Nonlinear flows, entropy methods and applications

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Outline

1. Stability results based on entropy methods
    - Subcritical inequalities on $\mathbb{R}^d$ and on $\mathbb{S}^d$
    - Constructive stability results for GNS on $\mathbb{R}^d$ and $\mathbb{S}^d$
    - Constructive stability results for (log-)Sobolev inequalities
2. Interpolation inequalities, phase transitions and symmetry
    - Subcritical inequalities, bifurcation results and phase transition
    - Caffarelli-Kohn-Nirenberg inequalities
    - Symmetry results for spinors in dimension $d = 2$

Rényi entropy powers, inequalities and flow, a formal approach on $\mathbb{R}^d$

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

▷ *How do we relate Gagliardo-Nirenberg-Sobolev inequalities on* $\mathbb{R}^d$

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\ge\mathcal{C}_{\mathrm{GNS}}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\tag{GNS}$$

and the fast diffusion equation

$$\frac{\partial u}{\partial t}=\Delta u^m\tag{FDE}$$

Entropy growth rate as a consequence of (GNS)

- *Entropy functional* and *Fisher information functional*

$$\mathsf{E}[u]:=\int_{\mathbb{R}^d}u^m\,dx\quad\text{and}\quad\mathsf{I}[u]:=\frac{m}{1-m}\int_{\mathbb{R}^d}u\,|\nabla u^{m-1}|^2\,dx$$

If u solves (FDE) $\frac{\partial u}{\partial t}=\Delta u^m$, then $\mathsf{E}'=\mathsf{I}$

- *Gagliardo-Nirenberg-Sobolev inequalities*, $p=\frac{1}{2\,m-1}\iff m=\frac{p+1}{2\,p}$

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\ge\mathcal{C}_{\mathrm{GNS}}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\tag{GNS}$$

$u=f^{2\,p}$ so that $u^m=f^{p+1}$ and $u\,|\nabla u^{m-1}|^2=(p-1)^2\,|\nabla f|^2$

$$\mathcal{M}=\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{2\,p}\,,\quad\mathsf{E}[u]=\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{p+1}\,,\quad\mathsf{I}[u]=(p+1)^2\,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

- Best growth estimate in (FDE) means best constant in (GNS)

$$\mathsf{E}'\ge\frac{p-1}{2\,p}\,(p+1)^2\,\mathcal{C}_{\mathrm{GNS}}^{\frac{2}{\theta}}\,\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{\frac{2}{\theta}}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{-\frac{2\,(1-\theta)}{\theta}}=C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d}u^m(t,x)\,dx\ge\left(\int_{\mathbb{R}^d}u_0^m\,dx+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\quad\forall\,t\ge0$$

Exponents

- *Gagliardo-Nirenberg-Sobolev inequalities*

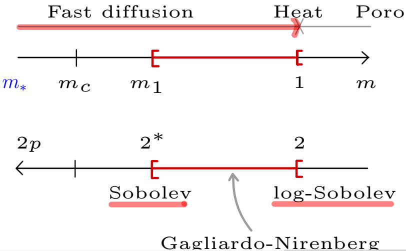
$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\ge\mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\qquad\text{(GNS)}$$

$u=f^{2\,p}$ so that $u^m=f^{p+1}$ and $u\,|\nabla v^{m-1}|^2=(p-1)^2\,|\nabla f|^2$

$$p=\frac{1}{2\,m-1}\quad\Longleftrightarrow\quad m=\frac{p+1}{2\,p}\in[m_1,1)$$

- Fast diffusion equation

$$\frac{\partial u}{\partial t}=\Delta u^m\qquad\text{(FDE)}$$



$m_*=\frac{d-2}{d+2}$

$m_1=\frac{d-1}{d}$

$m_c=\frac{d-2}{d}$

Self-similar solutions

If u solves (FDE) $\frac{\partial u}{\partial t} = \Delta u^m$

$$\int_{\mathbb{R}^d} u^m(t,x)\,dx \ge \left(\int_{\mathbb{R}^d} u_0^m\,dx + \frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}} \quad \forall\,t \ge 0$$

Equality case is achieved if and only if, up to a translation

$$u(t,x) = \frac{c_1}{R(t)^d}\,\mathcal{B}\left(\frac{c_2\,x}{R(t)}\right)$$

where $\mathcal{B}$ is the ***Barenblatt self-similar solution***

$$\mathcal{B}(x) := \left(1 + |x|^2\right)^{\frac{1}{m-1}}$$

The ***Aubin-Talenti profile*** $\varphi(x) = \left(1 + |x|^2\right)^{-\frac{1}{p-1}}$ such that $\varphi^{2p} = \mathcal{B}$ is optimal for (GNS)

A proof of (GNS) by entropy estimates

- Integrations by parts and completion of squares: with $m_1 = 1 - \frac{1}{d}$

$$-\frac{\mathsf{I}}{2\,\theta}\,\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^\theta\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)$$
$$=\int_{\mathbb{R}^d} u^m\left\|\mathrm{D}^2\mathsf{P}-\frac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right\|^2 dx+(m-m_1)\int_{\mathbb{R}^d} u^m\left|\Delta\mathsf{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2 dx$$

where $\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$ is the *pressure variable*

