Singular limit problems related to the Onofri inequality

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Singular limit problems in nonlinear PDEs
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1. Weighted Onofri inequality and Caffarelli-Kohn-Nirenberg inequalities in two space dimensions, symmetry breaking issues
2. The Onofri inequality as an endpoint of Gagliardo-Nirenberg inequalities
3. Legendre duality through flows: from Onofri to log HLS (Hardy-Littlewood-Sobolev) inequalities (critical case)
4. Subcritical log HLS inequalities and the subcritical Keller-Segel model

Weighted Onofri inequality and Caffarelli-Kohn-Nirenberg inequalities in two space dimensions

Onofri's inequality

[E. Onofri. On the positivity of the effective action in a theory of random surfaces. Comm. Math. Phys., 86 (3): 321-326, 1982]

$$\log\left(\int_{\mathbb{S}^2} e^{2\,u}\,d\sigma\right)-2\int_{\mathbb{S}^2} u\,d\sigma\;\le\;\|\nabla u\|^2_{L^2(\mathbb{S}^2,d\sigma)}$$

for all $u\in\mathcal{E}=\{u\in L^1(\mathbb{S}^2,d\sigma)\,:\,|\nabla u|\in L^2(\mathbb{S}^2,d\sigma)\}$

By the stereographic projection from $\mathbb{S}^2$ onto $\mathbb{R}^2$, we get an Onofri type inequality in $\mathbb{R}^2$

$$\log\left(\int_{\mathbb{R}^2} e^{v}\,d\mu\right)-\int_{\mathbb{R}^2} v\,d\mu\;\le\;\frac{1}{16\,\pi}\,\|\nabla v\|^2_{L^2(\mathbb{R}^2,dx)}$$

for all $v\in\mathcal{D}=\{v\in L^1(\mathbb{R}^2,d\mu)\,:\,|\nabla v|\in L^2(\mathbb{R}^2,dx)\}$ and

$$d\mu=\frac{dx}{\pi\,(1+|x|^2)^2}$$

A first result: generalized Onofri inequalities

On $\mathbb{R}^2$ for $\alpha > -1$, consider the family of probability measures

$$d\mu_\alpha = \frac{\alpha+1}{\pi} \frac{|x|^{2\alpha}\,dx}{(1+|x|^{2\,(\alpha+1)})^2}$$

Theorem (JD, M. Esteban, G. Tarantello)

$$\log\left(\int_{\mathbb{R}^2} e^v\, d\mu_\alpha\right) - \int_{\mathbb{R}^2} v\, d\mu_\alpha \le \frac{1}{16\,\pi\,(\alpha+1)} \|\nabla v\|^2_{L^2(\mathbb{R}^2,\,dx)}$$

holds in the space $\mathcal{E}_\alpha = \left\{v \in L^1(\mathbb{R}^2, d\mu_\alpha) : |\nabla v| \in L^2(\mathbb{R}^2, dx)\right\}$ *restricted to radially symmetric functions* $\forall\,\alpha > -1$, *and without restriction if* $\alpha \in (-1, 0]$

Proof

Lemma

Let $\alpha > -1$. For all $v \in \mathcal{E}_\alpha$, there holds

$$\int_{\mathbb{R}^2} e^{v - \int_{\mathbb{R}^2} v \, d\mu_\alpha} \, d\mu_\alpha \le e^{\frac{1}{16\,\pi\,(\alpha+1)} \left(\|\nabla v\|_2^2 + \alpha\,(\alpha+2)\, \| \frac{1}{r}\, \partial_\theta v \,\|_2^2 \right)}$$

$\mathbb{C} \approx \mathbb{R}^2 \ni x = r\, e^{i\theta}$, $r \ge 0$, $\theta \in [0, 2\pi)$. Stereographic projection: Σ_0
Let $\alpha > -1$ and define the inverse of a dilated stereographic projection

$$\Sigma_\alpha^{-1}(r\, e^{i\theta}) := \left(\frac{2\, r^{\alpha+1}\, e^{i\theta}}{1 + r^{2(\alpha+1)}} \,,\, \frac{r^{2(\alpha+1)} - 1}{1 + r^{2(\alpha+1)}} \right) = \Sigma_0^{-1}(r^{1+\alpha}\, e^{i\theta})$$

If $f \in C(\mathbb{R})$, $f(u)$, $|\nabla u|^2 \in L^1(\mathbb{S}^2)$ and $v = u \circ \Sigma_\alpha^{-1}$, then

