Résultats de stabilité pour des inégalités fonctionnelles: flots paraboliques non-linéaires et méthodes d'entropie

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Introduction

- Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \mathscr{D}^{1,2}(\mathbb{R}^d) \qquad \text{(S)}$$

Equality holds on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

[Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d \, \|f\|_{2^*}^2 \geq c_{\mathrm{BE}} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

- How do we estimate c_{BE} ? as $d \to +\infty$?

Stability & improved entropy – entropy production inequalities
Improved inequalities & faster decay rates for entropies

Outline

1. Explicit stability for Sobolev and LSI on $\mathbb{R}^d$
   - Main results, optimal dimensional dependence; history
   - Proofs and other stability results
2. Results based on entropy methods and fast diffusion equations
   - Sobolev and HLS inequalities: duality and Yamabe flow
   - Rényi entropy powers & Stability for Gagliardo-Nirenberg-Sobolev inequalities
   - Stability, fast diffusion equation and entropy methods
3. Sphere, Gaussian measure, interpolation and log-Sobolev inequalities
   - Subcritical interpolation inequalities on the sphere
   - Gaussian interpolation inequalities on $\mathbb{R}^n$
   - More results on LSI and Caffarelli-Kohn-Nirenberg inequalities

Explicit stability results for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

Joint papers with M.J. Esteban, A. Figalli, R. Frank, M. Loss

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence
arXiv: 2209.08651, Cambridge J. Math. 2025

A short review on improvements and stability for some interpolation inequalities
arXiv: 2402.08527, Proc. ICIAM 2023

An explicit stability result for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

Theorem (JD, Esteban, Figalli, Frank, Loss)

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \to +\infty$

A stability result for the logarithmic Sobolev inequality

- Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{1}{4}\,d\,(d-2)\left(\|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$
$$\geq \frac{\beta}{d}\inf_{G\in\mathcal{M}(\mathbb{S}^d)}\left(\|\nabla F-\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\,d\,(d-2)\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

- Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d\to+\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi\,|x|^2}\,dx$

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta>0$ as in the result for the Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi\int_{\mathbb{R}^n} u^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)d\gamma$$
$$\geq \frac{\beta\,\pi}{2}\inf_{a\in\mathbb{R}^n,\,c\in\mathbb{R}}\int_{\mathbb{R}^n}|u-c\,e^{a\cdot x}|^2\,d\gamma$$

Stability for the Sobolev inequality: the history

▷ [Rodemich, 1969], [Aubin, 1976], [Talenti, 1976]
In the inequality $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, the optimal constant is

$$\mathsf{S}_d = \tfrac{1}{4}\,d\,(d-2)\,|\mathbb{S}^d|^{1-2/d}$$

with equality on the manifold $\mathcal{M} = \{g_{a,b,c}\}$ of the *Aubin-Talenti functions*

▷ [Lions] a qualitative stability result

$$\textit{if}\ \lim_{n\to\infty}\|\nabla f_n\|_2^2/\|f_n\|_{2^*}^2 = \mathsf{S}_d\,,\ \textit{then}\ \lim_{n\to\infty}\inf_{g\in\mathcal{M}}\|\nabla f_n - \nabla g\|_2^2/\|\nabla f_n\|_2^2 = 0$$

▷ [Brezis, Lieb, 1985] a quantitative stability result ?
▷ [Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|_2^2 \geq \mathsf{S}_d\,\|f\|_{2^*}^2 + c_{\mathrm{BE}}\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|_2^2$$

- The strategy of Bianchi & Egnell involves two steps:

– a local (spectral) analysis: the *neighbourhood* of $\mathcal{M}$
– a local-to-global extension based on concentration-compactness : the *far away regime*

- The constant c_{BE} is not explicit

Stability for the logarithmic Sobolev inequality

▷ [Gross, 1975] *Gaussian logarithmic Sobolev inequality* for $n \ge 1$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \ge \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$

▷ [Weissler, 1979] scale invariant (but dimension-dependent) version of the Euclidean form of the inequality

▷ [Stam, 1959], [Federbush, 1969], [Costa, 1985] *Cf.* [Villani, 2008]

▷ [Bakry, Emery, 1984], [Carlen, 1991] equality iff

$$u \in \mathscr{M} := \left\{w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R}\right\} \quad \text{where} \quad w_{a,c}(x) = c\, e^{a\cdot x} \quad \forall\, x \in \mathbb{R}^n$$

[Carlen, 1991] reinforcement of the inequality (Wiener transform)

▷ [McKean, 1973], [Beckner, 92] (LSI) as a large d limit of Sobolev

▷ [Bobkov, Gozlan, Roberto, Samson, 2014], [Indrei et al., 2014-23] stability in Wasserstein distance, in $\mathrm{W}^{1,1}$, *etc.*

▷ [JD, Toscani, 2016] Comparison with Weissler's form, a (dimension dependent) improved inequality

▷ [Fathi, Indrei, Ledoux, 2016] improved inequality assuming a Poincaré inequality (Mehler formula)