- Analysis of the asymptotic regime as $t \to +\infty$

$$\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^\theta\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{\mathcal{M}^{\frac{2\,\theta}{p}}}=\frac{\mathsf{I}[\mathcal{B}]^\theta\,\mathsf{E}[\mathcal{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathcal{B}\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\frac{2\,\theta}{p}}}=(p+1)^{2\,\theta}\,\mathcal{C}_{\mathrm{GNS}}^{2\,\theta}$$

$m \ge m_1$: we recover the Gagliardo-Nirenberg-Sobolev inequality

$$\mathsf{I}[u]^\theta\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\ge(p+1)^{2\,\theta}\,(\mathcal{C}_{\mathrm{GNS}})^{2\,\theta}\,\mathcal{M}^{\frac{2\,\theta}{p}}$$

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$

Joint papers with M. Bonforte, B. Nazaret and N. Simonov

Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method

arXiv:2007.03674, *Memoirs of the AMS 308 (2025)*

Constructive stability results in interpolation inequalities and explicit improvements of decay rates of fast diffusion equations

DCDS, 43 (3&4): 10701089, 2023

Entropy – entropy production inequality

The fast diffusion equation on $\mathbb{R}^d$ in ***self-similar variables***

$$\frac{\partial v}{\partial t}+\nabla\cdot\Big[v\left(\nabla v^{m-1}-2\,x\right)\Big]=0 \qquad \text{(FDE)}$$

admits a stationary Barenblatt solution $\mathcal{B}(x):=\big(1+|x|^2\big)^{\frac{1}{m-1}}$

$$\frac{d}{dt}\mathcal{F}[v(t,\cdot)]=-\,\mathcal{I}[v(t,\cdot)]$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v]:=-\,\frac{1}{m}\int_{\mathbb{R}^d}\left(v^m-\mathcal{B}^m-m\,\mathcal{B}^{m-1}\,(v-\mathcal{B})\right)\,dx$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\,\left|\nabla v^{m-1}-\nabla\mathcal{B}^{m-1}\right|^2\,dx$$

are such that $\mathcal{I}[v]\geq 4\,\mathcal{F}[v]\iff$ (GNS) [del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)]\leq\mathcal{F}[v_0]\,e^{-\,4\,t}$$

Asymptotic regime as $t \to +\infty$, spectral gap

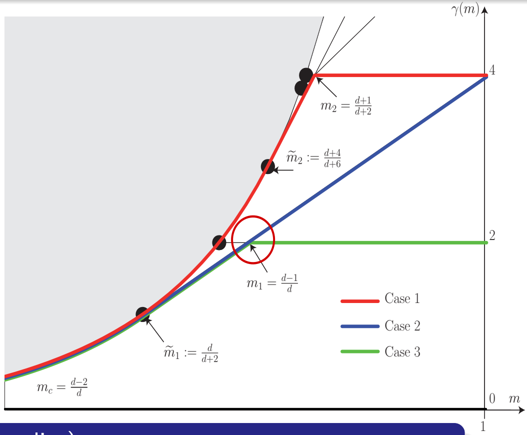
Linearized free energy
& linearized Fisher information

With $f_\varepsilon := \mathcal{B}\,(1 + \varepsilon\,\mathcal{B}^{1-m}\,w)$, expand $\mathcal{F}[f_\varepsilon]$, $\mathcal{I}[f_\varepsilon]$ at order $O(\varepsilon^2)$

$$\mathsf{F}[w] := \frac{m}{2}\int_{\mathbb{R}^d} w^2\,\mathcal{B}^{2-m}\,dx$$
$$I[w] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla w|^2\,\mathcal{B}\,dx$$

[Denzler, McCann, 2005], [BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015], [Denzler, Koch, McCann, 2015]

$m \in [m_1, 1)$ if $d \ge 3$, $m \in (1/2, 1)$ if $d = 2$, $m \in (1/3, 1)$ if $d = 1$



Proposition (Hardy-Poincaré inequality)

If $w \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^{2-m}\,dx)$ *with* $\nabla w \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}\,dx)$, $\int_{\mathbb{R}^d} w\,\mathcal{B}^{2-m}\,dx = 0$

$$\mathsf{I}[w] \ge 4\,\alpha\,\mathsf{F}[w]$$

with $\alpha = 1$, *or* $\alpha = 2 - d\,(1-m)$ *if* $\int_{\mathbb{R}^d} x\,w\,\mathcal{B}^{2-m}\,dx = 0$

Asymptotic and initial time layers

- The asymptotic time layer improvement

Proposition

Let $\chi = m/(266 + 56\,m)$. *If* $\int_{\mathbb{R}^d} v\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ *and*

$$(1-\varepsilon)\,\mathcal{B} \le v \le (1+\varepsilon)\,\mathcal{B} \text{ for some } \varepsilon \in (0, \chi\,\eta) \,,$$

then

$$\mathcal{I}[v] \ge (4+\eta)\,\mathcal{F}[v]$$

- The initial time layer improvement. As a consequence of the *carré du champ* method, if v solves (FDE), $\mathcal{Q}[v] := \mathcal{I}[v]/\mathcal{F}[v]$, then

$$\frac{d\mathcal{Q}}{dt} \le \mathcal{Q}\,(\mathcal{Q}-4)$$

Lemma

If for some $\eta > 0$ *and* $t_\star > 0$, *we have* $\mathcal{Q}[v(t_\star,\cdot)] \ge 4 + \eta$, *then*

