Self-similar solutions, relative entropy and applications

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Outline

A brief introduction to entropy methods

- ▷ Sobolev's inequality and the Bianchi-Egnell stability result
- ▷ Gagliardo-Nirenberg-Sobolev inequalities
- ▷ The Bakry-Emery method: Fokker-Planck equation on $\mathbb{R}^d$ (linear case)
- ▷ The fast diffusion equation (nonlinear case)

Sobolev inequality and Aubin-Talenti profiles

In ***Sobolev's inequality*** (with optimal constant S_d),

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge 0$$

The manifold $\mathscr{M}$ of the *optimal functions* is generated by the multiples, translates, scalings of the ***Aubin-Talenti*** functions

$$\mathsf{g}(x) := \left(1+|x|^2\right)^{-\frac{d-2}{2}} \quad \forall\, x \in \mathbb{R}^d$$

A question raised in [Brezis, Lieb (1985)]: *is there a natural way to bound the l.h.s. from below in terms of a "distance" to the set of optimal [Aubin-Talenti] functions* when $d \ge 3$?

▷ [Bianchi, Egnell (1991)] There is a positive constant α such that

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \alpha \inf_{\varphi\in\mathscr{M}} \|\nabla f - \nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ Various improvements, *e.g.*, [Cianchi, Fusco, Maggi, Pratelli (2009)] but the question of ***constructive*** estimates is still widely open

Improved inequalities and stability results

Entropy – entropy production inequality

$$\boxed{\mathscr{I}[u] \geq \Lambda\,\mathscr{F}[u]}$$

▷ ***Improved entropy – entropy production inequality*** (weaker form)

$$\mathscr{I} \geq \Lambda\,\psi(\mathscr{F})$$

for some ψ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi'' > 0$

$$\mathscr{I} - \Lambda\,\mathscr{F} \geq \Lambda\big(\psi(\mathscr{F}) - \mathscr{F}\big) \geq 0$$

▷ ***Improved constant*** means ***stability***
Under some restrictions on the functions, there is some $\Lambda_\star > \Lambda$ such that

$$\mathscr{I} - \Lambda\,\mathscr{F} \geq (\Lambda_\star - \Lambda)\,\mathscr{F} \geq 0 \quad \text{or} \quad \mathscr{I} - \Lambda\,\mathscr{F} \geq \left(1 - \frac{\Lambda}{\Lambda_\star}\right)\mathscr{I} \geq 0$$

Gagliardo-Nirenberg-Sobolev inequalities

We consider the inequalities

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\geq\mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p} \tag{GNS}$$

$$\theta=\frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}\,,\quad p\in(1,+\infty)\text{ if }d=1\text{ or }2\,,\quad p\in(1,p^*]\text{ if }d\geq3\,,\quad p^*=\frac{d}{d-2}$$

Theorem (del Pino, JD)

Equality case in (GNS) is achieved if and only if

$$f\in\mathfrak{M}:=\left\{g_{\lambda,\mu,y}\,:\,(\lambda,\mu,y)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\right\}$$
$$\mathrm{g}(x)=\left(1+|x|^2\right)^{-\frac{1}{p-1}}$$

Aubin-Talenti functions: $g_{\lambda,\mu,y}(x):=\mu\,\mathrm{g}\left((x-y)/\lambda\right)$

[del Pino, JD, 2002], [Gunson, 1987, 1991]

Related inequalities

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p}\qquad\text{(GNS)}$$

▷ *Sobolev's inequality*: $d\ge3$, $p=p^*=d/(d-2)$, $\theta=1$

$$\|\nabla f\|_2^2\ge\mathsf{S}_d\,\|f\|_{2p^*}^2$$

▷ *Euclidean Onofri inequality*: $d=2$, $p\to+\infty$, $\theta=1$

$$\int_{\mathbb{R}^2}e^{h-\overline{h}}\,\frac{dx}{\pi\,(1+|x|^2)^2}\le e^{\frac{1}{16\pi}\int_{\mathbb{R}^2}|\nabla h|^2\,dx}$$

$p\to+\infty$ with $f_p(x):=\mathrm{g}(x)\left(1+\frac{1}{2p}\,(h(x)-\overline{h})\right)$, $\overline{h}=\int_{\mathbb{R}^2}h(x)\,\frac{dx}{\pi\,(1+|x|^2)^2}$