Proofs and other stability results

Colloquium of Rupert FRANK
(Ludwig Maximilians Universität München)
on
Sharp functional inequalities and their stability
(May 6, 2025)

Results based on entropy methods and fast diffusion equations

Sobolev and Hardy-Littlewood-Sobolev inequalities

- ▷ Stability in a weaker norm, with explicit constants
- ▷ From duality to improved estimates
- ▷ Fast diffusion equation with Yamabe's exponent
- ▷ Explicit stability constants

Joint paper with G. Jankowiak
Sobolev and Hardy–Littlewood–Sobolev inequalities
J. Differential Equations, 257, 2014

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d) \qquad \text{(S)}$$

and the Hardy-Littlewood-Sobolev inequality

$$\|g\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \mathsf{S}_d \int_{\mathbb{R}^d} g\,(-\Delta)^{-1} g\,dx \quad \forall\, g \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \qquad \text{(HLS)}$$

are dual of each other. Here S_d is the Aubin-Talenti constant, $2^* = \frac{2d}{d-2}$, $(2^*)' = \frac{2d}{d+2}$ and by the Legendre transform

$$\sup_{f \in \mathscr{D}^{1,2}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} f\,g\,dx - \frac{1}{2}\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right) = \frac{1}{2}\,\|g\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

$$\sup_{f \in \mathscr{D}^{1,2}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} f\,g\,dx - \frac{1}{2}\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \right) = \frac{1}{2} \int_{\mathbb{R}^d} g\,(-\Delta)^{-1} g\,dx$$

Improved Sobolev inequality by duality

Theorem

[JD, Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$
There exists a positive constant $\mathcal{C} \in [\frac{d}{d+4}, 1)$ such that

$$\|f^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \mathsf{S}_d \int_{\mathbb{R}^d} f^q\,(-\Delta)^{-1} f^q\,dx \\ \leq \mathcal{C}\,\mathsf{S}_d^{-1}\,\|f\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left(\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right)$$

for any $f \in \mathscr{D}^{1,2}(\mathbb{R}^d)$

$\mathcal{C} = 1$: "completion" of the square

$$0 \leq \int_{\mathbb{R}^d} \left| \|f\|^{\frac{4}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \nabla f - \mathsf{S}_d\,\nabla(-\Delta)^{-1} g \right|^2 dx$$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m\,, \quad t>0\,, \quad x\in\mathbb{R}^d \tag{Y}$$

Choice $m=\frac{d-2}{d+2}$ (Yamabe flow): $m+1=\frac{2\,d}{d+2}$

Proposition

Assume that $d\ge 3$ *and* $m=\frac{d-2}{d+2}$*. If* $u=v^m$ *and* v *is a solution of* (Y) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$*, then*

$$\frac12\,\frac{d}{dt}\left(\mathsf{S}_d^{-1}\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}-\int_{\mathbb{R}^d}v\,(-\Delta)^{-1}v\,dx\right)$$
$$=-\left(\int_{\mathbb{R}^d}v^{m+1}\,dx\right)^{\frac2d}\left(\mathsf{S}_d^{-1}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right)\le 0$$

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Lemma

[del Pino, Saez] For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\left|\frac{v(t,x)}{\overline{v}(t,x)}-1\right| = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

A convexity improvement

$$\mathsf{J}[v] := \int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\,dx \quad \text{and} \quad \mathsf{H}[v] := \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx$$

Theorem

[JD, Jankowiak] Assume that $d \ge 3$. Then we have

$$0 \le \mathsf{H}[v] + \mathsf{S}_d\,\mathsf{J}[v]^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}[v]^{\frac{2}{d}-1}\left(\mathsf{S}_d^{-1}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right)\right)$$

where $\varphi(x) := \sqrt{1+2\,x} - 1$ for any $x \ge 0$

Proof: with $\kappa_0 := -\,\mathsf{H}_0'/\mathsf{J}_0$ and $\mathsf{H} = \mathsf{Y}(\mathsf{J})$, consider the differential inequality

$$\mathsf{Y}'\left(\mathcal{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \le \frac{d+2}{2\,d}\,\mathcal{C}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,, \quad \mathsf{Y}(0) = 0\,, \quad \mathsf{Y}(\mathsf{J}_0) = \mathsf{H}_0$$

Rényi entropy powers, inequalities and flow, a formal approach

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

▷ *How do we relate Gagliardo-Nirenberg-Sobolev inequalities on* $\mathbb{R}^d$

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\geq\mathcal{C}_{\mathrm{GNS}}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\qquad\text{(GNS)}$$

and the fast diffusion equation

$$\frac{\partial u}{\partial t}=\Delta u^m\qquad\text{(FDE)}$$

Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With $m \geq m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\,dx = 0$$

(ii) *Second moment.* With $m > d/(d+2)$ and $u_0 \in \mathrm{L}^1_+\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2\,u(t,x)\,dx = 2\,d\int_{\mathbb{R}^d} u^m(t,x)\,dx$$