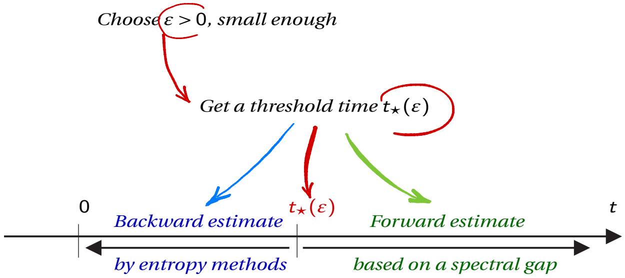
$$\mathcal{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}} \quad \forall\, t \in [0, t_\star]$$

Uniform convergence in relative error: threshold time

Theorem

[Bonforte, JD, Nazaret, Simonov, 2023-25] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$ and $A > 0$ be given. There exists an explicit threshold time $t_\star \geq 0$ such that, if v solves (FDE) with $\mathsf{A}[v(t=0,\cdot)] = \sup_{r>0} \frac{\int_{|x|>r} v(t=0,\cdot)\,dx}{\int_{|x|>r} \mathcal{B}\,dx} \leq A < \infty$, then

$$(1-\varepsilon)\,\mathcal{B} \leq v(t,\cdot) \leq (1+\varepsilon)\,\mathcal{B} \quad \text{for any } t \geq t_\star$$



Choose $\varepsilon > 0$, small enough

Get a threshold time $t_\star(\varepsilon)$

Backward estimate
by entropy methods

Forward estimate
based on a spectral gap

Two consequences (subcritical case)

▷ Improved decay rate for (FDE) in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, and $A > 0$. If v solves (FDE) with initial datum $v_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} v_0 \, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x \, v_0 \, dx = 0$ and v_0 satisfies (H_A), then

$$\mathcal{F}[v(t,.)] \leq \mathcal{F}[v_0] \, e^{-\,(4+\zeta)\,t} \quad \forall \, t \geq 0$$

▷ ***Stability of the entropy - entropy production inequality***

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

▷ Critical case (***Sobolev inequality***

… a refinement is needed to take care of the scale invariance

Constructive stability results in subcritical Gagliardo-Nirenberg inequalities on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2,2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* = +\infty$ if $d = 1$ or 2

Joint paper with G. Brigati and N. Simonov
Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results
Annales IHP, Analyse non linéaire, 362, 2023
arXiv: 2211.13180

Gagliardo-Nirenberg inequalities: stability

An improved inequality under orthogonality constraint (Π_1 is a projection on some positive spherical harmonic functions) and the stability inequality arising from the *carré du champ* method can be combined *in the subcritical case* as follows

Theorem (Brigati, JD, Simonov)

Let $d \geq 1$ *and* $p \in (1, 2^*)$*. For any* $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$*, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d\,\mathcal{E}_p[F] \geq \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla(\mathrm{Id} - \Pi_1)\,F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

▷ The result holds true for the logarithmic Sobolev inequality ($p = 2$), again with an explicit constant $\mathscr{S}_{d,2}$, for any finite dimension d

▷ A ***spectral*** estimate based on harmonic analysis (Funk-Hecke)

▷ The ***far away*** regime: use an improved interpolation inequality
If $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,/\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \ge \vartheta_0 > 0$, by the convexity of ψ

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right) \ge \frac{d}{\vartheta_0}\,\psi\left(\frac{\vartheta_0}{d}\right)\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

▷ The ***local*** case: $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta_0\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$
Take $\|F\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$, assume that $\frac{d\,\vartheta_0}{d-(p-2)\,\vartheta_0} > 0$ and deduce from the Poincaré inequality that

$$1 - \frac{\vartheta}{d} < \left(\int_{\mathbb{S}^d} F\,d\mu\right)^2 \le 1$$

+ a Taylor expansion using a partial decomposition on spherical harmonics

Constructive stability results for the Sobolev inequality

Joint papers with M.J. Esteban, A. Figalli, R. Frank, M. Loss

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence
arXiv: 2209.08651, Cambridge J. Math. 2025

A short review on improvements and stability for some interpolation inequalities
arXiv: 2402.08527, Proc. ICIAM 2023

+ Stability results for Sobolev, logarithmic Sobolev, and related inequalities
Proceedings of the Summer School “Direct and Inverse Problems with Applications, and Related Topics” August 19-23, 2024
arXiv: 2411.13271

Stability for the Sobolev inequality: the history

▷ [Rodemich, 1969], [Aubin, 1976], [Talenti, 1976] In the Sobolev inequality $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d\ \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$, the optimal constant is

$$\mathsf{S}_d = \tfrac{1}{4}\,d\,(d-2)\,|\mathbb{S}^d|^{1-2/d}$$

Equality holds on the manifold $\mathcal{M}$ of the *Aubin-Talenti functions*

▷ *Concentration compactness* [Lions]: a qualitative stability result

$$\textit{if}\ \lim_{n\to\infty}\|\nabla f_n\|_2^2/\|f_n\|_{2^*}^2 = \mathsf{S}_d\,,\ \textit{then}\ \lim_{n\to\infty}\inf_{g\in\mathcal{M}}\|\nabla f_n-\nabla g\|_2^2/\|\nabla f_n\|_2^2 = 0$$

▷ [Brezis, Lieb, 1985] a quantitative stability result ?