▷ *Euclidean logarithmic Sobolev inequality in scale invariant form*

$$\frac{d}{2}\log\left(\frac{2}{\pi\,d\,e}\int_{\mathbb{R}^d}|\nabla f|^2\,dx\right)\ge\int_{\mathbb{R}^d}|f|^2\log|f|^2\,dx$$

or $\int_{\mathbb{R}^d}|\nabla f|^2\,dx\ge\frac12\int_{\mathbb{R}^d}|f|^2\log\left(\frac{|f|^2}{\|f\|_2^2}\right)dx+\frac{d}{4}\log\left(2\pi\,e^2\right)\int_{\mathbb{R}^d}|f|^2\,dx$

The Fokker-Planck equation (domain in $\mathbb{R}^d$)

The linear Fokker-Planck (FP) equation

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla \phi)$$

on a domain $\Omega \subset \mathbb{R}^d$, with no-flux boundary conditions

$$(\nabla u + u \nabla \phi) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

is equivalent to the Ornstein-Uhlenbeck (OU) equation

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \phi \cdot \nabla v =: \mathscr{L} v$$

[Bakry, Emery, 1985], [Arnold, Markowich, Toscani, Unterreiter, 2001]
With mass normalized to 1, the unique stationary solution of (FP) is

$$u_s = \frac{e^{-\phi}}{\int_\Omega e^{-\phi}\, dx} \iff v_s = 1$$

The Bakry-Emery method (domain in $\mathbb{R}^d$)

With $d\gamma = u_s\,dx$ and v such that $\int_\Omega v\,d\gamma = 1$, $q \in (1,2]$, the *q-entropy* is defined by

$$\mathcal{E}_q[v] := \frac{1}{q-1}\int_\Omega \left(v^q - 1 - q\,(v-1)\right) d\gamma$$

Under the action of (OU), with $w = v^{q/2}$, $\mathcal{I}_q[v] := \frac{4}{q}\int_\Omega |\nabla w|^2\,d\gamma$,

$$\frac{d}{dt}\mathcal{E}_q[v(t,\cdot)] = -\,\mathcal{I}_q[v(t,\cdot)] \quad \text{and} \quad \frac{d}{dt}\Big(\mathcal{I}_q[v] - 2\,\lambda\,\mathcal{E}_q[v]\Big) \le 0$$

$$\text{with} \quad \lambda := \inf_{w\in H^1(\Omega,d\gamma)\setminus\{0\}} \frac{\int_\Omega \left(2\,\frac{q-1}{q}\,\|\mathrm{Hess}\,w\|^2 + \mathrm{Hess}\,\phi : \nabla w\otimes\nabla w\right) d\gamma}{\int_\Omega |\nabla w|^2\,d\gamma}$$

Proposition

[Bakry, Emery, 1984] [JD, Nazaret, Savaré, 2008] *Let Ω be convex. If $\lambda > 0$ and v is a solution of (OU), then $\mathcal{I}_q[v(t,\cdot)] \le \mathcal{I}_q[v(0,\cdot)]\,e^{-2\lambda t}$ and $\mathcal{E}_q[v(t,\cdot)] \le \mathcal{E}_q[v(0,\cdot)]\,e^{-2\lambda t}$ for any $t \ge 0$ and, as a consequence,*

$$\mathcal{I}_q[v] \ge 2\,\lambda\,\mathcal{E}_q[v] \quad \forall\, v \in \mathrm{H}^1(\Omega, d\gamma) \quad \text{(Entropy-entropy production ineq.)}$$

From the carré du champ method to stability results

$\mathscr{F}$ denotes a *relative entropy* or *free energy*
$\mathscr{I}$ denotes the Fisher information
Entropy – entropy production inequality

$$\mathscr{I}[u] \geq \Lambda \mathscr{F}[u]$$

▷ ***Carré du champ method*** (adapted from D. Bakry and M. Emery)

$$\frac{\partial u}{\partial t} = \Delta u^m \,, \quad \frac{d\mathscr{F}}{dt} = -\mathscr{I} \,, \quad \frac{d\mathscr{I}}{dt} \leq -\Lambda \mathscr{I}$$

deduce that $\mathscr{I} - \Lambda \mathscr{F}$ is monotone non-increasing with limit 0
▷ Using remainder terms and constraints, we look for *entropy – entropy production inequalities* that are reinterpreted as stability results