(iii) *Entropy estimate.* With $m \geq m_1 := (d-1)/d$, $u_0^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \in \mathrm{L}^1_+\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u^m(t,x)\,dx = \frac{m^2}{1-m}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\,dx \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy growth rate as a consequence of (GNS)

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\ge\mathcal{C}_{\mathrm{GNS}}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\qquad\text{(GNS)}$$

$$p=\frac{1}{2\,m-1}\quad\Longleftrightarrow\quad m=\frac{p+1}{2\,p}\in[m_1,1)$$

$u=f^{2\,p}$ so that $u^m=f^{p+1}$ and $u\,|\nabla u^{m-1}|^2=(p-1)^2\,|\nabla f|^2$

$$\mathcal{M}=\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{2\,p}\,,\quad\mathsf{E}[u]=\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{p+1}\,,\quad\mathsf{I}[u]=(p+1)^2\,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

If u solves (FDE) $\frac{\partial u}{\partial t}=\Delta u^m$, then $\mathsf{E}'=m\,\mathsf{I}$

$$\mathsf{E}'\ge\frac{p-1}{2\,p}\,(p+1)^2\,\mathcal{C}_{\mathrm{GNS}}^{\frac{2}{\theta}}\,\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{\frac{2}{\theta}}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{-\frac{2\,(1-\theta)}{\theta}}=C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d}u^m(t,x)\,dx\ge\left(\int_{\mathbb{R}^d}u_0^m\,dx+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\qquad\forall\,t\ge0$$

Self-similar solutions

$$\int_{\mathbb{R}^d} u^m(t,x)\,dx \ge \left(\int_{\mathbb{R}^d} u_0^m\,dx + \frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}} \quad \forall\, t \ge 0$$

Equality case is achieved if and only if, up to a normalisation and a translation

$$u(t,x) = \frac{c_1}{R(t)^d}\,\mathcal{B}\left(\frac{c_2\,x}{R(t)}\right)$$

where $\mathcal{B}$ is the ***Barenblatt self-similar solution***

$$\mathcal{B}(x) := \left(1+|x|^2\right)^{\frac{1}{m-1}}$$

Notice that $\mathcal{B} = \varphi^{2p}$ means that

$$\varphi(x) = \left(1+|x|^2\right)^{-\frac{1}{p-1}}$$

is an ***Aubin-Talenti profile***

Pressure variable and decay of the Fisher information

The derivative of the *Rényi entropy power* $\mathsf{E}^{\frac{2}{d}\,\frac{1}{1-m}-1}$ is proportional to

$$\mathsf{I}^\theta\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

- *Pressure variable*

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

- *Fisher information*

$$\mathsf{I}[u] = \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx$$

If u solves (FDE), then

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla \mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta \mathsf{P} + |\nabla \mathsf{P}|^2\Big)\,dx$$
$$= -\,2\int_{\mathbb{R}^d} u^m\,\Big(\|\mathrm{D}^2\mathsf{P}\|^2 - (1-m)\,(\Delta \mathsf{P})^2\Big)\,dx$$

Rényi entropy powers and interpolation inequalities

▷ Integrations by parts and completion of squares: with $m_1 = \frac{d-1}{d}$

$$-\frac{\mathsf{I}}{2\,\theta}\,\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^\theta\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)$$
$$=\int_{\mathbb{R}^d} u^m\left\|\mathrm{D}^2\mathsf{P}-\frac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right\|^2 dx+(m-m_1)\int_{\mathbb{R}^d} u^m\left|\Delta\mathsf{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2 dx$$

▷ Analysis of the asymptotic regime as $t\to+\infty$

$$\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^\theta\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{\mathcal{M}^{\frac{2\,\theta}{p}}}=\frac{\mathsf{I}[\mathcal{B}]^\theta\,\mathsf{E}[\mathcal{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathcal{B}\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\frac{2\,\theta}{p}}}=(p+1)^{2\,\theta}\,\mathcal{C}_{\mathrm{GNS}}^{2\,\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$\mathsf{I}[u]^\theta\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\geq(p+1)^{2\,\theta}\,\left(\mathcal{C}_{\mathrm{GNS}}\right)^{2\,\theta}\,\mathcal{M}^{\frac{2\,\theta}{p}}$$

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

Joint papers with M. Bonforte, B. Nazaret and N. Simonov

Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method
arXiv:2007.03674, *Memoirs of the AMS 308 (2025)*

Constructive stability results in interpolation inequalities and explicit improvements of decay rates of fast diffusion equations
DCDS, 43 (3&4): 1070–1089, 2023