▷ [Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \geq c_{\mathrm{BE}}\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|_2^2$$

- The strategy of Bianchi & Egnell involves two steps:

– a local (spectral) analysis: the *neighbourhood* of $\mathcal{M}$

– a local-to-global extension based on concentration-compactness : the *far away regime*

- The constant c_{BE} is not explicit

An explicit stability result for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\,\big(a + |x-b|^2\big)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

Theorem (JD, Esteban, Figalli, Frank, Loss)

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathscr{D}^{1,2}(\mathbb{R}^d)$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d}\,\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \to +\infty$

▷ Using the inverse stereographic projection: same result on $\mathbb{S}^d$

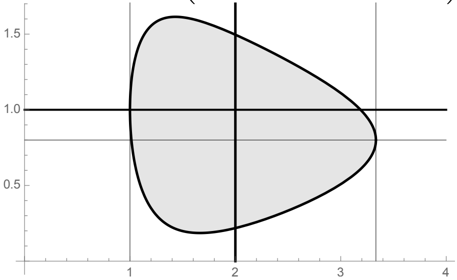
Carré du champ & admissible parameters on $\mathbb{S}^d$

[JD, Esteban, Kowalczyk, Loss, 2014] Monotonicity of the deficit

$$\delta[u] := \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[u] \quad \text{with} \quad \mathcal{E}_p[u] := \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

along the nonlinear diffusion flow

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)}\left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u}\right)$$



$d=5$: admissible $p \in (2, 2^*)$ and m (horizontal axis: p, vertical axis: m)

The global and the local problem

$$\mathsf{d}(u,v)^2 := \mathsf{q}[u-v] \quad \text{where} \quad \mathsf{q}[w] := \|\nabla w\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|w\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

- *deficit* : $\delta[u] := \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right)$, $\quad p = \frac{2\,d}{d-2}$
- *distance* to the set $\mathscr{M}$ of the Aubin-Talenti (optimal) functions

$$\mathsf{d}(u,\mathscr{M}) := \inf_{v\in\mathscr{M}} \mathsf{d}(u,v)$$

$\lim_{t\to+\infty}\mathsf{d}\big(u(t,\cdot),\mathscr{M}\big) = 0$ and $\delta[u(t,\cdot)]$ is monotone non-increasing if

$$\frac{\partial u}{\partial t} = m\,u^{(m-1)\,p}\left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u}\right)$$

For a given $\varepsilon\in(0,1)$, u is in the ***far away*** regime if

$$\mathsf{d}(u,\mathscr{M})^2 > \varepsilon\,\mathsf{q}[u]$$

and in the neighbourhood of $\mathscr{M}$ if $\mathsf{d}(u,\mathscr{M})^2 \le \varepsilon\,\mathsf{q}[u]$

Local stability : $\mathcal{I}(\varepsilon) := \inf\left\{\frac{\delta[u]}{\mathsf{d}(u,\mathscr{M})^2}\ :\ u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)\,,\ \mathsf{d}(u,\mathscr{M})^2 \le \varepsilon\,\mathsf{q}[u]\right\}$

A new proof for the global to local reduction

[Bonforte, JD], [JD, Esteban, Figalli, Frank, Loss], based on an idea by Christ. If we start in the ***far away*** regime, which means

$$\mathsf{d}(u_{|t=0},\mathscr{M})^2>\varepsilon\,\mathsf{q}[u_{|t=0}]$$

using $\mathsf{d}(u_{|t=0},\mathscr{M})\le\mathsf{d}(u_{|t=0},0)=\mathsf{q}[u_{|t=0}]$, $\|u(t,\cdot)\|_{\mathrm{L}^p(\mathbb{S}^d)}=1$ we obtain

$$\frac{\delta[u_{|t=0}]}{\mathsf{d}(u_{|t=0},\mathscr{M})^2}\ge\frac{\mathsf{q}[u_{|t=0}]-\frac{d}{p-2}}{\mathsf{q}[u_{|t=0}]}\ge1-\frac{\frac{d}{p-2}}{\mathsf{q}[u(t,\cdot)]}=\frac{\delta[u(t,\cdot)]}{\mathsf{q}[u(t,\cdot)]}$$

We know that

$$\lim_{t\to+\infty}\mathsf{q}[u(t,\cdot)]=\frac{d}{p-2}\quad\text{and}\quad\lim_{t\to+\infty}\mathsf{d}\big(u(t,\cdot),\mathscr{M}\big)^2=0$$

so that for some $t_*>0$ we have

$$\mathsf{q}[u(t_*,\cdot)]=\tfrac{1}{\varepsilon}\,\mathsf{d}\big(u(t_*,\cdot),\mathscr{M}\big)^2$$

$$\frac{\delta[u_{|t=0}]}{\mathsf{d}(u_{|t=0},\mathscr{M})^2}\ge\frac{\delta[u(t_*,\cdot)]}{\mathsf{q}[u(t_*,\cdot)]}=\varepsilon\,\frac{\delta[u(t_*,\cdot)]}{\mathsf{d}\big(u(t_*,\cdot),\mathscr{M}\big)^2}\ge\varepsilon\,\mathcal{I}(\varepsilon)$$

Large dimensional limit of (GNS) on $\mathbb{S}^d$

... based on the Maxwell-Poincaré lemma [McKean, 1973]
Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)$$

Proposition

Let $v \in \mathrm{H}^1(\mathbb{R}^n,dx)$ *with compact support,* $d \ge n$ *and*

$$u_d(\omega) = v\left(\omega_1/\sqrt{d},\omega_2/\sqrt{d},\ldots,\omega_n/\sqrt{d}\right)$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. *With* $d\gamma(y) := (2\,\pi)^{-n/2}\,e^{-\frac{1}{2}\,|y|^2}\,dy$*,*

