $d = 2$ and $\lambda_\star > 0$. If u is a smooth solution to

$$-\frac{1}{2}\,\Delta_g u + \lambda = e^u$$

then u is a constant function if $\lambda \in (0, \lambda_\star)$

The Moser-Trudinger-Onofri inequality on $\mathfrak{M}$

$$\frac{1}{4}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \lambda \int_{\mathfrak{M}} u\,d\,v_g \geq \lambda\,\log\left(\int_{\mathfrak{M}} e^u\,d\,v_g\right) \quad \forall\,u \in \mathrm{H}^1(\mathfrak{M})$$

for some constant $\lambda > 0$. Let us denote by λ_1 the first positive eigenvalue of $-\Delta_g$

Corollary

If $d = 2$, then the MTO inequality holds with $\lambda = \Lambda := \min\{4\,\pi, \lambda_\star\}$. Moreover, if Λ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than Λ

The flow

$$\frac{\partial f}{\partial t} = \Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}$$

$$\mathcal{G}_\lambda[f] := \int_{\mathfrak{M}} \|\,\mathrm{L}_g f - \tfrac{1}{2}\,\mathrm{M}_g f\,\|^2\,e^{-f/2}\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla f,\nabla f)\,e^{-f/2}\,d\,v_g \\ - \lambda \int_{\mathfrak{M}} |\nabla f|^2\,e^{-f/2}\,d\,v_g$$

Then for any $\lambda \le \lambda_\star$ we have

$$\frac{d}{dt}\mathcal{F}_\lambda[f(t,\cdot)] = \int_{\mathfrak{M}} \left(-\tfrac{1}{2}\,\Delta_g f + \lambda\right)\left(\Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}\right)d\,v_g \\ = -\,\mathcal{G}_\lambda[f(t,\cdot)]$$

Since $\mathcal{F}_\lambda$ is nonnegative and $\lim_{t\to\infty}\mathcal{F}_\lambda[f(t,\cdot)] = 0$, we obtain that

$$\mathcal{F}_\lambda[u] \ge \int_0^\infty \mathcal{G}_\lambda[f(t,\cdot)]\,dt$$

Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

On the Euclidean space $\mathbb{R}^2$, given a general probability measure μ does the inequality

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \ge \lambda\left[\log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right)-\int_{\mathbb{R}^2} u\,d\mu\right]$$

hold for some $\lambda > 0$? Let

$$\Lambda_\star := \inf_{x\in\mathbb{R}^2}\frac{-\Delta\log\mu}{8\,\pi\,\mu}$$

Theorem

Assume that μ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda < \Lambda_\star$ and the inequality holds with $\lambda = \Lambda_\star$ if equality is achieved among radial functions

Caffarelli-Kohn-Nirenberg inequalities

Work in progress with M.J. Esteban and M. Loss

Caffarelli-Kohn-Nirenberg inequalities and the symmetry breaking issue

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d - 2 + 2\,(b-a)}$$

▷ *With*

$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c - a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

do we have $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry)*
or $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

The Emden-Fowler transformation and the cylinder

$$v(r,\omega) = r^{a-a_c}\,\varphi(s,\omega) \quad \text{with} \quad r = |x|\,, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r}$$

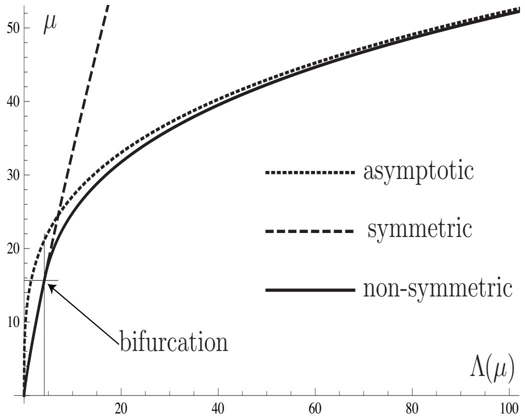
With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C}_1)} + \|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C}_1)} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C}_1)} \geq \mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C}_1)} \quad \forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda := (a_c - a)^2$, $\mathcal{C} = \mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda) = \frac{1}{\mathsf{C}_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda}$$