Entropy – entropy production inequality

The fast diffusion equation on $\mathbb{R}^d$ in self-similar variables

$$\frac{\partial v}{\partial t} + \nabla \cdot \Big[v \left(\nabla v^{m-1} - 2\,x \right) \Big] = 0 \qquad \text{(FDE)}$$

admits a stationary Barenblatt solution $\mathcal{B}(x) := \left(1 + |x|^2\right)^{\frac{1}{m-1}}$

$$\frac{d}{dt} \mathcal{F}[v(t,\cdot)] = - \mathcal{I}[v(t,\cdot)]$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1} (v - \mathcal{B}) \right) dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} - \nabla \mathcal{B}^{m-1} \right|^2 dx$$

are such that $\mathcal{I}[v] \ge 4\,\mathcal{F}[v]$ [del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)] \le \mathcal{F}[v_0]\, e^{-4\,t}$$

Entropy growth rate

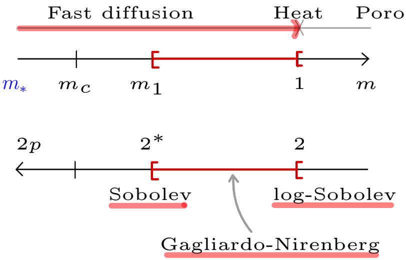
$\mathcal{I}[v] \ge 4\,\mathcal{F}[v] \Longleftrightarrow$ *Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta} \ge \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

with optimal constant. Under appropriate mass normalization

$v = f^{2\,p}$ so that $v^m = f^{p+1}$ and $v\,|\nabla v^{m-1}|^2 = (p-1)^2\,|\nabla f|^2$

$$p = \frac{1}{2\,m-1} \quad \Longleftrightarrow \quad m = \frac{p+1}{2\,p} \in [m_1, 1)$$



$m_* = \frac{d-2}{d+2}$

$m_1 = \frac{d-1}{d}$

$m_c = \frac{d-2}{d}$

Asymptotic regime as $t \to +\infty$

Take $f_\varepsilon := \mathcal{B}\,(1 + \varepsilon\,\mathcal{B}^{1-m}\,w)$ and expand $\mathcal{F}[f_\varepsilon]$ and $\mathcal{I}[f_\varepsilon]$ at order $O(\varepsilon^2)$

linearized free energy and *linearized Fisher information*

$$\mathsf{F}[w] := \frac{m}{2}\int_{\mathbb{R}^d} w^2\,\mathcal{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[w] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla w|^2\,\mathcal{B}\,dx$$

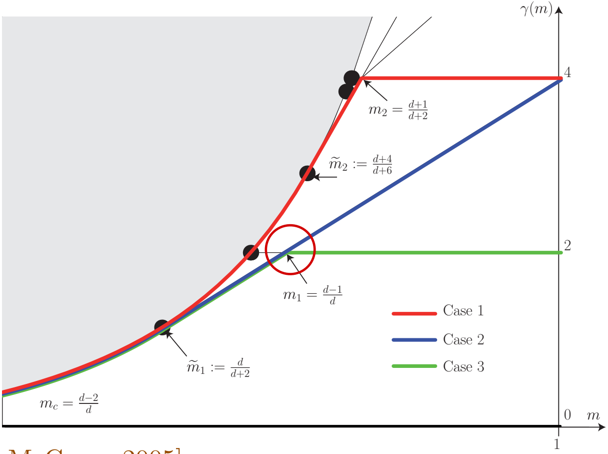
Proposition (Hardy-Poincaré inequality)

[BBDGV ,BDNS] Let $m \in [m_1, 1)$ if $d \ge 3$, $m \in (1/2, 1)$ if $d = 2$, and $m \in (1/3, 1)$ if $d = 1$. If $w \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^{2-m}\,dx)$ is such that $\nabla w \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}\,dx)$, $\int_{\mathbb{R}^d} w\,\mathcal{B}^{2-m}\,dx = 0$, then

$$\mathsf{I}[w] \ge 4\,\alpha\,\mathsf{F}[w]$$

with $\alpha = 1$, or $\alpha = 2 - d\,(1-m)$ if $\int_{\mathbb{R}^d} x\,w\,\mathcal{B}^{2-m}\,dx = 0$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

The asymptotic time layer improvement

Proposition

Let $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$, $\eta = 2\,(d\,m - d + 1)$ *and* $\chi = m/(266 + 56\,m)$. *If* $\int_{\mathbb{R}^d} v\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ *and*

$$(1-\varepsilon)\,\mathcal{B} \leq v \leq (1+\varepsilon)\,\mathcal{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, *then*

$$\mathcal{I}[v] \geq (4+\eta)\,\mathcal{F}[v]$$

Uniform convergence in relative error: threshold time

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $t_\star \geq 0$ such that, if u is a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \Big[v \left(\nabla v^{m-1} - 2\,x \right) \Big] = 0 \qquad \text{(FDE)}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\mathsf{A}[u_0] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \qquad (\mathrm{H}_A)$$

$\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} B\,dx = \mathcal{M}$, *then*

$$\sup_{x\in\mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq t_\star$$

The initial time layer improvement: backward estimate

By the *carré du champ* method, we have
Away from the Barenblatt solutions, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt} \le \mathcal{Q}\,(\mathcal{Q}-4)$$