- $\int_{\mathbb{S}^2} f(u)\, d\sigma = \int_{\mathbb{R}^2} f(v)\, d\mu_\alpha$
- $4\pi \int_{\mathbb{S}^2} |\nabla u|^2\, d\sigma = \frac{1}{\alpha+1} \int_{\mathbb{R}^2} \left(|\nabla v|^2 + \alpha\,(\alpha+2) \left| \frac{1}{r}\, \partial_\theta v \right|^2 \right) dx$

The result follows from Onofri's inequality

Application to Caffarelli-Kohn-Nirenberg inequalities

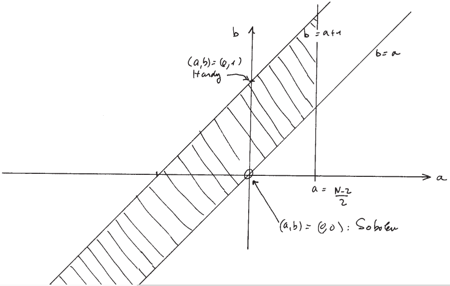
$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx \qquad \forall\, u \in \mathcal{D}_{a,b}$$

with $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, and $a \ne \frac{d-2}{2} =: a_c$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

$$\mathcal{D}_{a,b} := \left\{\, |x|^{-b}\,u \in L^p(\mathbb{R}^d, dx) \;:\; |x|^{-a}\,|\nabla u| \in L^2(\mathbb{R}^d, dx)\right\}$$

[Catrina, Wang]
[Felli, Schneider]
[Smets, Willem]
[Lin, Wang]
...



Approaching Onofri's inequality ($d = 2$)

Theorem (JD, Esteban, Tarantello)

For all $\varepsilon > 0$ $\exists\, \eta > 0$ *s.t. for* $a < 0$, $|a| < \eta$

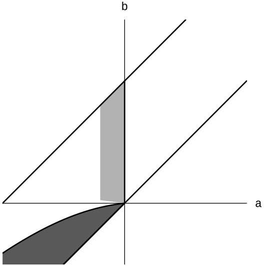
(i) *if* $|a| > \frac{2}{p-\varepsilon}\,(1+|a|^2)$, *then* $\mathsf{C}_{a,b} > \mathsf{C}^*_{a,b}$ *(symmetry breaking)*

(ii) *if* $|a| < \frac{2}{p+\varepsilon}\,(1+|a|^2)$, *then* $\mathsf{C}_{a,b} = \mathsf{C}^*_{a,b}$ *and* $u_{a,b} = u^*_{a,b}$

For $d = 2$, radial symmetry holds if $-\eta < a < 0$ and $-\varepsilon(\eta)\, a \le b < a+1$

$\varepsilon = \frac{2}{p}$ $\alpha = -1 + (1-\varepsilon)\,\frac{a}{\varepsilon}$

Blow-up + Liouville equation



The Onofri inequality as an endpoint of Gagliardo-Nirenberg inequalities

Gagliardo-Nirenberg inequalities

Consider the following sub-family of Gagliardo-Nirenberg inequalities

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \leq \mathsf{C}_{p,d}\,\|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)}$$

with $\theta = \theta(p) := \frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, $p = \frac{1}{2m-1}$

- $1 < p \leq \frac{d}{d-2}$ if $d \geq 3$, $\frac{d-1}{d} \leq m < 1$
- $1 < p < \infty$ if $d = 2$, $\frac{1}{2} < m < 1$

[M. del Pino, JD] equality holds if $f = F_p$ with

$$F_p(x) = (1 + |x|^2)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

All extremal functions are equal to F_p up to a multiplication by a constant, a translation and a scaling

- When $p \to 1$, we recover the euclidean logarithmic Sobolev inequality in optimal scale invariant form [F. Weissler]
- If $d \geq 3$, the limit case $p = d/(d-2)$ corresponds to Sobolev's inequality [T. Aubin, G. Talenti]
- If $d = 2$ and $p \to \infty$...

Onofri's inequality as a limit case

When $d=2$, Onofri's inequality can be seen as an endpoint case of the family of the Gagliardo-Nirenberg inequalities

Proposition (JD)

Assume that $g \in \mathcal{D}(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^2} g\, d\mu = 0$ and let $f_p := F_p\left(1 + \frac{g}{2p}\right)$. With $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$, and $d\mu(x) := \mu(x)\,dx$, we have

$$1 \le \lim_{p\to\infty} \mathsf{C}_{p,2}\, \frac{\|\nabla f_p\|_{\mathrm{L}^2(\mathbb{R}^2)}^{\theta(p)}\,\|f_p\|_{\mathrm{L}^{p+1}(\mathbb{R}^2)}^{1-\theta(p)}}{\|f_p\|_{\mathrm{L}^{2p}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla g|^2\,dx}}{\int_{\mathbb{R}^2} e^g\,d\mu}$$

The standard form of the euclidean version of Onofri's inequality is

$$\log\left(\int_{\mathbb{R}^2} e^g\,d\mu\right) - \int_{\mathbb{R}^2} g\,d\mu \le \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla g|^2\,dx$$

The Onofri inequality in higher dimensions ?