Numerical results



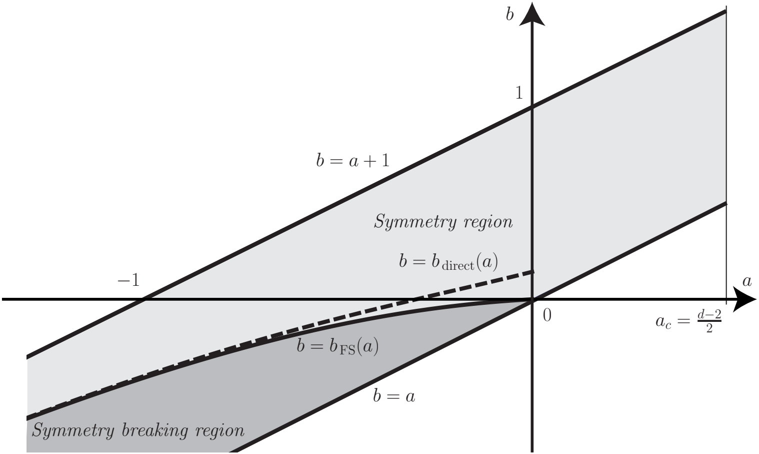
Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$, $\theta = 1$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as predicted by F. Catrina and Z.-Q. Wang*

The symmetry result

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$

Theorem

Let $d \geq 2$ and $p \leq 4$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the Caffarelli-Kohn-Nirenberg inequalities are radially symmetric



The Felli-Schneider region, or symmetry breaking region, appears in dark grey and is defined by $a < 0$, $a \leq b < b_{\mathrm{FS}}(a)$. *We prove that symmetry holds in the light grey region defined by* $b \geq b_{\mathrm{FS}}(a)$ *when* $a < 0$ *and for any* $b \in [a, a+1]$ *if* $a \in [0, a_c)$

Sketch of a proof

A change of variables

With $(r=|x|,\,\omega=x/r)\in\mathbb{R}^+\times\mathbb{S}^{d-1}$, the Caffarelli-Kohn-Nirenberg inequality is

$$\left(\int_0^\infty\int_{\mathbb{S}^{d-1}}|v|^p\;r^{d-b\,p}\,\frac{dr}{r}\;d\omega\right)^{\frac{2}{p}}\le C_{a,b}\int_0^\infty\int_{\mathbb{S}^{d-1}}|\nabla v|^2\;\;r^{d-2\,a}\,\frac{dr}{r}\;d\omega$$

Change of variables $r\mapsto r^\alpha$, $v(r,\omega)=w(r^\alpha,\omega)$

$$\alpha^{1-\frac{2}{p}}\left(\int_0^\infty\int_{\mathbb{S}^{d-1}}|w|^p\;r^{\frac{d-b\,p}{\alpha}}\,\frac{dr}{r}\;d\omega\right)^{\frac{2}{p}}$$
$$\le C_{a,b}\int_0^\infty\int_{\mathbb{S}^{d-1}}\left(\alpha^2\left|\frac{\partial w}{\partial r}\right|^2+\frac{1}{r^2}\,|\nabla_\omega w|^2\right)\;r^{\frac{d-2\,a-2}{\alpha}+2}\,\frac{dr}{r}\;d\omega$$

Choice of α

$$n=\frac{d-b\,p}{\alpha}=\frac{d-2\,a-2}{\alpha}+2$$

Then $p=\frac{2\,n}{n-2}$ is the critical Sobolev exponent associated with n

A Sobolev type inequality

The parameters α and n vary in the ranges $0<\alpha<\infty$ and $d<n<\infty$ and the *Felli-Schneider curve* in the (α,n) variables is given by

$$\alpha=\sqrt{\frac{d-1}{n-1}}=:\alpha_{\rm FS}$$

With

$$\mathsf{D}w=\left(\alpha\,\frac{\partial w}{\partial r},\frac{1}{r}\,\nabla_\omega w\right)\,,\quad d\mu:=r^{n-1}\,dr\,d\omega$$

the inequality becomes

$$\alpha^{1-\frac{2}{p}}\left(\int_{\mathbb{R}^d}|w|^p\,d\mu\right)^{\frac{2}{p}}\le\mathsf{C}_{a,b}\int_{\mathbb{R}^d}|\mathsf{D}w|^2\,d\mu$$

Proposition

Let $d\ge4$. Optimality is achieved by radial functions and $\mathsf{C}_{a,b}=\mathsf{C}^\star_{a,b}$ if $\alpha\le\alpha_{\rm FS}$