Lemma

Assume that $m > m_1$ and v is a solution to (FDE) *with nonnegative initial datum v_0. If for some $\eta > 0$ and $t_\star > 0$, we have $\mathcal{Q}[v(t_\star,\cdot)] \ge 4+\eta$, then*

$$\mathcal{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}} \quad \forall\, t \in [0,t_\star]$$

Stability in Gagliardo-Nirenberg-Sobolev inequalities

Our strategy

Choose $\varepsilon > 0$*, small enough*

Get a threshold time $t_\star(\varepsilon)$

0 $\qquad$ $t_\star(\varepsilon)$ $\qquad$ t

Backward estimate	*Forward estimate*
by entropy methods	*based on a spectral gap*
Initial time layer	Asymptotic time layer

Two consequences (subcritical case)

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (FDE) *with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ and v_0 satisfies (H_A), then*

$$\mathcal{F}[v(t,.)] \leq \mathcal{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\, t \geq 0$$

▷ The ***stability of the entropy - entropy production inequality*** $\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

A constructive stability result (critical case)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \ge 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\ |x|\,f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Deficit of the Sobolev inequality: $\delta[f] := \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$

Theorem

Let $d \ge 3$ and $A > 0$. Then for any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, dx \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \le A$$

we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \frac{\mathcal{C}_\star(A)}{4+\mathcal{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathcal{C}_\star(A) = \mathcal{C}_\star(0)\left(1 + A^{1/(2\,d)}\right)^{-1}$ and $\mathcal{C}_\star(0) > 0$ depends only on d

From interpolation inequalities on the sphere to Gaussian interpolation inequalities

Joint work with G. Brigati and N. Simonov

Gaussian interpolation inequalities

arXiv:2302.03926

C. R. Math. Acad. Sci. Paris 41, 2024

Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results

Annales IHP, Analyse non linéaire, 362, 2023

arXiv: 2211.13180

Subcritical interpolation inequalities on the sphere

- ***Gagliardo-Nirenberg-Sobolev inequality***

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2,2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* = +\infty$ if $d = 1$ or 2

- Limit $p \to 2$: the ***logarithmic Sobolev inequality***

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu \ge \frac{d}{2}\,\mathcal{E}_2[F] := \frac{d}{2}\int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

[Bakry, Emery, 1984], [Bidaut-Véron, Véron, 1991], [Beckner, 1993]

Gagliardo-Nirenberg inequalities: stability

An improved inequality under orthogonality constraint and the stability inequality arising from the *carré du champ* method can be combined *in the subcritical case* as follows

Theorem

Let $d \geq 1$ and $p \in (1, 2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\,\mathcal{E}_p[F] \geq \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

▷ The result holds true for the logarithmic Sobolev inequality ($p = 2$), again with an explicit constant $\mathscr{S}_{d,2}$, for any finite dimension d

Carré du champ – admissible parameters on $\mathbb{S}^d$

[JD, Esteban, Kowalczyk, Loss] Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u} \right)$$

$$m_\pm(d,p) := \tfrac{1}{(d+2)\,p} \left(d\,p + 2 \pm \sqrt{d\,(p-1)\,(2\,d-(d-2)\,p)} \right)$$

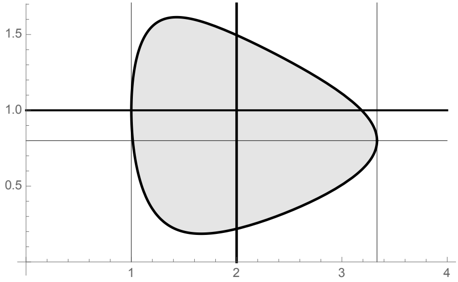


Figure: Case $d=5$: admissible parameters $1 \leq p \leq 2^* = 10/3$ and m (horizontal axis: p, vertical axis: m). Improved inequalities inside !

Admissible parameters

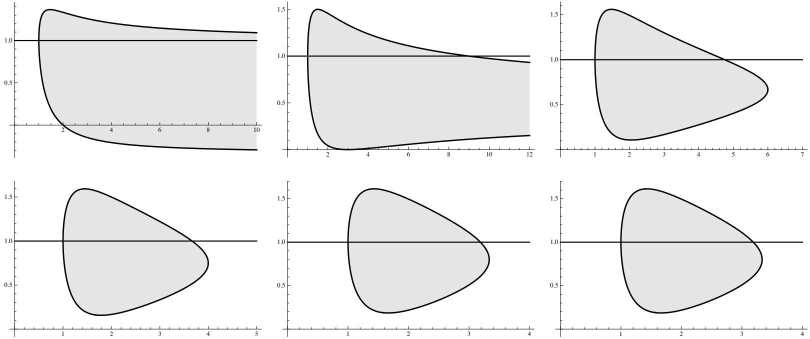


Figure: $d = 1, 2, 3$ (first line) and $d = 4, 5$ and 10 (second line): the curves $p \mapsto m_\pm(p)$ determine the admissible parameters (p, m)
[JD, Esteban, Kowalczyk, Loss 2014] [JD, Esteban, 2019]

$$m_\pm(d,p) := \frac{1}{(d+2)\,p}\left(d\,p + 2 \pm \sqrt{d\,(p-1)\,(2\,d-(d-2)\,p)}\right)$$