Three points of view

- decay rates in diffusion equations
- entropy - entropy production inequalities and functional inequalities
- rigidity problems in elliptic equations, bifurcation problemss

Bakry-Emery

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

Stability, a joint project with M. Bonforte, B. Nazaret and N. Simonov

Joint work on ***Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method*** arXiv:2007.03674, to appear in *Memoirs of the AMS*, in collaboration with

Matteo Bonforte
▷ *Universidad Autónoma de Madrid and ICMAT*

Bruno Nazaret
▷ *Université Paris 1 Panthéon-Sorbonne and Mokaplan team*

Nikita Simonov
▷ *Ceremade, Université Paris-Dauphine (PSL)*

Fast diffusion equation and entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad \text{(FDE)}$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy – entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)
- Initial time layer: improved entropy – entropy production estimates

Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

Mass, moment, entropy and Fisher information

(i) *Mass conservation*. With $m \ge m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} u(t,x)\,dx = 0$$

(ii) *Second moment*. With $m > d/(d+2)$ and $u_0 \in \mathrm{L}^1_+\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x|^2\,u(t,x)\,dx = 2\,d \int_{\mathbb{R}^d} u^m(t,x)\,dx$$

(iii) *Entropy estimate*. With $m \ge m_1 := (d-1)/d$, $u_0^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \in \mathrm{L}^1_+\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} u^m(t,x)\,dx = \frac{m^2}{1-m} \int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\,dx \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2} \int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy growth rate

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\rm GNS}(p)\, \|f\|_{2p} \qquad \text{(GNS)}$$

$$p=\frac{1}{2\,m-1} \quad \Longleftrightarrow \quad m=\frac{p+1}{2\,p}\in[m_1,1)$$

$u=f^{2p}$ so that $u^m=f^{p+1}$ and $u\,|\nabla u^{m-1}|^2=(p-1)^2\,|\nabla f|^2$

$$\mathscr{M}=\|f\|_{2p}^{2p}\,,\quad \mathsf{E}[u]=\|f\|_{p+1}^{p+1}\,,\quad \mathsf{I}[u]=(p+1)^2\,\|\nabla f\|_2^2$$

If u solves (FDE) $\frac{\partial u}{\partial t}=\Delta u^m$

$$\mathsf{E}'\ge\frac{p-1}{2\,p}\,(p+1)^2\left(\mathscr{C}_{\mathrm{GNS}(p)}\right)^{\frac{2}{\theta}}\,\|f\|_{2p}^{\frac{2}{\theta}}\,\|f\|_{p+1}^{-\frac{2(1-\theta)}{\theta}}=C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d}u^m(t,x)\,dx\ge\left(\int_{\mathbb{R}^d}u_0^m\,dx+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\quad\forall\,t\ge0$$

Equality case: $u(t,x)=\frac{c_1}{R(t)^d}\,\mathscr{B}\left(\frac{c_2\,x}{R(t)}\right)$, $\mathscr{B}(x):=\left(1+|x|^2\right)^{\frac{1}{m-1}}$

Pressure variable and decay of the Fisher information

The t-derivative of the *Rényi entropy power* $\mathsf{E}^{\frac{2}{d}\frac{1}{1-m}-1}$ is proportional to

$$\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

▷ *Pressure variable*

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

▷ *Fisher information*

$$\mathsf{I}[u] = \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx$$

If u solves (FDE), then

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla \mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta \mathsf{P} + |\nabla \mathsf{P}|^2\Big)\,dx$$

$$= -2\int_{\mathbb{R}^d} u^m\left(\|\mathrm{D}^2\mathsf{P}\|^2 - (1-m)\,(\Delta \mathsf{P})^2\right)dx$$

Rényi entropy powers and interpolation inequalities

▷ Integrations by parts and completion of squares: with $m_1 = \frac{d-1}{d}$

$$-\frac{\mathsf{I}}{2\theta}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)$$
$$=\int_{\mathbb{R}^d} u^m \left\|\mathrm{D}^2\mathrm{P}-\frac{1}{d}\,\Delta\mathrm{P}\,\mathrm{Id}\right\|^2 dx+(m-m_1)\int_{\mathbb{R}^d} u^m \left|\Delta\mathrm{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2 dx$$