Gagliardo-Nirenberg inequalities on general cylinders: similar

Notations

When there is no ambiguity, we will omit the index $_\omega$ and from now on write that $\nabla = \nabla_\omega$ denotes the gradient with respect to the angular variable $\omega \in \mathbb{S}^{d-1}$ and that Δ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. We define the self-adjoint operator $\mathcal{L}$ by

$$\mathcal{L}\, w := -\,\mathsf{D}^*\,\mathsf{D}\, w = \alpha^2\, w'' + \alpha^2\, \frac{n-1}{r}\, w' + \frac{\Delta\, w}{r^2}$$

The fundamental property of $\mathcal{L}$ is the fact that

$$\int_{\mathbb{R}^d} w_1\, \mathcal{L}\, w_2\, d\mu = -\int_{\mathbb{R}^d} \mathsf{D} w_1 \cdot \mathsf{D} w_2\, d\mu \quad \forall\, w_1,\, w_2 \in \mathcal{D}(\mathbb{R}^d)$$

$\triangleright$ Heuristics: we look for a monotonicity formula along a well chosen nonlinear flow, based on the analogy with the decay of the Fisher information along the fast diffusion flow in $\mathbb{R}^d$

Fisher information

Let $u^{\frac{1}{2}-\frac{1}{n}} = |w| \quad \Longleftrightarrow \quad u = |w|^p,\ p = \frac{2\,n}{n-2}$

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u\,|\mathrm{Dp}|^2\,d\mu\,, \quad \mathrm{p} = \frac{m}{1-m}\,u^{m-1} \quad \text{and} \quad m = 1-\frac{1}{n}$$

Here $\mathcal{I}$ is the *Fisher information* and p is the *pressure function*

Proposition

With $\Lambda = 4\,\alpha^2/(p-2)^2$ and for some explicit numerical constant κ, we have

$$\kappa\,\mu(\Lambda) = \inf\left\{\mathcal{I}[u]\ :\ \|u\|_{\mathrm{L}^1(\mathbb{S}^d,d\nu_n)}\right\}$$

The fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{L}\, u^m\,, \quad m = 1 - \frac{1}{n}$$

Barenblatt self-similar solutions

$$u_\star(t,r,\omega) = t^{-n}\left(\mathsf{c}_\star + \frac{r^2}{2\,(n-1)\,\alpha^2\,t^2}\right)^{-n}$$

Lemma

$$\kappa\,\mu_\star(\Lambda) = \mathcal{I}[u_\star(t,\cdot)] \quad \forall\, t > 0$$

▷ Strategy:
1) prove that $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] \le 0$,
2) prove that $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = 0$ means that $u = u_\star$ up to a time shift

Decay of the Fisher information along the flow ?

$$\frac{\partial \mathsf{p}}{\partial t} = \frac{1}{n}\, \mathsf{p}\, \mathcal{L}\, \mathsf{p} - |\mathsf{D}\mathsf{p}|^2$$

$$\mathcal{Q}[\mathsf{p}] := \frac{1}{2}\, \mathcal{L}\, |\mathsf{D}\mathsf{p}|^2 - \mathsf{D}\mathsf{p} \cdot \mathsf{D}\mathcal{L}\, \mathsf{p}$$

$$\mathcal{K}[\mathsf{p}] := \int_{\mathbb{R}^d} \left(\mathcal{Q}[\mathsf{p}] - \frac{1}{n}\, (\mathcal{L}\, \mathsf{p})^2 \right) \mathsf{p}^{1-n}\, d\mu$$

Lemma

$$\frac{d}{dt} \mathcal{I}[u(t,\cdot)] = -\, 2\, (n-1)^{n-1}\, \mathcal{K}[\mathsf{p}]$$

If u is a critical point, then $\mathcal{K}[\mathsf{p}] = 0$

Boundary terms ! Regularity !

Proving decay (1/2)

$$\mathsf{k}[\mathsf{p}] := \mathcal{Q}(\mathsf{p}) - \frac{1}{n}(\mathcal{L}\,\mathsf{p})^2 = \frac{1}{2}\,\mathcal{L}\,|\mathsf{D}\mathsf{p}|^2 - \mathsf{D}\mathsf{p}\cdot\mathsf{D}\,\mathcal{L}\,\mathsf{p} - \frac{1}{n}(\mathcal{L}\,\mathsf{p})^2$$

$$\mathsf{k}_{\mathfrak{M}}[\mathsf{p}] := \frac{1}{2}\,\Delta\,|\nabla\mathsf{p}|^2 - \nabla\mathsf{p}\cdot\nabla\Delta\,\mathsf{p} - \tfrac{1}{n-1}\,(\Delta\,\mathsf{p})^2 - (n-2)\,\alpha^2\,|\nabla\mathsf{p}|^2$$