The "far away" regime and the "neighborhood" of $\mathcal{M}$

▷ If $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} / \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \ge \vartheta_0 > 0$, by the convexity of ψ

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right)$$
$$\ge \frac{d}{\vartheta_0}\,\psi\left(\frac{\vartheta_0}{d}\right)\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

▷ From now on, we assume that $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta_0\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$, take $\|F\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$, learn that

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta := \frac{d\,\vartheta_0}{d - (p-2)\,\vartheta_0} > 0$$

from the standard interpolation inequality and deduce from the Poincaré inequality that

$$\frac{d-\vartheta}{d} < \left(\int_{\mathbb{S}^d} F\,d\mu\right)^2 \le 1$$

Partial decomposition on spherical harmonics

$\mathscr{M} = \Pi_0 F$ and $\Pi_1 F = \varepsilon\,\mathscr{Y}$ where $\mathscr{Y}(x) = \sqrt{\frac{d+1}{d}}\,x\cdot\nu$, $\nu\in\mathbb{S}^d$

$$F = \mathscr{M}\,(1+\varepsilon\,\mathscr{Y}+\eta\,G)$$

Apply $c_{p,d}^{(-)}\,\varepsilon^6 \le \|1+\varepsilon\,\mathscr{Y}\|_{\mathrm{L}^p(\mathbb{S}^d)}^p - \left(1+a_{p,d}\,\varepsilon^2+b_{p,d}\,\varepsilon^4\right) \le c_{p,d}^{(+)}\,\varepsilon^6$ (with explicit constants) to $u = 1+\varepsilon\,\mathscr{Y}$ and $r = \eta\,G$ the estimate

$$\begin{aligned}
&\|u+r\|_{\mathrm{L}^p(\mathbb{S}^d)}^2 - \|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^2\\
&\quad\le \tfrac{2}{p}\,\|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^{2-p}\,\Big(p\int_{\mathbb{S}^d} u^{p-1}\,r\,d\mu + \tfrac{p}{2}\,(p-1)\int_{\mathbb{S}^d} u^{p-2}\,r^2\,d\mu\\
&\qquad\qquad + \textstyle\sum_{2<k<p} C_k^p\int_{\mathbb{S}^d} u^{p-k}\,|r|^k\,d\mu + K_p\int_{\mathbb{S}^d}|r|^p\,d\mu\Big)
\end{aligned}$$

Estimate $\int_{\mathbb{S}^d}(1+\varepsilon\,\mathscr{Y})^{p-1}\,G\,d\mu$, $\int_{\mathbb{S}^d}(1+\varepsilon\,\mathscr{Y})^{p-k}\,|G|^k\,d\mu$, etc. to obtain (under the condition that $\varepsilon^2+\eta^2\sim\vartheta$)

$$\begin{aligned}
\int_{\mathbb{S}^d}|\nabla F|^2\,d\mu - d\,\mathcal{E}_p[F] &\ge \mathscr{M}^2\left(A\,\varepsilon^4 - B\,\varepsilon^2\,\eta + C\,\eta^2 - \mathcal{R}_{p,d}\left(\vartheta^p+\vartheta^{5/2}\right)\right)\\
&\ge \mathscr{C}\left(\frac{\varepsilon^4}{\varepsilon^2+\eta^2+1}+\eta^2\right)
\end{aligned}$$

Large dimensional limit

Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)$$

Theorem

Let $v \in \mathrm{H}^1(\mathbb{R}^n, dx)$ with compact support, $d \geq n$ and

$$u_d(\omega) = v\left(\omega_1/\sqrt{d}, \omega_2/\sqrt{d}, \ldots, \omega_n/\sqrt{d}\right)$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. With $d\gamma(y) := (2\,\pi)^{-n/2}\, e^{-\frac{1}{2}\,|y|^2}\, dy$,

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

Gaussian interpolation inequalities on $\mathbb{R}^n$

- ***Beckner interpolation inequalities***

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

▷ $1 \leq p < 2$ [Beckner, 1989], [Bakry, Emery, 1984]
▷ Poincaré inequality corresponding: $p = 1$
▷ *Gaussian logarithmic Sobolev inequality* $p \to 2$

- ***Gaussian logarithmic Sobolev inequality*** $p \to 2$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \frac{1}{2}\int_{\mathbb{R}^n} |v|^2 \log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$