▷ Analysis of the asymptotic regime as $t \to +\infty$

$$\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^{\theta}\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{\mathcal{M}^{\frac{2\theta}{p}}}=\frac{\mathsf{I}[\mathcal{B}]^{\theta}\,\mathsf{E}[\mathcal{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathcal{B}\|_1^{\frac{2\theta}{p}}}=(p+1)^{2\theta}\,(\mathscr{C}_{\mathrm{GNS}}(p))^{2\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$\mathsf{I}[u]^{\theta}\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\ge(p+1)^{2\theta}\,(\mathscr{C}_{\mathrm{GNS}}(p))^{2\theta}\,\mathcal{M}^{\frac{2\theta}{p}}$$

The fast diffusion equation in self-similar variables

- ▷ Rescaling and self-similar variables
- ▷ Relative entropy and the entropy – entropy production inequality
- ▷ Large time asymptotics and spectral gaps

Entropy – entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x)=\frac{1}{\kappa^d\,R^d}\,v\left(\tau,\frac{x}{\kappa\,R}\right)\quad\text{where}\quad\frac{dR}{dt}=R^{1-\mu}\,,\quad\tau(t):=\tfrac12\log R(t)$$

$\frac{\partial u}{\partial t}=\Delta u^m$ is changed into *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial\tau}+\nabla\cdot\left[v\left(\nabla v^{m-1}-2\,x\right)\right]=0\qquad(r\,\mathrm{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathscr{F}[v]:=-\frac{1}{m}\int_{\mathbb{R}^d}\left(v^m-\mathscr{B}^m-m\,\mathscr{B}^{m-1}\,(v-\mathscr{B})\right)dx$$

$$\mathscr{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}+2\,x\right|^2dx$$

are such that $\mathscr{I}[v]\geq4\,\mathscr{F}[v]$ by (GNS) [del Pino, JD, 2002] so that

$$\mathscr{F}[v(t,\cdot)]\leq\mathscr{F}[v_0]\,e^{-4\,t}$$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \qquad \text{(H)}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,, \quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,dx$, for $\alpha<0$

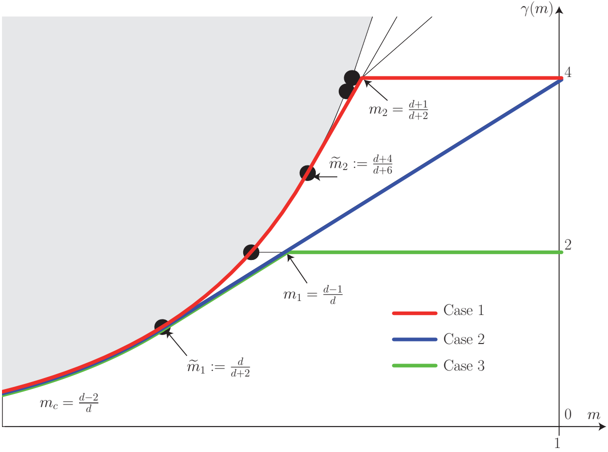
Lemma

Under assumption (H),

$$\mathscr{F}[v(t,\cdot)] \le C\,e^{-2\gamma(m)\,t} \quad \forall\, t\ge 0\,, \quad \gamma(m):=(1-m)\,\Lambda_{1/(m-1),d}$$

Moreover $\gamma(m):=2$ *if* $\frac{d-1}{d}=m_1\le m<1$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Initial and asymptotic time layers

▷ Asymptotic time layer: constraint, spectral gap and improved entropy – entropy production inequality

▷ Initial time layer: the carré du champ inequality and a backward estimate

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathscr{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,dx$$

Hardy-Poincaré inequality. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}^{2-m}\,dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,dx = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,dx = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2\,(d\,m - d + 1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\,dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ and

$$(1-\varepsilon)\,\mathscr{B} \le v \le (1+\varepsilon)\,\mathscr{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then

$$\mathscr{I}[v] \ge (4+\eta)\,\mathscr{F}[v]$$

The initial time layer improvement: backward estimate

Hint: for some strictly convex function ψ with $\psi(0)=\psi'(0)=0$, we have

$$\mathscr{I}-4\,\mathscr{F}\ge 4\left(\psi(\mathscr{F})-\mathscr{F}\right)\ge 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions) for (FDE), we expect some improvement