On $\mathbb{R}^d$ with $d \geq 3$, let us define

$$\mathcal{Q}_d[u] := \frac{\int_{\mathbb{R}^d} \mathsf{H}_d(x,\nabla u)\,dx}{\log\left(\int_{\mathbb{R}^d} e^u\,d\mu_d\right) - \int_{\mathbb{R}^2} u\,d\mu}\,, \quad d\mu_d(x) := \frac{d}{|\mathbb{S}^{d-1}|}\,\frac{dx}{\left(1+|x|^{\frac{d}{d-1}}\right)^d}$$

$$\mathsf{R}_d(X,Y) := |X+Y|^d - |X|^d - d\,|X|^{d-2}\,X\cdot Y\,, \quad (X,Y)\in\mathbb{R}^d\times\mathbb{R}^d$$

$$\mathsf{H}_d(x,p) := \mathsf{R}_d\left(-\frac{d\,|x|^{-\frac{d-2}{d-1}}}{1+|x|^{\frac{d}{d-1}}}\,x,\,\frac{d-1}{d}\,p\right)\,, \quad (x,p)\in\mathbb{R}^d\times\mathbb{R}^d$$

Theorem (JD, del Pino)

$$\log\left(\int_{\mathbb{R}^d} e^u\,d\mu_d\right) - \int_{\mathbb{R}^2} u\,d\mu \leq \alpha_d \int_{\mathbb{R}^d} \mathsf{H}_d(x,\nabla u)\,dx \qquad (1)$$

The optimal constant α_d is explicit and given by

$$\alpha_d = \frac{d^{1-d}\,\Gamma(d/2)}{2\,(d-1)\,\pi^{d/2}}$$

Comments

Let $\mathsf{e} \in \mathbb{S}^{d-1}$

$$\lim_{\varepsilon\to 0} \mathcal{Q}_d[\varepsilon\, v] = \frac{1}{\alpha_d} \quad \text{if} \quad v(x) = -\,d\, \frac{x \cdot \mathsf{e}}{|x|^{\frac{d-2}{d-1}} \left(1 + |x|^{\frac{d}{d-1}}\right)}$$

Example

- If $d = 2$, $\int_{\mathbb{R}^d} \mathsf{H}_2(x, \nabla u)\, dx = \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^2\, dx$: Onofri's inequality with optimal constant $1/\alpha_2 = 4\,\pi$.
- If $d = 4$, we find that $H_4(x, \nabla u)$ is a fourth order polynomial and $\mathsf{R}_4(X, Y) = 4\,(X \cdot Y)^2 + |Y|^2(|Y|^2 + 4\,X \cdot Y + 2\,|X|^2)$.

Extensions of the inequality to higher dimensions have been obtained by [Carlen & Loss, 1992] and [Beckner, 1993] in the case of $\mathbb{S}^d$ but there natural conformally invariant, non-local generalizations of the Laplacian were used

A general family of Gagliardo-Nirenberg inequalities

Theorem (JD, del Pino)

Let $p \in (1, d]$, $a > 1$ such that $a \leq \frac{p\,(d-1)}{d-p}$ if $p < d$, and $b = p\,\frac{a-1}{p-1}$. There exists a positive constant $\mathsf{C}_{p,a}$ such that, for any function $f \in \mathrm{L}^a(\mathbb{R}^d, dx)$ with $\nabla f \in \mathrm{L}^p(\mathbb{R}^d, dx)$, we have

$$\|f\|_{\mathrm{L}^b(\mathbb{R}^2)} \leq \mathsf{C}_{p,a}\,\|\nabla f\|^{\theta}_{\mathrm{L}^p(\mathbb{R}^2)}\,\|f\|^{1-\theta}_{\mathrm{L}^a(\mathbb{R}^2)} \quad \text{with } \theta = \frac{(a-p)\,d}{(a-1)\,(d\,p-(d-p)\,a)} \tag{2}$$

if $a > p$. A similar inequality also holds if $a < p$, namely

$$\|f\|_{\mathrm{L}^a(\mathbb{R}^2)} \leq \mathsf{C}_{p,a}\,\|\nabla f\|^{\theta}_{\mathrm{L}^p(\mathbb{R}^2)}\,\|f\|^{1-\theta}_{\mathrm{L}^b(\mathbb{R}^2)} \quad \text{with } \theta = \frac{(p-a)\,d}{a\,(d\,(p-a)+p\,(a-1))}$$