$d\gamma(y) := (2\,\pi)^{-n/2}\, e^{-\frac{1}{2}\,|y|^2}\, dy$

Gaussian carré du champ and nonlinear diffusion

$$\frac{\partial v}{\partial t} = v^{-p\,(1-m)} \left(\mathcal{L} v + (m\,p-1)\,\frac{|\nabla v|^2}{v} \right) \quad \text{on} \quad \mathbb{R}^n$$

Ornstein-Uhlenbeck operator: $\mathcal{L} = \Delta - x \cdot \nabla$

$$m_\pm(p) := \lim_{d\to+\infty} m_\pm(d,p) = 1 \pm \frac{1}{p}\sqrt{(p-1)\,(2-p)}$$

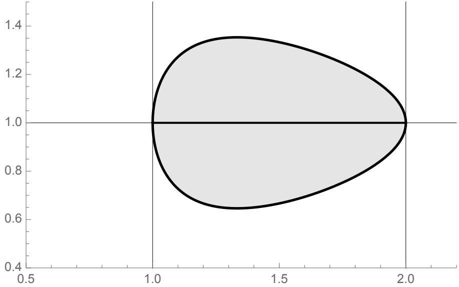


Figure: The admissible parameters $1 \le p \le 2$ and m are independent of n

A stability result for Gaussian interpolation inequalities

Theorem

For all $n \geq 1$, and all $p \in (1,2)$, there is an explicit constant $c_{n,p} > 0$ such that, for all $v \in \mathrm{H}^1(d\gamma)$,

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{p-2}\left(\|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\right)$$
$$\geq c_{n,p}\left(\|\nabla(\mathrm{Id}-\Pi_1)v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \frac{\|\nabla \Pi_1 v\|^4_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)$$

L^2 stability of LSI: comments

[JD, Esteban, Figalli, Frank, Loss]

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi \int_{\mathbb{R}^n} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma \\ \ge \frac{\beta\,\pi}{2} \inf_{a\in\mathbb{R}^d,\, c\in\mathbb{R}} \int_{\mathbb{R}^n} |u - c\, e^{a\cdot x}|^2\, d\gamma$$

- One dimension is lost (for the manifold of invariant functions) in the limiting process
- Euclidean forms of the stability
- The $\dot{\mathrm{H}}^1(\mathbb{R}^n)$ does not appear, it gets lost in the limit $d \to +\infty$
- $\int_{\mathbb{R}^n} |\nabla(u - c\, e^{a\cdot x})|^2\, d\gamma$? False, but makes sense under additional assumptions. Some results based on the Ornstein-Uhlenbeck flow and entropy methods: [Fathi, Indrei, Ledoux, 2016], [JD, Brigati, Simonov]
- Taking the limit is difficult because of the lack of compactness

More results on logarithmic Sobolev inequalities

Joint work with G. Brigati and N. Simonov

Stability for the logarithmic Sobolev inequality

Journal of Functional Analysis, 287, oct. 2024

arXiv: 2411.13271

Logarithmic Sobolev inequalities: a review on stability and instability results

arXiv: 2504.08658

▷ *Entropy methods, with constraints*

Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon\, x$ in the limit as $\varepsilon \to 0$

$$d(u_\varepsilon, 1)^2 = \|u_\varepsilon'\|^2_{\mathrm{L}^2(\mathbb{R}, d\gamma)} = \varepsilon^2 \quad \text{and} \quad \inf_{w \in \mathscr{M}} \mathrm{d}(u_\varepsilon, w)^\alpha \le \frac{1}{2}\,\varepsilon^4 + O(\varepsilon^6)$$

$\mathscr{M} := \left\{ w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R} \right\}$ where $w_{a,c}(x) = c\, e^{-a \cdot x}$

Proposition

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ and $\|x\, u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le d$, we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge \frac{1}{2\, d} \left(\int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \right)^2$$

and, with $\psi(s) := s - \frac{d}{4} \log\left(1 + \frac{4}{d}\, s\right)$,

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge \psi\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \right)$$

Stability under log-concavity

[Bakry, Ledoux '06], [Toscani '14], [JD, Toscani '16]

Lemma

Let $d \geq 1$*. With* $\varphi(t) := \frac{d}{4}\,\left[\exp\left(\frac{2\,t}{d}\right) - 1 - \frac{2\,t}{d}\right]$

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma - \frac{1}{2}\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma \\ \geq \varphi\left(\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma + \frac{d}{2} - \frac{1}{2}\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma\right)$$

for any $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ *such that* $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$

Counter-examples to the H^1 stability if $\|x\,u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > d$
[Indrei, Marcon '14], [Kim '18], [Kim, Indrei '21], [Indrei '21-'23], [Brigati, JD, Simonov '25]

Stability under log-concavity

Theorem

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $u^2\,\gamma$ is log-concave and such that

$$\int_{\mathbb{R}^d} (1,x)\,|u|^2\,d\gamma = (1,0) \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2\,|u|^2\,d\gamma \le \mathsf{K}$$

we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \frac{\mathscr{C}_\star}{2}\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,d\gamma \ge 0$$

$$\mathscr{C}_\star = 1 + \frac{1}{432\,\mathsf{K}} \approx 1 + \frac{0.00231481}{\mathsf{K}}$$