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

A stability result for the logarithmic Sobolev inequality

- Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{1}{4}\,d\,(d-2)\left(\|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$
$$\geq \frac{\beta}{d}\inf_{G\in\mathcal{M}(\mathbb{S}^d)}\left(\|\nabla F-\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\,d\,(d-2)\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

- Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d\to+\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi\,|x|^2}\,dx$

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta>0$ as in the result for the Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi\int_{\mathbb{R}^n} u^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)d\gamma$$
$$\geq \frac{\beta\,\pi}{2}\inf_{a\in\mathbb{R}^n,\,c\in\mathbb{R}}\int_{\mathbb{R}^n}|u-c\,e^{a\cdot x}|^2\,d\gamma$$

Further results on logarithmic Sobolev inequalities

Joint papers with G. Brigati and N. Simonov
Stability for the logarithmic Sobolev inequality
Journal of Functional Analysis, 287, oct. 2024
arXiv: 2411.13271
Logarithmic Sobolev inequalities: a review on stability and instability results
La Matematica 5, 2026
arXiv: 2504.08658

▷ *Entropy methods, with constraints*

Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon\, x$ in the limit as $\varepsilon \to 0$

$$d(u_\varepsilon, 1)^2 = \|u_\varepsilon'\|^2_{\mathrm{L}^2(\mathbb{R}, d\gamma)} = \varepsilon^2 \quad \text{and} \quad \inf_{w \in \mathscr{M}} \mathrm{d}(u_\varepsilon, w)^\alpha \le \frac{1}{2}\,\varepsilon^4 + O\big(\varepsilon^6\big)$$

$\mathscr{M} := \big\{ w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R} \big\}$ where $w_{a,c}(x) = c\, e^{-a\cdot x}$

Proposition

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ and $\|x\, u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le d$, we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2}\int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge \frac{1}{2\,d}\left(\int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma\right)^2$$

and, with $\psi(s) := s - \frac{d}{4}\log\left(1 + \frac{4}{d}\, s\right)$,

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{1}{2}\int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge \psi\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)}\right)$$

Unconditional stability with worse exponents

[Bakry, Ledoux '06], [Toscani '14], [JD, Toscani '16]

Lemma

Let $d \geq 1$*. With* $\varphi(t) := \frac{d}{4}\,\left[\exp\left(\frac{2\,t}{d}\right) - 1 - \frac{2\,t}{d}\right]$

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma - \frac{1}{2}\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma$$
$$\geq \varphi\left(\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma + \frac{d}{2} - \frac{1}{2}\int_{\mathbb{R}^d} |x|^2\,|v|^2\,d\gamma\right)$$

for any $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ *such that* $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$

Counter-examples to the H^1 stability if $\|x\,u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > d$
[Indrei, Marcon '14], [Kim '18], [Kim, Indrei '21], [Indrei '21-'23], [Brigati, JD, Simonov '25]

Stability under log-concavity

Theorem

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $u^2\,\gamma$ is log-concave and such that

$$\int_{\mathbb{R}^d} (1,x)\,|u|^2\,d\gamma = (1,0) \quad \textit{and} \quad \int_{\mathbb{R}^d} |x|^2\,|u|^2\,d\gamma \le \mathsf{K}$$

we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \frac{\mathscr{C}_\star}{2} \int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,d\gamma \ge 0$$

$$\mathscr{C}_\star = 1 + \frac{1}{432\,\mathsf{K}} \approx 1 + \frac{0.00231481}{\mathsf{K}}$$

Self-improving Poincaré inequality and stability for LSI

[Fathi, Indrei, Ledoux, '16]

Stability under weak decay conditions

Theorem

Let $d \ge 1$. For any $\varepsilon > 0$, there is some explicit $\mathscr{C} > 1$ depending only on ε such that, for any $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ with

$$\int_{\mathbb{R}^d} (1, x)\, |u|^2\, d\gamma = (1, 0)\,,\; \int_{\mathbb{R}^d} |u|^2\, e^{\,\varepsilon\,|x|^2}\, d\gamma < \infty$$

for some $\varepsilon > 0$, then we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \ge \frac{\mathscr{C}}{2} \int_{\mathbb{R}^d} |u|^2\, \log |u|^2\, d\gamma$$

with $\mathscr{C} = 1 + \frac{\mathscr{C}_\star(\mathsf{K}_\star)-1}{1+R^2\,\mathscr{C}_\star(\mathsf{K}_\star)}$, $\mathsf{K}_\star := \max\left(d, \frac{(d+1)\,R^2}{1+R^2}\right)$ *if* $\mathrm{supp}(u) \subset B(0, R)$

Compact support: [Lee, Vázquez, '03]; [Chen, Chewi, Niles-Weed, '21]

Interpolation inequalities, phase transitions and symmetry

▷ Subcritical Gagliardo-Nirenberg-Sobolev inequalities on the sphere and classical bifurcation results

Other mechanisms of phase transition; the *carré du champ* method for the pressure variable

▷ Caffarelli-Kohn-Nirenberg inequalities: a proof of symmetry by the parabolic *carré du champ* method