In both cases, equality holds for any function taking the form

$$f(x) = A\,\left(1 + B\,|x-x_0|^{\frac{p}{p-1}}\right)_+^{-\frac{p-1}{a-p}} \quad \forall\, x \in \mathbb{R}^d$$

for some $(A,\,B,\,x_0) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d$, where B has the sign of $a-p$

Remark: the other endpoint of the family

For $a=p$, inequality (2) degenerates into an equality. By substracting it to the inequality, dividing by $a-p$ and taking the limit as $a \to p_+$, we obtain an *optimal Euclidean L^p-Sobolev logarithmic inequality* which goes as follows. Assume that $1<p \le d$. Then for any $u \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |u|^p\,dx = 1$ we have

$$\int_{\mathbb{R}^d} |u|^p \log |u|^p\,dx \le \frac{d}{p}\,\log\left[\beta_{p,d}\int_{\mathbb{R}^d} |\nabla u|^p\,dx\right]$$

where the optimal constant $\beta_{p,d}$ is explicit. Equality holds if and only if for some $\sigma>0$ and $x_0 \in \mathbb{R}^d$

$$u(x) = \left[\frac{1}{2\,\pi^{\frac{d}{2}}}\,\frac{p}{p-1}\,\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(d\,\frac{p-1}{p}\right)}\,\left(\frac{p}{\sigma}\right)^{d\,\frac{p-1}{p}}\right]^{\frac{1}{p}} e^{-\frac{1}{\sigma}|x-x_0|^{\frac{p}{p-1}}} \quad \forall\, x \in \mathbb{R}^d$$

[del Pino, JD], [del Pino, JD, Gentil]
[Cordero-Erausquin, Gangbo, Houdré]

Legendre duality through flows

From Onofri to logarithmic Hardy-Littlewood-Sobolev (log HLS) inequalities: the critical case

Legendre duality

To a convex functional F, we may associate the functional F^* defined by Legendre's duality as

$$F^*[v] := \sup \left(\int_{\mathbb{R}^d} u\, v\, dx - F[u] \right)$$

- To $F_1[u] = \frac{1}{2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}$, we associate $F_1^*[v] = \frac{1}{2}\,\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}$ where p and q are Hölder conjugate exponents: $1/p + 1/q = 1$
- To $F_2[u] = \frac{1}{2}\,\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, we associate

$$F_2^*[v] = \frac{1}{2}\,\mathsf{S}_d^{-1} \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\, dx$$

where $(-\Delta)^{-1} v = G_d * v$, $G_d(x) = \frac{1}{d-2}\,|\mathbb{S}^{d-1}|^{-1}\,|x|^{2-d}$ if $d \ge 3$

As a straightforward consequence of Legendre's duality, if we have a functional inequality of the form $F_1[u] \le F_2[u]$, then we have the dual inequality $F_1^*[v] \ge F_2^*[v]$

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall \, u \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx \quad \forall \, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2\,d}{d-2}$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{3}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\;dx + \mathsf{S}_d\left(\int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\;dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\;dx$$

where $v = v(t,\cdot)$ is a solution of (3). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2\,d}{d+2}$

Relating Sobolev and HLS inequalities through a flow

Theorem (JD)

Assume that $d \geq 3$ *and* $m = \frac{d-2}{d+2}$. *If* v *is a solution of* (3) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right] = \left(\int_{\mathbb{R}^d} v^{m+1}\;dx\right)^{2/d}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

Assume that $d \geq 5$ *and let* $q = \frac{d+2}{d-2}$. *There exists a positive constant* $\mathcal{C} \leq \left(1+\frac{2}{d}\right)\left(1-e^{-d/2}\right)\mathsf{S}_d$ *such that*

$$\mathsf{S}_d\,\|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q\,(-\Delta)^{-1} w^q\;dx \leq \mathcal{C}\,\|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]$$

The two-dimensional case: Legendre duality

Onofri's inequality

$$F_1[u] := \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) \le \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \int_{\mathbb{R}^2} u\,\mu\,dx =: F_2[u]$$

By duality: $F_i^*[v] = \sup\left(\int_{\mathbb{R}^2} v\,u\,d\mu - F_i[u]\right)$ we can relate Onofri's inequality with (logHLS)

Proposition (E. Carlen, M. Loss & V. Calvez, L. Corrias)