Self-improving Poincaré inequality and stability for LSI

[Fathi, Indrei, Ledoux, '16]

Theorem

Let $d \geq 1$. For any $\varepsilon > 0$, there is some explicit $\mathscr{C} > 1$ depending only on ε such that, for any $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ with

$$\int_{\mathbb{R}^d} (1, x)\, |u|^2\, d\gamma = (1, 0)\,, \quad \int_{\mathbb{R}^d} |u|^2\, e^{\,\varepsilon\,|x|^2}\, d\gamma < \infty$$

for some $\varepsilon > 0$, then we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \geq \frac{\mathscr{C}}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma$$

with $\mathscr{C} = 1 + \frac{\mathscr{C}_\star(\mathsf{K}_\star) - 1}{1 + R^2\, \mathscr{C}_\star(\mathsf{K}_\star)}$, $\mathsf{K}_\star := \max\left(d, \frac{(d+1)\, R^2}{1+R^2}\right)$ *if* $\operatorname{supp}(u) \subset B(0, R)$

Compact support: [Lee, Vázquez, '03]; [Chen, Chewi, Niles-Weed, '21]

Stability in Caffarelli-Kohn-Nirenberg inequalities ?

in collaboration with M. Bonforte, B. Nazaret and N. Simonov

Constructive stability results in interpolation inequalities and explicit improvements of decay rates of fast diffusion eq.

DCDS, 43 (3 & 4): 1070-1089, 2023

Subcritical Caffarelli-Kohn-Nirenberg inequalities

On $\mathbb{R}^d$ with $d \geq 1$, let us consider the *Caffarelli-Kohn-Nirenberg interpolation inequalities*

$$\|f\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \leq \mathcal{C}_{\beta,\gamma,p} \, \|\nabla f\|_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}^{\theta} \, \|f\|_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}^{1-\theta}$$

$$\gamma-2 < \beta < \frac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with} \quad p_\star := \frac{d-\gamma}{d-\beta-2}\,,$$

with $\theta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$

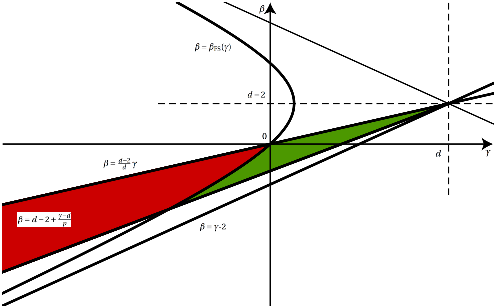
and $\|f\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f|^q \, |x|^{-\gamma} \, dx\right)^{1/q}$

Symmetry: equality is achieved by the *Aubin-Talenti type functions*

$$\mathsf{g}(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-\frac{1}{p-1}}$$

[JD, Esteban, Loss, Muratori, 2017] Symmetry holds if and only if

$$\gamma < d\,, \quad \text{and} \quad \gamma - 2 < \beta < \frac{d-2}{d}\,\gamma \quad \text{and} \quad \beta \leq \beta_{\mathrm{FS}}(\gamma)$$



$d = 4$ and $p = 6/5$: (γ, β) admissible region

V

An improved decay rate along the flow

In self-similar variables, with $m=(p+1)/(2\,p)$

$$|x|^{-\gamma}\,\frac{\partial v}{\partial t}+\nabla\cdot\left(|x|^{-\beta}\,v\,\nabla v^{m-1}\right)=\sigma\,\nabla\cdot\left(x\,|x|^{-\gamma}\,v\right)$$

$$\mathcal{F}[v]=\frac{2\,p}{1-p}\int_{\mathbb{R}^d}\left(v^{\frac{p+1}{2\,p}}-\mathrm{g}^{p+1}-\frac{p+1}{2\,p}\mathrm{g}^{1-p}\left(v-\mathrm{g}^{2p}\right)\right)|x|^{-\gamma}\,dx$$

Theorem

In the symmetry region, if $v\geq 0$ is a solution with a initial datum v_0 s.t.

$$A[v_0]:=\sup_{R>0}R^{\frac{2+\beta-\gamma}{1-m}-(d-\gamma)}\int_{|x|>R}v_0(x)\,|x|^{-\gamma}\,dx<\infty$$

then there are some $\zeta>0$ and some $T>0$ such that

$$\mathcal{F}[v(t,.)]\leq\mathcal{F}[v_0]\,e^{-\,(4\,\alpha^2+\zeta)\,t}\quad\forall\,t\geq 2\,T$$

[Bonforte, JD, Nazaret, Simonov, 2022]

Stability results for Sobolev, logarithmic Sobolev, and related inequalities
Proceedings of the Summer School "Direct and Inverse Problems with Applications, and Related Topics" August 19-23, 2024
arXiv: 2411.13271

These slides can be found at

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Thank you for your attention !