Rephrasing the *carré du champ* method, $\mathcal{Q}[v]:=\frac{\mathscr{I}[v]}{\mathscr{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt}\le\mathcal{Q}\,(\mathcal{Q}-4)$$

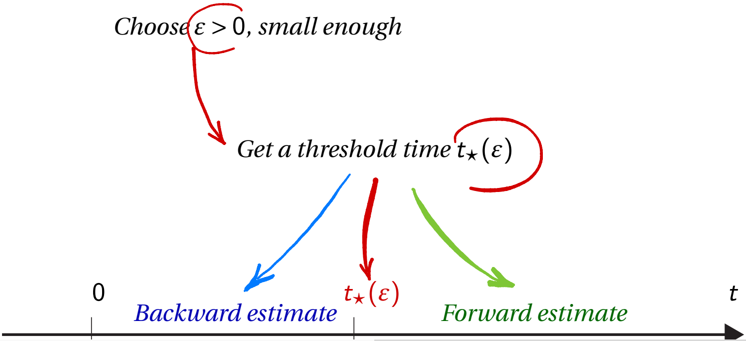
Lemma

Assume that $m>m_1$ and v is a solution to $(r\,\mathrm{FDE})$ with nonnegative initial datum v_0. If for some $\eta>0$ and $t_\star>0$, we have $\mathcal{Q}[v(t_\star,\cdot)]\ge 4+\eta$, then

$$\mathcal{Q}[v(t,\cdot)]\ge 4+\frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}}\quad\forall\,t\in[0,t_\star]$$

Stability in Gagliardo-Nirenberg-Sobolev inequalities

Our strategy



The threshold time and the uniform convergence in relative error

▷ The regularity results allow us to glue the initial time layer estimates with the asymptotic time layer estimates

The improved entropy – entropy production inequality holds for any time along the evolution along (*r* FDE)

(and in particular for the initial datum)

If v is a solves (r FDE) for some nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} v_0 \, dx \le A < \infty \qquad (\mathrm{H}_A)$$

then

$$(1-\varepsilon)\mathscr{B} \le v(t,\cdot) \le (1+\varepsilon)\mathscr{B} \quad \forall\, t \ge t_\star$$

for some ***explicit*** $t_\star$ depending only on ε and A

Global Harnack Principle

The *Global Harnack Principle* holds if for some $t>0$ large enough

$$\mathscr{B}_{M_1}(t-\tau_1,x)\le u(t,x)\le \mathscr{B}_{M_2}(t+\tau_2,x) \tag{GHP}$$

[Vázquez, 2003], [Bonforte, Vázquez, 2006]: (GHP) holds if $u_0\lesssim |x|^{-\frac{2}{1-m}}$
[Vázquez, 2003], [Bonforte, Simonov, 2020]: (GHP) holds if

$$\mathsf{A}[u_0]:=\sup_{R>0} R^{\frac{2}{1-m}-d}\int_{\mathbb{R}^d\setminus B_R(0)}|u_0|\,dx<\infty$$

Theorem

[Bonforte, Simonov, 2020] If $M+\mathsf{A}[u_0]<\infty$, then

$$\lim_{t\to\infty}\left\|\frac{u(t)-B(t)}{B(t)}\right\|_\infty=0$$

Uniform convergence in relative error

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m\in(m_1,1)$ if $d\ge2$, $m\in(1/3,1)$ if $d=1$ and let $\varepsilon\in(0,1/2)$, small enough, $A>0$, and $G>0$ be given. There exists an explicit threshold time $T\ge0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t}=\Delta u^m \tag{FDE}$$

with nonnegative initial datum $u_0\in\mathrm L^1(\mathbb R^d)$ satisfying

$$\mathsf A[u_0]=\sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}}\int_{|x|>r} u_0\,dx\le A<\infty \tag{$\mathsf H_A$}$$

$\int_{\mathbb R^d} u_0\,dx=\int_{\mathbb R^d} B\,dx=\mathscr M$ *and* $\mathscr F[u_0]\le G$, *then*

$$\sup_{x\in\mathbb R^d}\left|\frac{u(t,x)}{B(t,x)}-1\right|\le\varepsilon\quad\forall\,t\ge T$$