For any $v \in \mathrm{L}^1_+(\mathbb{R}^2)$ *with* $\int_{\mathbb{R}^2} v\,dx = 1$, *such that* $v \log v$ *and* $(1 + \log|x|^2)\,v \in \mathrm{L}^1(\mathbb{R}^2)$, *we have*

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)dx - 4\,\pi\int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx \ge 0$$

The two-dimensional case: (logHLS) and flows

[E. Carlen, J. Carrillo, M. Loss]

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx$$

is related to Gagliardo-Nirenberg inequalities if $v_t = \Delta\sqrt{v}$

- Alternatively, assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta\log\left(\frac{v}{\mu}\right) \quad t>0\,, \quad x\in\mathbb{R}^2$$

Proposition (JD)

If v is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\,dx = 1$, $v_0\,\log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu \in \mathrm{L}^1(\mathbb{R}^2)$, then

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu \geq F_2[u] - F_1[u]$$

with $\log(v/\mu) = u/2$

The two-dimensional case: the sphere setting

The image w of v by the inverse stereographic projection on the sphere $\mathbb{S}^2$, up to a scaling, solves the equation

$$\frac{\partial w}{\partial t} = \Delta_{\mathbb{S}^2} \log w \quad t > 0\,, \quad y \in \mathbb{S}^2$$

More precisely, if $x = (x_1, x_2) \in \mathbb{R}^2$, then u and w are related by

$$w(t,y) = \frac{u(t,x)}{4\,\pi\,\mu(x)}\,, \quad y = \left(\tfrac{2\,(x_1,x_2)}{1+|x|^2}, \tfrac{1-|x|^2}{1+|x|^2}\right) \in \mathbb{S}^2$$

The loss of mass of the solution of

$$\frac{\partial v}{\partial t} = \Delta \log v \quad t > 0\,, \quad x \in \mathbb{R}^2$$

is compensated in case of

$$\frac{\partial v}{\partial t} = \Delta \log \left(\frac{v}{\mu}\right) \quad t > 0\,, \quad x \in \mathbb{R}^2$$

by the source term $-\Delta \log \mu$

Subcritical log HLS inequalities and the subcritical Keller-Segel model

Optimal inequalities in the subcritical case provide a functional framework for large time asymptotics of the Keller-Segel model

When mass increases to its critical value ($M = 8\,\pi$), the log HLS inequality is recovered as a singular limit

The parabolic-elliptic Keller and Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \; t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \; t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| \, u(t,y) \, dy$$

and observe that

$$\nabla v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \, u(t,y) \, dy$$

Mass conservation: $\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^2} u(t,x) \, dx = 0$

Finite time blow-up if $M = \int_{\mathbb{R}^2} n_0 \, dx > 8\pi$

Existence and free energy

$M = \int_{\mathbb{R}^2} n_0 \, dx \leq 8\pi$: global existence [W. Jäger, S. Luckhaus], [JD, B. Perthame], [A. Blanchet, JD, B. Perthame], [A. Blanchet, J.A. Carrillo, N. Masmoudi]

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u \, (\nabla \, (\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u \; dx - \frac{1}{2} \int_{\mathbb{R}^2} u \, v \; dx$$

satisfies

$$\frac{d}{dt} F[u(t,\cdot)] = - \int_{\mathbb{R}^2} u \left| \nabla \, (\log u) - \nabla v \right|^2 \, dx$$

(log HLS) inequality [E. Carlen, M. Loss]:
F is bounded from below if $M < 8\pi$

The existence setting

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2\,,\; t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2\,,\; t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\, dx)\,, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)\,, \quad M := \int_{\mathbb{R}^2} n_0(x)\, dx < 8\,\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\, dx$ for any $t \geq 0$, see [W. Jäger, S. Luckhaus], [A. Blanchet, JD, B. Perthame]

$v = -\frac{1}{2\pi} \log|\cdot| * u$

Time-dependent rescaling

$$u(x,t)=\frac{1}{R^2(t)}\,n\left(\frac{x}{R(t)},\tau(t)\right)\quad\text{and}\quad v(x,t)=c\left(\frac{x}{R(t)},\tau(t)\right)$$

with $R(t)=\sqrt{1+2t}$ and $\tau(t)=\log R(t)$

$$\begin{cases}\dfrac{\partial n}{\partial t}=\Delta n-\nabla\cdot\left(n\left(\nabla c-x\right)\right) & x\in\mathbb{R}^2\,,\ t>0\\ c=-\dfrac{1}{2\pi}\log|\cdot|*n & x\in\mathbb{R}^2\,,\ t>0\\ n(\cdot,t=0)=n_0\ge 0 & x\in\mathbb{R}^2\end{cases}$$