▷ Magnetic rings, 2d spinors and problems with magnetic fields

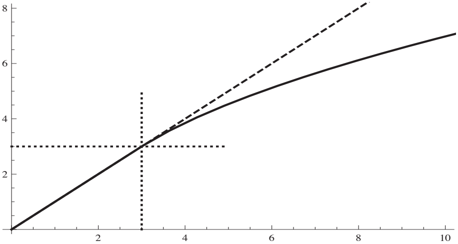
Critical and subcritical Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, classical bifurcation results and phase transition

Joint paper with E. Bou Dagher

Interpolation inequalities on the sphere and phase transition: rigidity, symmetry and symmetry breaking

arXiv:2210.16878

Bifurcation and phase transition in GNS inequalities



$\lambda \mapsto \mu(\lambda)$ *on* $\mathbb{S}^d$ *with* $d = 3$

$$\boxed{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{\mu(\lambda)}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > d$$

▷ The inequality holds with $\mu(\lambda) = \lambda = d$ [Bakry, Emery, 1985] [Beckner, 1993], [Bidaut-Véron, Véron, 1991, Corollary 6.1]

GNS as entropy-entropy production inequalities on $\mathbb{S}^d$

- ***(subcritical) Gagliardo-Nirenberg-Sobolev inequality***

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[F] := \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2,2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* = +\infty$ if $d = 1$ or 2

- Limit $p \to 2$: the ***logarithmic Sobolev inequality***

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu \ge \frac{d}{2}\,\mathcal{E}_2[F] := \frac{d}{2}\int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$

- $p = 1$: ***Poincaré inequality***

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_1[F] := d\left(\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^1(\mathbb{S}^d)}\right)$$

Another *Gagliardo-Nirenberg-Sobolev* inequality

$$\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\tfrac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^\theta\,\|u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}\ge\left(\tfrac{\mu(p,\theta,\lambda)}{p-2}\right)^\theta\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

- *Symmetry* holds if $\mu(p,\theta,\lambda)=\lambda$, optimal functions are constant
- *Symmetry breaking* if $\lambda>d\,\theta$: take $u_\varepsilon:=1+\varepsilon\,\varphi$, $\Delta\varphi+d\,\varphi=0$

Bakry-Emery exponent : $2^\#:=+\infty$ if $d=1$, $2^\#:=(2\,d^2+1)/(d-1)^2$ if $d\ge2$

and take $p\in(2,2^\#]$

$$\theta^\#:=3\,\frac{p-2}{4\,p-7}\quad\text{if } d=1\,,\qquad\frac{1}{\theta^\#}:=1+\frac{(p-1)\,(2^\#-p)}{p-2}\,\left(\frac{d-1}{d+2}\right)^2\ \text{if } d\ge2$$

Proposition (JD, Esteban)

Let $d\ge1$, $p\in(2,2^\#)$, and $\theta^\#\le\theta\le1$. The function $\lambda\mapsto\mu(p,\theta,\lambda)$ is monotone increasing, concave and $\mu(p,\theta,\lambda)<\lambda$ if and only if $\lambda>d\,\theta$

Parameter range

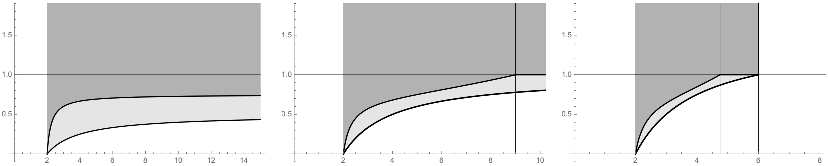
Theorem (Bou Dagher, JD)

Let $d \geq 1$, $p \in (2, 2^*)$ and $\theta_\star := d\,(p-2)/(2\,p) < \theta < \infty$
The function $\lambda \mapsto \mu(p,\theta,\lambda)$ is monotone increasing, concave

$$\mu(p,\theta,\lambda) \sim \kappa\,\lambda^{1-\theta_\star/\theta} \quad \text{as} \quad \lambda \to +\infty$$

$$\mu(p,\theta,\lambda) \leq \lambda \text{ and } \mu(p,\theta,\lambda) < \lambda \text{ if } \lambda > d\,\theta$$

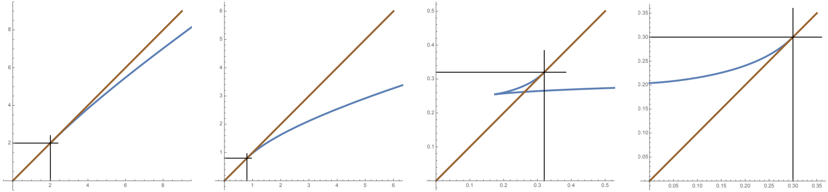
$$\mu(p,\theta,\lambda) = \lambda \text{ if } \lambda \leq d\,\theta\,,\ \theta^\# \leq \theta \leq 1\,,\ p \in (2, 2^\#] \text{ or } p > 2 \text{ if } d = 1$$



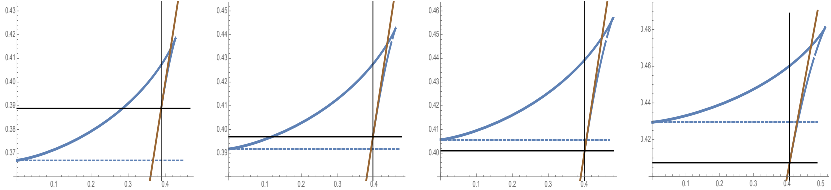
horizontal axis: p, vertical axis: θ

in dimensions $d=1$, $d=2$ and $d=3$ (from left to right)