Lemma

Let $n\neq 1$ be any real number, $d\in\mathbb{N}$, $d\geq 2$, and consider a function $\mathsf{p}\in C^3((0,\infty)\times\mathfrak{M})$, where $(\mathfrak{M},g)$ is a smooth, compact Riemannian manifold. Then we have

$$\mathsf{k}[\mathsf{p}] = \alpha^4\left(1-\frac{1}{n}\right)\left[\mathsf{p}''-\frac{\mathsf{p}'}{r}-\frac{\Delta\,\mathsf{p}}{\alpha^2\,(n-1)\,r^2}\right]^2 + 2\,\alpha^2\,\frac{1}{r^2}\left|\nabla\mathsf{p}'-\frac{\nabla\mathsf{p}}{r}\right|^2 + \frac{1}{r^4}\,\mathsf{k}_{\mathfrak{M}}[\mathsf{p}]$$

Proving decay (2/2)

Lemma

Assume that $d \geq 3$, $n > d$ and $\mathfrak{M} = \mathbb{S}^{d-1}$. There is a positive constant $\zeta_\star$ such that

$$\int_{\mathbb{S}^{d-1}} \mathrm{k}_{\mathfrak{M}}[\mathrm{p}]\,\mathrm{p}^{1-n}\,d\omega \geq \left(\lambda_\star - (n-2)\,\alpha^2\right) \int_{\mathbb{S}^{d-1}} |\nabla \mathrm{p}|^2\,\mathrm{p}^{1-n}\,d\omega + \zeta_\star\,(n-d) \int_{\mathbb{S}^{d-1}} |\nabla \mathrm{p}|^4\,\mathrm{p}^{1-n}\,d\omega$$

Proof based on the Bochner-Lichnerowicz-Weitzenböck formula

Corollary

Let $d \geq 2$ and assume that $\alpha \leq \alpha_{\mathrm{FS}}$. Then for any nonnegative function $u \in \mathrm{L}^1(\mathbb{R}^d)$ with $\mathcal{I}[u] < +\infty$ and $\int_{\mathbb{R}^d} u\,d\mu = 1$, we have

$$\mathcal{I}[u] \geq \mathcal{I}_\star$$

A perturbation argument

- If u is a critical point of $\mathcal{I}$ under the mass constraint $\int_{\mathbb{R}^d} u\,d\mu = 1$, then

$$o(\varepsilon) = \mathcal{I}[u + \varepsilon\,\mathcal{L}\,u^m] - \mathcal{I}[u] = -\,2\,(n-1)^{n-1}\,\varepsilon\,\mathcal{K}[\mathsf{p}] + o(\varepsilon)$$

because $\varepsilon\,\mathcal{L}\,u^m$ is an admissible perturbation. Indeed, we know that

$$\int_{\mathbb{R}^d} (u + \varepsilon\,\mathcal{L}\,u^m)\,d\mu = \int_{\mathbb{R}^d} u\,d\mu = 1$$

and, as we take the limit as $\varepsilon \to 0$, $u + \varepsilon\,\mathcal{L}\,u^m$ makes sense and, in particular, is positive

- If $\alpha \le \alpha_{\rm FS}$, then $\mathcal{K}[\mathsf{p}] = 0$ implies that $u = u_\star$

A summary

- the sphere: the flow tells us what to do, and provides a simple proof (*choice of the exponents / of the nonlinearity*) once the problem is reduced to the ultraspherical setting + improvements

- [not presented here: Keller-Lieb-Thirring estimates] the spectral point of view on the inequality: how to measure the deviation with respect to the *semi-classical* estimates, a nice example of bifurcation (and *symmetry breaking*)

- *Riemannian manifolds:* no sign is required on the Ricci tensor and an improved integral criterion is established. We extend the theory from pointwise criteria to a non-local Schrödinger type estimate (Rayleigh quotient). The method generically shows the non-optimality of the improved criterion

- the flow is a nice way of exploring an energy space: it explain how to produce a good test function at *any* critical point. A *rigidity* result tells you that a local result is actually global because otherwise the flow would relate (far away) extremal points while keeping the energy minimal

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These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
▷ Lectures

Thank you for your attention !