[A. Blanchet, JD, B. Perthame] Convergence in self-similar variables

$$\lim_{t\to\infty}\|n(\cdot,\cdot+t)-n_\infty\|_{L^1(\mathbb{R}^2)}=0\quad\text{and}\quad\lim_{t\to\infty}\|\nabla c(\cdot,\cdot+t)-\nabla c_\infty\|_{L^2(\mathbb{R}^2)}=0$$

means *intermediate asymptotics* in original variables:

$$\left\|u(x,t)-\tfrac{1}{R^2(t)}\,n_\infty\left(\tfrac{x}{R(t)},\tau(t)\right)\right\|_{L^1(\mathbb{R}^2)}\searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\, dx} = -\Delta c_\infty \,, \qquad c_\infty = -\frac{1}{2\pi} \log |\cdot| * n_\infty$$

- Radial symmetry [Y. Naito]
- Uniqueness [P. Biler, G. Karch, P. Laurençot, T. Nadzieja]
- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon \in (0,1)$ [A. Blanchet, JD, B. Perthame]
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M:

$$\lim_{M \to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[D.D. Joseph, T.S. Lundgren] [JD, R. Stańczy]

The stationary solution when mass varies

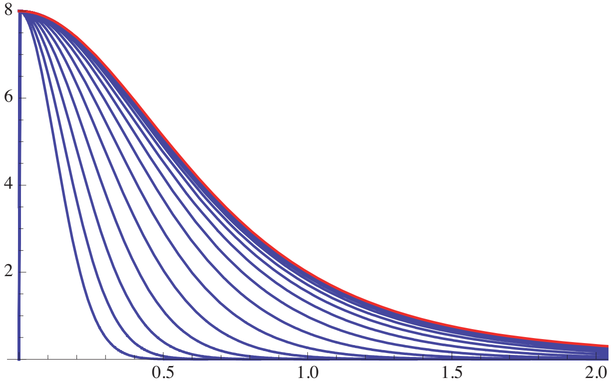


Figure: Representation of the solution appropriately scaled so that the 8π case appears as a limit (in red)

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla \Big[n \left(\log n - x + \nabla c \right) \Big]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \, dx + \int_{\mathbb{R}^2} \frac{1}{2} |x|^2 \, n \, dx - \frac{1}{2} \int_{\mathbb{R}^2} n \, c \, dx$$

satisfies

$$\frac{d}{dt} F[n(t,\cdot)] = - \int_{\mathbb{R}^2} n \left| \nabla \left(\log n \right) + x - \nabla c \right|^2 \, dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2 \, n(\lambda \, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2) \, dx + \int_{\mathbb{R}^2} \frac{\lambda^{-2}-1}{2} \, |x|^2 \, n \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) \, n(y) \, \log \frac{1}{\lambda} \, dx \, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi} \right)}_{>0 \text{ if } M < 8\pi} \log \lambda + \frac{\lambda^{-2} - 1}{2} \int_{\mathbb{R}^2} |x|^2 \, n \, dx$$

Keller-Segel with subcritical mass in self-similar variables

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t = 0) = n_0 \ge 0 & x \in \mathbb{R}^2 \end{cases}$$

$$\lim_{t\to\infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

$$n_\infty = M\,\frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

A parametrization of the solutions and the linearized operator

[J. Campos, JD]

$$-\Delta c = M \frac{e^{-\frac{1}{2}\,|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}\,|x|^2+c}\,dx}$$

Solve

$$-\phi'' - \frac{1}{r}\,\phi' = e^{-\frac{1}{2}\,r^2+\phi}\,,\quad r>0$$

with initial conditions $\phi(0) = a$, $\phi'(0) = 0$ and get with $r = |x|$

$$M(a) := 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2}\,r^2+\phi_a}\,dx$$

$$n_a(x) = M(a)\,\frac{e^{-\frac{1}{2}\,r^2+\phi_a(r)}}{2\pi\int_{\mathbb{R}^2} r\,e^{-\frac{1}{2}\,r^2+\phi_a}\,dx} = e^{-\frac{1}{2}\,r^2+\phi_a(r)}$$