Second and first order phase transitions



$d = 1$, $p = 5$, $\theta = 2$: $\theta = 0.8$, $\theta = 0.32$ and $\theta = \theta_\star = 0.3$

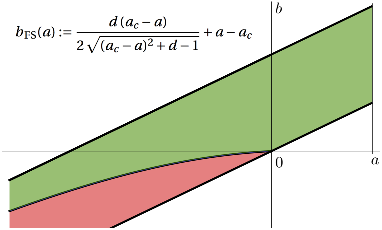


Critical case: $d = 1$, $\theta = \theta_\star$, for $p = 9.0$, 9.7, 10.1 and 10.8

Caffarelli-Kohn-Nirenberg inequalities

Joint paper with E. Bou Dagher
Caffarelli-Kohn-Nirenberg inequalities, parabolic carré du champ estimates and symmetry results for weighted interpolation inequalities on the sphere
(work in progress)

The critical Caffarelli-Kohn-Nirenberg inequality



$$b_{\rm FS}(a):=\frac{d\,(a_c-a)}{2\sqrt{(a_c-a)^2+d-1}}+a-a_c$$

$$\left(\int_{\mathbb R^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}\le\mathsf C_{a,b}\int_{\mathbb R^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

$a\le b\le a+1$, $a<a_c$, $d\ge3$

$p=\frac{2\,d}{d-2+2\,(b-a)}>0$, $a_c=\frac12\,(d-2)$

$\triangleright$ *A radial optimal function:*
$v_\star(x)=\left(1+|x|^{(p-2)\,(a_c-a)}\right)^{-2/(p-2)}$
among radially symmetric functions

Theorem (JD, Esteban, Loss, 2015)

There is *symmetry*, *i.e.*, $\mathsf C_{a,b}=\mathsf C^\star_{a,b}$, and all optimal functions are radially symmetric if $b_{\rm FS(a)}\le b<a+1$. If $a<b<b_{\rm FS}(a)$, then there is *symmetry breaking*, $\mathsf C_{a,b}>\mathsf C^\star_{a,b}$, and optimal functions are not radially symmetric.

[Caffarelli, Kohn, Nirenberg (1984)], [F. Catrina, Z.-Q. Wang (2001)]
[Smets, Willem], [Catrina, Wang], [Felli, Schneider]
[Bonforte, JD, Nazaret, Muratori]

A new proof: rewriting of CKN

1) Change of variables: $v(r,\omega)=u\big(r^\alpha,\omega\big)$, $\mathsf{D}_\alpha u=(\alpha\,\partial_r u,\nabla_\omega u)$

$$\int_{\mathbb{R}^d}|\mathsf{D}_\alpha u|^2\,|x|^{n-d}\,dx\ge\mathcal{C}_{\alpha,n}\left(\int_{\mathbb{R}^d}|u|^p\,|x|^{n-d}\,dx\right)^{2/p}$$

with $n=2\,p/(p-2)$. Symmetry means that the Aubin-Talenti function $u_*(x):=(1+|x|^2)^{-(n-2)/2}$ realizes the equality case

2) Relative measure: with $w=u/u_*$ and $d\mu_q(x)=|u_*(x)|^q\,|x|^{n-d}\,dx$

$$\int_{\mathbb{R}^d}|\mathsf{D}_\alpha w|^2\,d\mu_2\,dx+\tfrac14\,\alpha^2\,n\,(n-2)\int_{\mathbb{R}^d}|w|^2\,d\mu_p\,dx\ge\mathcal{C}_{\alpha,n}\left(\int_{\mathbb{R}^d}|w|^p\,d\mu_p\,dx\right)^{2/p}$$

3) Stereographic projection: $w(x)=f(z,\omega)$ with $z=\frac{1-|x|^2}{1+|x|^2}$, $\omega=\frac{2\,x}{1+|x|^2}$

$$\int_{\mathbb{S}^d}\left(\alpha^2\big(1-z^2\big)|f'|^2+\frac{|\nabla_\omega f|^2}{1-z^2}\right)d\sigma_n+\frac{\alpha^2}{4}\,n\,(n-2)\int_{\mathbb{S}^d}|f|^2\,d\sigma_n$$

$$\ge\mathcal{K}_{\alpha,n}\left(\int_{\mathbb{S}^d}|f|^p\,d\sigma_n\right)^{2/p}$$

$$d\sigma_n=Z_n^{-1}\left(1-z^2\right)^{(n-2)/2}dz\,d\omega,\;z\in[-1,+1],\;\omega\in\mathbb{S}^{d-1}$$

A new proof: fast diffusion equation and *carré du champ*

Let $'$ and ∇ denote the derivatives with respect to $z \in [-1,1]$ and $\omega \in \mathbb{S}^{d-1}$, $\Delta = \nabla \cdot \nabla$ and

$$\mathbf{D}v := \left(\alpha\sqrt{1-z^2}\,v', \frac{1}{\sqrt{1-z^2}}\nabla v\right), \quad \mathbf{L}v := \mathbf{D}\cdot\mathbf{D}v$$

$$\mathbf{L}v = \alpha^2\,\mathcal{L}v + \frac{1}{1-z^2}\,\Delta v, \quad \mathcal{L}v := \left(1-z^2\right)v'' - n\,z\,v'$$