The threshold time

Proposition

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\varepsilon \in (0, \varepsilon_{m,d})$, $A > 0$ and $G > 0$

$$T = \mathsf{c}_\star \frac{1 + A^{1-m} + G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a} = \frac{\alpha}{\vartheta} \frac{2-m}{1-m}$, $\alpha = d\,(m - m_c)$ *and* $\vartheta = \nu/(d+\nu)$

$$\mathsf{c}_\star = \mathsf{c}_\star(m,d) = \sup_{\varepsilon \in (0, \varepsilon_{m,d})} \max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\, \varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\, \varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m) := \max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m} - 1}, \frac{2^{3-m}\,\kappa_\star}{1 - (1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m) := \frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}} \quad \text{and} \quad \kappa_3(\varepsilon,m) := \frac{8\,\alpha^{-1}}{1 - (1-\varepsilon)^{1-m}}$$

Improved entropy – entropy production inequality (subcritical case)

Theorem

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. Then there is a positive number ζ such that

$$\mathscr{I}[v]\ge(4+\zeta)\,\mathscr{F}[v]$$

for any nonnegative function $v\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v]=G$, $\int_{\mathbb{R}^d}v\,dx=\mathscr{M}$, $\int_{\mathbb{R}^d}x\,v\,dx=0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon\in(0,2\varepsilon_\star)\,,\quad \varepsilon_\star:=\frac12\min\{\varepsilon_{m,d},\chi\,\eta\}\quad\text{with}\quad t_\star=t_\star(\varepsilon)=\frac12\log R(T)$$

$$(1-\varepsilon)\,\mathscr{B}\le v(t,\cdot)\le(1+\varepsilon)\,\mathscr{B}\quad\forall\,t\ge t_\star$$

and, as a consequence, the *initial time layer estimate*

$$\mathscr{I}[v(t,.)]\ge(4+\zeta)\,\mathscr{F}[v(t,.)]\quad\forall\,t\in[0,t_\star]\quad\text{where}\quad\zeta=\frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}}$$

Two consequences

$$\zeta = \mathsf{Z}\big(A, \mathscr{F}[u_0]\big)\,, \quad \mathsf{Z}(A,G) := \frac{\zeta_\star}{1 + A^{(1-m)\,\frac{2}{\alpha}} + G}\,, \quad \zeta_\star := \frac{4\,\eta\,c_\alpha}{4+\eta}\left(\frac{\varepsilon_\star^{\mathsf{a}}}{2\,\alpha\,\mathsf{c}_\star}\right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ *if* $d \ge 2$, $m \in (1/2, 1)$ *if* $d = 1$, $A > 0$ *and* $G > 0$. *If* v *is a solution of* (r FDE) *with nonnegative initial datum* $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *such that* $\mathscr{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ *and* v_0 *satisfies* (H_A), *then*

$$\mathscr{F}[v(t,.)] \le \mathscr{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\, t \ge 0$$

▷ The ***stability in the entropy - entropy production estimate*** $\mathscr{I}[v] - 4\,\mathscr{F}[v] \ge \zeta\,\mathscr{F}[v]$ also holds in a stronger sense

$$\mathscr{I}[v] - 4\,\mathscr{F}[v] \ge \frac{\zeta}{4+\zeta}\,\mathscr{I}[v]$$

Stability results (subcritical case)

▷ We rephrase the results obtained by entropy methods in the language of stability *à la* Bianchi-Egnell

Subcritical range

$$p^* = +\infty \text{ if } d = 1 \text{ or } 2, \qquad p^* = \frac{d}{d-2} \text{ if } d \geq 3$$

$$\lambda[f]:=\left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2p}{d-p(d-4)}}\,,\quad \kappa[f]:=\frac{\mathscr{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

$$\mathsf{A}[f]:=\frac{\mathscr{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}}\,\|f\|_{2p}^{2p}}\,\sup_{r>0}r^{\frac{d-p(d-4)}{p-1}}\int_{|x|>r}|f(x+x_f)|^{2p}\,dx$$

$$\mathsf{E}[f]:=\frac{2p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}}\,f^{p+1}-\mathsf{g}^{p+1}-\frac{1+p}{2p}\,\mathsf{g}^{1-p}\left(\frac{\kappa[f]^{2p}}{\lambda[f]^2}\,f^{2p}-\mathsf{g}^{2p}\right)\right)dx$$

$$\mathfrak{S}[f]:=\frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf{Z}\big(\mathsf{A}[f],\mathsf{E}[f]\big)$$