Mass

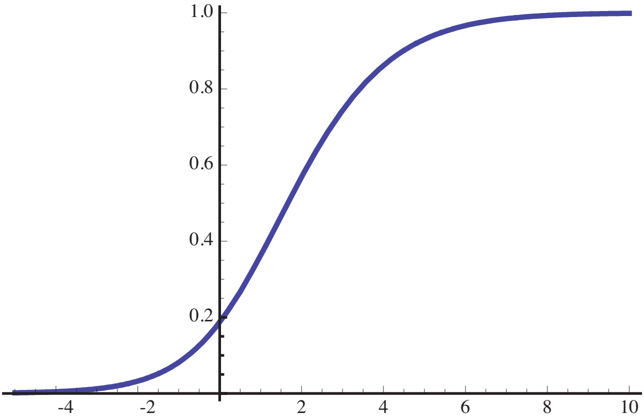


Figure: *The mass can be computed as* $M(a) = 2\pi \int_0^\infty n_a(r)\, r\, dr$. *Plot of* $a \mapsto M(a)/8\pi$

Bifurcation diagram

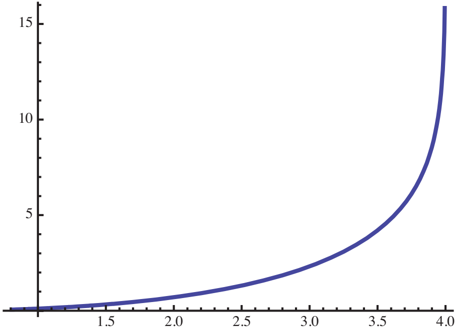


Figure: *The bifurcation diagram can be parametrized by* $a \mapsto \left(\frac{1}{2\pi} M(a), \|c_a\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\right)$ *with* $\|c_a\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = c_a(0) = a - b(a)$ *(cf. Keller-Segel system in a ball with no flux boundary conditions)*

Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

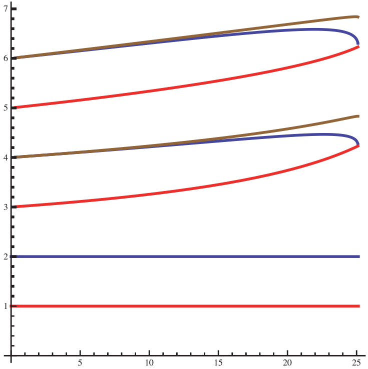


Figure: The lowest eigenvalues of $-\mathcal{L} = (-\Delta)^{-1}(n_a\, f)$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

[A. Blanchet, JD, M. Escobedo, J. Fernández], [J. Campos, JD], [V. Calvez, J.A. Carrillo], [J. Bedrossian, N. Masmoudi]

Spectral analysis in the functional framework determined by the relative entropy method

Simple eigenfunctions

Kernel Let $f_0 = \frac{\partial}{\partial M} c_\infty$ be the solution of

$$-\Delta\, f_0 = n_\infty\, f_0$$

and observe that $g_0 = f_0/c_\infty$ is such that

$$\frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f_0 - c_\infty\, g_0)\big) =: \mathcal{L}\, f_0 = 0$$

Lowest non-zero eigenvalues $f_1 := \frac{1}{n_\infty} \frac{\partial n_\infty}{\partial x_1}$ associated with $g_1 = \frac{1}{c_\infty} \frac{\partial c_\infty}{\partial x_1}$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L}\, f_1 = f_1$

With $D := x \cdot \nabla$, let$f_2 = 1 + \frac{1}{2}\, D \log n_\infty = 1 + \frac{1}{2\, n_\infty}\, D\, n_\infty$. Then

$$-\Delta\,(D\, c_\infty) + 2\,\Delta c_\infty = D\, n_\infty = 2\,(f_2 - 1)\, n_\infty$$

and so $g_2 := \frac{1}{c_\infty}\,(-\Delta)^{-1}(n_\infty\, f_2)$ is such that $-\mathcal{L}\, f_2 = 2\, f_2$

Functional setting...

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality

$$F[n]:=\int_{\mathbb{R}^2} n\,\log\left(\frac{n}{n_\infty}\right)dx-\frac{1}{2}\int_{\mathbb{R}^2}(n-n_\infty)\,(c-c_\infty)\,dx\geq 0$$

achieves its minimum for $n=n_\infty$

$$\mathsf{Q}_1[f]=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}F[n_\infty(1+\varepsilon\,f)]\geq 0$$

if $\int_{\mathbb{R}^2} f\,n_\infty\,dx=0$. Notice that f_0 generates the kernel of Q_1

Lemma (J. Campos, JD)

HLS-Poincaré type inequality For any $f\in H^1(\mathbb{R}^2,n_\infty\,dx)$ *such that* $\int_{\mathbb{R}^2} f\,n_\infty\,dx=0$, *we have*

$$\int_{\mathbb{R}^2}|\nabla(-\Delta)^{-1}(f\,n_\infty)|^2\,n_\infty\,dx=\int_{\mathbb{R}^2}|\nabla(g\,c_\infty)|^2\,n_\infty\,dx\leq\int_{\mathbb{R}^2}|f|^2\,n_\infty\,dx$$