Weighted fast diffusion equation

$$\frac{\partial v}{\partial t} = \mathbf{L}v^m = -\,\mathbf{D}\cdot(v\,\mathbf{D}\mathrm{P}), \quad \mathrm{P} = \frac{m}{1-m}\,v^{m-1}, \quad m = \frac{n-1}{n}, \quad p = \frac{2\,n}{n-2}$$

$$v = u^p \quad \text{and} \quad \mathcal{D}(t) := \int_{\mathbb{S}^d} |\mathbf{D}u(t,\cdot)|^2\,d\sigma_n + \frac{n\,\alpha^2}{p-2}\int_{\mathbb{S}^d} |u(t,\cdot)|^2\,d\sigma_n$$

Proposition (Bou Dagher, JD)

$$\mathcal{D}'(t) \le 0 \text{ if } \alpha \le \alpha_{\rm FS} := \sqrt{\tfrac{d-1}{n-1}}$$

- Nonlinear *carré du champ* techniques and Felli & Schneider (FS)
- [JD, Zhang, 2021]: weights $(1+\varepsilon-z^2)$ $\Longrightarrow$ First parabolic proof

Symmetry results for spinors in dimension two

Symmetry results for spinors in dimension $d = 2$

- the $d = 2$ *spinorial Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}^2} \frac{|\sigma \cdot \nabla \psi|^2}{|x|^{2\alpha}}\,dx \ge C_{\alpha,p} \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^{\beta p}}\,dx \right)^{2/p} \qquad \text{(SCKN)}$$

for spinor valued functions $\psi : \mathbb{R}^2 \to \mathbb{C}^2$

▷ the *logarithmic Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}} \int_{\mathbb{S}^1} \left(|\partial_s \phi(s,\theta)|^2 + |(\alpha - i\sigma_3 \partial_\theta)\phi(s,\theta)|^2 \right) ds\,d\theta \ge C_{\alpha,p} \left(\int_{\mathbb{R}} \int_{\mathbb{S}^1} |\phi(s,\theta)|^p\,ds\,d\theta \right)^{2/p}$$

- *Interpolation inequalities for Aharonov–Bohm magnetic fields* $A(x) = (x_2, -x_1)/|x|^2$

$$\int_{\mathbb{R}^2} |(-i\nabla - \alpha A)\psi|^2\,dx \ge C^{\mathrm{AB}}_{\alpha,p} \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2}\,dx \right)^{2/p} \qquad \text{(AB)}$$

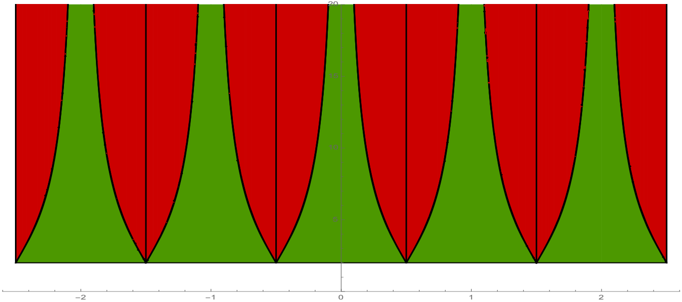
Theorem (JD, Frank, Weixler)

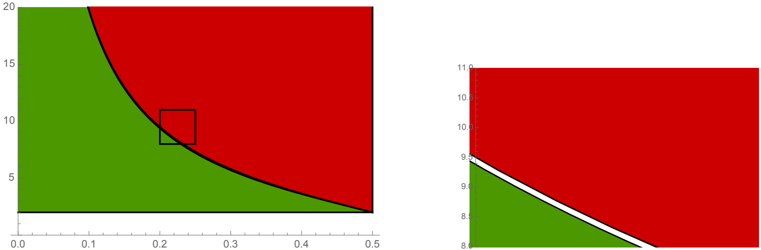
$C_{\alpha,p} = C^{\mathrm{AB}}_{\alpha,p}$ for any $(\alpha, p) \in (0, 1/2) \times (2, +\infty)$

Symmetry *versus* symmetry breaking

Theorem (JD, Frank, Weixler)

- For every $\alpha \in (0, 1/2)$ and $p > 2$, there is an optimizer with $C_{\alpha,p} > 0$ and $\lim_{\alpha\to 0_+} C_{\alpha,p} = 0$. Symmetry holds if and only if $\alpha \in \big(0, \alpha(p)\big]$ for some function $p \mapsto \alpha(p) : (2, \infty) \to (0, 1/2)$
- The symmetry and symmetry breaking regions are symmetric with respect to $\alpha = 0$ and 1-periodic





(SCKN) *with* $d = 2$. *Horizontal axis:* $\alpha \in (0, 1/2)$. *Vertical axis:* $p \in (2, \infty)$

- *Symmetry range: green, by the equivalence with Aharonov-Bohm problem and entropy methods for flows associated to (CKN) inequalities*
- *Symmetry breaking range: red and blue; Undecided in the tiny white gap*

- magnetic ring: an interpolation inequality on $\mathbb{S}^1$
[JD, Esteban, Laptev, Loss]
- Aharonov-Bohm and Caffarelli-Kohn-Nirenberg inequalities
[Bonheure, JD, Esteban, Laptev, Loss]
- a Gegenbauer polynomial basis to study linear instability
[JD, Frank, Weixler]

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Thank you for your attention !