Theorem

Let $d\ge1$, $p\in(1,p^*)$

If $f\in\mathscr{W}_p(\mathbb{R}^d):=\left\{f\in\mathrm{L}^{2p}(\mathbb{R}^d)\,:\,\nabla f\in\mathrm{L}^2(\mathbb{R}^d)\,,\ |x|\,f^p\in\mathrm{L}^2(\mathbb{R}^d)\right\}$,

$$\left(\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\right)^{2p\gamma}-\left(\mathscr{C}_{\rm GN}\,\|f\|_{2p}\right)^{2p\gamma}\ge\mathfrak{S}[f]\,\|f\|_{2p}^{2p\gamma}\,\mathsf{E}[f]$$

With $\mathscr{K}_{\mathrm{GNS}} = C(p,d)\,\mathscr{C}_{\mathrm{GNS}}^{2p\gamma}$, $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GNS}}\,\|f\|_{2p}^{2p\gamma}$$

Theorem

Let $d \ge 1$ and $p \in (1, p^*)$. There is an explicit $\mathscr{C} = \mathscr{C}[f]$ such that, for any $f \in \mathrm{L}^{2p}(\mathbb{R}^d, (1+|x|^2)\,dx)$ such that $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathsf{A}[f^{2p}] < \infty$,

$$\delta[f] \ge \mathscr{C}[f] \inf_{\varphi \in \mathfrak{M}} \int_{\mathbb{R}^d} \left|(p-1)\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 dx$$

▷ The dependence of $\mathscr{C}[f]$ on $\mathsf{A}[f^{2p}]$ and $\mathscr{F}[f^{2p}]$ is explicit and does not degenerate if $f \in \mathfrak{M}$

▷ Can we remove the condition $\mathsf{A}[f^{2p}] < \infty$?

Stability in Sobolev's inequality (critical case)

- ▷ A constructive stability result
- ▷ The main ingredient of the proof

A constructive stability result

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \geq 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\ |x|\,f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Theorem

Let $d \geq 3$ and $A > 0$. Then for any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, g\, dx \quad \textit{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \leq A$$

we have

$$\delta[f] := \|\nabla f\|_2^2 - \mathsf{S}_d^2\, \|f\|_{2^*}^2 \geq \frac{\mathscr{C}_\star(A)}{4+\mathscr{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla g^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathscr{C}_\star(A) = \mathfrak{C}_\star \left(1 + A^{1/(2d)}\right)^{-1}$ *and $\mathfrak{C}_\star > 0$ depends only on d*

Peculiarities of the critical case

▷ We can remove the normalization of f, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x+x_f) \quad \text{and} \quad Z[f] := \Big(1+\mu[f]^{-d}\,\lambda[f]^d\,A[f]\Big)$$

the *Bianchi-Egnell type result*

$$\delta[f] \ge \frac{\mathfrak{C}_\star\,Z[f]}{4+Z[f]} \inf_{g\in\mathfrak{M}} \mathscr{J}[f|g]$$

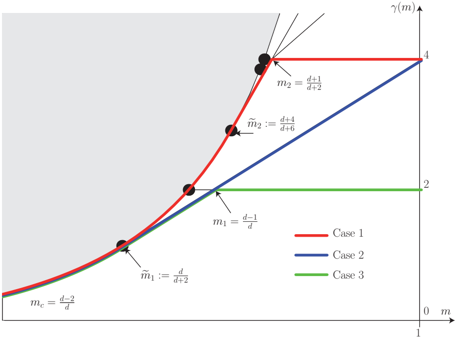
with x_f, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

▷ Notion of time delay [JD, Toscani, 2014 & 2015]

Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathscr{B}_\lambda(x) = \lambda^{-d/2}\,\mathscr{B}\big(x/\sqrt{\lambda}\big)$ in order to match $\int_{\mathbb{R}^d}|x|^2\,v\,dx$ where the function v solves $(r\,\mathrm{FDE})$ or to further rescale v according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\,w\left(t+\tau(t),\frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d}|x|^2\,v\,dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,,\quad \tau(0)=0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\,\tau(t)}$$