... and eigenvalues

With g such that $-\Delta(g\,c_\infty) = f\,n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1\,f_2\,n_\infty\;dx - \int_{\mathbb{R}^2} f_1\,n_\infty\,(g_2\,c_\infty)\;dx$$

on the orthogonal space to f_0 in $L^2(n_\infty\,dx)$

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\,c_\infty)|^2\,n_\infty\;dx \quad \text{with} \quad g = -\frac{1}{c_\infty}\,\frac{1}{2\pi}\,\log|\cdot| * (f\,n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L}\,f \rangle = \mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma (J. Campos, JD)

$\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is 1

Linearized Keller-Segel theory

$$\mathcal{L}\, f = \frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f - c_\infty\, g) \big)$$

Corollary (J. Campos, JD)

$$\langle f, f \rangle \le \langle \mathcal{L}\, f, f \rangle$$

The linearized problem takes the form

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f$$

where $\mathcal{L}$ is a self-adjoint operator on the orthogonal of f_0 equipped with $\langle \cdot, \cdot \rangle$. A solution of

$$\frac{d}{dt} \langle f, f \rangle = -\, 2 \, \langle \mathcal{L}\, f, f \rangle$$

has therefore exponential decay

More functional inequalities

A new Onofri type inequality

By Legendreduality, sub-critical HLS and HLS-Poincaré type inequalities decome

Theorem (J. Campos, JD)

For any $M \in (0, 8\pi)$, if $n_\infty = M \frac{e^{c_\infty - \frac{1}{2}|x|^2}}{\int_{\mathbb{R}^2} e^{c_\infty - \frac{1}{2}|x|^2} dx}$ with $c_\infty = (-\Delta)^{-1} n_\infty$, $d\mu_M = \frac{1}{M} n_\infty\, dx$, we have the inequality

$$\log\left(\int_{\mathbb{R}^2} e^{\phi}\, d\mu_M\right) - \int_{\mathbb{R}^2} \phi\, d\mu_M \le \frac{1}{2\,M} \int_{\mathbb{R}^2} |\nabla \phi|^2\, dx \quad \forall\, \phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$$

Corollary (J. Campos, JD)

The following Poincaré inequality holds

$$\int_{\mathbb{R}^2} \left|\psi - \overline{\psi}\right|^2 n_\infty\, dx \le \int_{\mathbb{R}^2} |\nabla \psi|^2\, dx \quad \text{where} \quad \overline{\psi} = \int_{\mathbb{R}^2} \psi\, d\mu_M$$

An improved interpolation inequality (coercivity estimate)

Lemma (J. Campos, JD)

For any $f \in L^2(\mathbb{R}^2, n_\infty\,dx)$ such that $\int_{\mathbb{R}^2} f\,f_0\,n_\infty\,dx = 0$ holds, we have

$$-\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,n_\infty(x)\,\log|x-y|\,f(y)\,n_\infty(y)\,dx\,dy \le (1-\varepsilon)\int_{\mathbb{R}^2}|f|^2\,n_\infty\,dx$$

for some $\varepsilon > 0$, where $g\,c_\infty = G_2 * (f\,n_\infty)$ and, if $\int_{\mathbb{R}^2} f\,n_\infty\,dx = 0$ holds,

$$\int_{\mathbb{R}^2}|\nabla(g\,c_\infty)|^2\,dx \le (1-\varepsilon)\int_{\mathbb{R}^2}|f|^2\,n_\infty\,dx$$

An application:

Rates of convergence for the subcritical Keller-Segel model (self-similar variables)

Exponential convergence for any mass $M \leq 8\pi$

If $n_{0,*}(\sigma)$ stands for the symmetrized function associated to n_0, assume that for any $s \geq 0$

$$(H) \quad \exists\, \varepsilon \in (0, 8\,\pi - M) \quad \text{such that} \quad \int_0^s n_{0,*}(\sigma)\, d\sigma \leq \int_{B\left(0,\sqrt{s/\pi}\right)} n_{\infty, M+\varepsilon}(x)\, dx$$

Theorem (J. Campos, JD)

Under the above assumption, if $n_0 \in L^2_+(n_\infty^{-1}\, dx)$ and $M := \int_{\mathbb{R}^2} n_0\, dx < 8\,\pi$, then any solution with initial datum n_0 is such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \frac{dx}{n_\infty(x)} \leq C\, e^{-\,2\,t} \quad \forall\, t \geq 0$$

for some positive constant C, where n_∞ is the unique stationary solution with mass M

Thank you for your attention !