Lemma

$t\mapsto\lambda(t)$ and $t\mapsto\tau(t)$ are bounded on $\mathbb{R}^+$

Symmetry and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities

Caffarelli-Kohn-Nirenberg

Caffarelli-Kohn-Nirenberg inequalities

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) : |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

▷ *An optimal function among radial functions:*

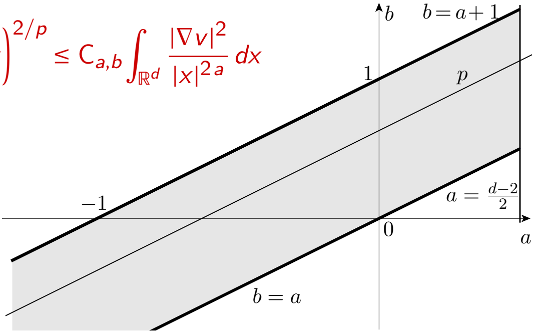
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

CKN: range of the parameters

Figure: $d=3$

$$\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}\le \mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a\le b\le a+1$ if $d\ge 3$

$a<b\le a+1$ if $d=2$, $a+1/2<b\le a+1$ if $d=1$

and $a<a_c:=(d-2)/2$

$$p=\frac{2\,d}{d-2+2\,(b-a)}$$

[Glaser, Martin, Grosse, Thirring (1976)]
[F. Catrina, Z.-Q. Wang (2001)]

▷ Proving symmetry breaking
[F. Catrina, Z.-Q. Wang], [V. Felli, M. Schneider (2003)]
[J.D., Esteban, Loss, Tarantello, 2009] There is a curve...

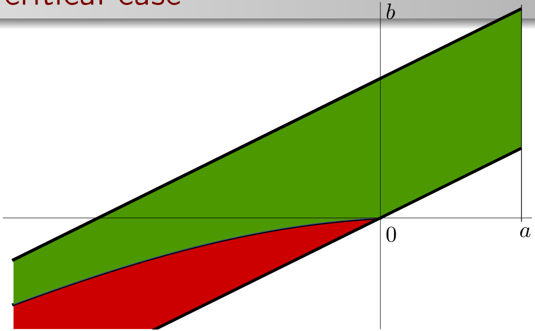
▷ Moving planes and symmetrization techniques
[Chou, Chu], [Horiuchi]
[Betta, Brock, Mercaldo, Posteraro]
+ Perturbation results: [CS Lin, ZQ Wang], [Smets, Willem], [JD, Esteban, Tarantello 2007], [J.D., Esteban, Loss, Tarantello, 2009]

▷ Linear instability of radial minimizers: the Felli-Schneider curve
[Catrina, Wang], [Felli, Schneider]

▷ Direct spectral estimates
[J.D., Esteban, Loss, 2011]: sharp interpolation on the sphere and a Keller-Lieb-Thirring spectral estimate on the line

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss, 2016]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The symmetry proof in one slide

- A change of variables: $v(|x|^{\alpha-1}x) = w(x)$, $\mathsf{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$

$$\|v\|_{2p,d-n} \le \mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|_{2,d-n}^{\vartheta}\,\|v\|_{p+1,d-n}^{1-\vartheta} \quad \forall\, v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

The Felli & Schneider condition becomes $\alpha > \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$ and $p = \frac{2n}{n-2}$

- Concavity of the Rényi entropy power: with

$\mathcal{L}_\alpha = -\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha = \alpha^2\left(u'' + \frac{n-1}{s}\,u'\right) + \frac{1}{s^2}\,\Delta_\omega\,u$ and $\frac{\partial u}{\partial t} = \mathcal{L}_\alpha\,u^m$

$$-\,\frac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\int_{\mathbb{R}^d} u^m\,|x|^{n-d}\,dx\right)^{1-\sigma}$$

$$\ge +\,2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\frac{1}{n}\right)\left|\mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}' - \frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right)u^m\,|x|^{n-d}\,dx$$

$$+\,2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\rm FS}^2 - \alpha^2\right)|\nabla_\omega\mathsf{P}|^2 + c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right)u^m\,|x|^{n-d}\,dx$$

- Elliptic regularity and the Emden-Fowler transformation: justifying the integrations by parts

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Thank you for your attention !