Entropy methods and hypocoercivity for large time asymptotics

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June 15, 2022

INDAM workshop on
Kolmogorov Operators and their Applications
Cortona
June 13-17, 2022

Outline

- **Introduction**
  ▷ Decay and convergence rates based on functional inequalities
- H^1 **Hypocoercivity**
  ▷ Entropy methods and *carré du champ*
- L^2 **Hypocoercivity**
  ▷ The diffusion limit
  ▷ Mode-by-mode analysis in Fourier variables
- **Functional inequalities and applications**
  ▷ Towards a systematic classification
  ▷ Some examples and extensions

Introduction

(Vlasov-)Fokker-Planck equation

Vlasov-Fokker-Planck equation without external potential

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f + \nabla_v (v\, f)$$

acting on a (probability) distribution function $f(t,x,v) \geq 0$
with time t, position x and velocity v

▷ ***Homogeneous case:*** no dependence in x, $\mathscr{M}(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}}$
standard Fokker-Planck equation

$$\|f(t,v) - \mathscr{M}(v)\|^2_{L^p(\mathbb{R}^d)} \leq \|f_0 - \mathscr{M}\|^2_{L^p(\mathbb{R}^d)}\, e^{-2t} \quad \forall\, t \geq 0$$

- *Beckner's inequalities* with Gaussian measure $d\mu = \mathscr{M}(v)\, dv$

$$\|h\|^2_{L^2(\mathbb{R}^d,d\mu)} - \|h\|^2_{L^q(\mathbb{R}^d,d\mu)} \leq (2-q)\ \|\nabla h\|^2_{L^2(\mathbb{R}^d,d\mu)} \quad \forall\, h \in H^1(\mathbb{R}^d, d\gamma)$$

applied to $h = (f/\mathscr{M})^p$, $q = 2/p \in (1,2]$

- $q = 2$: *Gaussian logarithmic Sobolev inequality* (Gross, 1975)

$$\int_{\mathbb{R}^d} h^2 \log\left(h^2/\|h\|^2_{L^q(\mathbb{R}^d,d\mu)}\right) d\mu \leq \|\nabla h\|^2_{L^2(\mathbb{R}^d,d\mu)} \quad \forall\, h \in H^1(\mathbb{R}^d, d\gamma)$$

▷ Inhomogeneous case: Green's function (Kolmogorov, 1934)

Vlasov-Fokker-Planck equation: methods

Vlasov-Fokker-Planck equation with external potential ψ

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \Delta_v f + \nabla_v (v\, f)$$

▷ Harmonic potential case: $\psi(x) = \frac{\kappa}{2}\,|x|^2$

- Decomposition on Hermite functions and spectral results
- Green's function as in Kolmogorov's computation

$$G(t,x,v) = \frac{\exp\left(-\frac{\gamma(t)\,|x|^2+\alpha(t)\,|v|^2+\beta(t)\,x\cdot v}{4\,\alpha(t)\,\gamma(t)-\beta^2(t)}\right)}{(2\,\pi)^d\,\left(4\,\alpha(t)\,\gamma(t)-\beta^2(t)\right)^{d/2}}$$

- Hypoelliptic methods (Hörmander, 1965)
- H^1 hypocoercivity (Villani, 2001 & 2005)
- L^2 hypocoercivity (Mouhot, Neumann, 2006), (Hérau, 2006), (JD, Mouhot, Schmeiser 2009 & 2015)
- H^{-1} hypocoercivity (Armstrong, Mourrat, 2019), (Brigati, 2021), (Cao, Lu, Wang, 2020), (Albritton-Armstrong-Mourrat-Novack, 2021)

A toy model

$$\frac{du}{dt} = (\mathsf{L}-\mathsf{T})\,u\,,\quad \mathsf{L} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},\quad \mathsf{T} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix},\quad k \neq 0$$

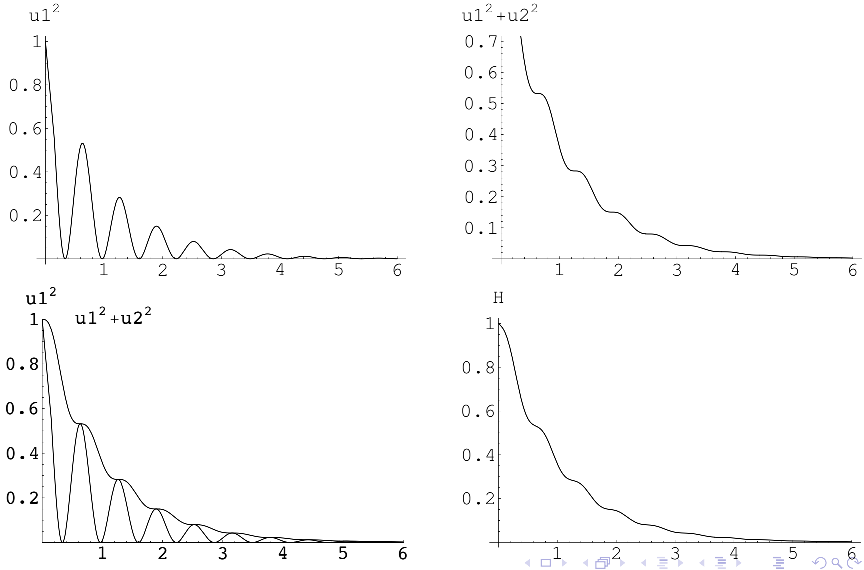
$$u = (u_1, u_2) \text{ and } |u|^2 = u_1^2 + u_2^2$$

Non-monotone decay, a well known picture:
see for instance (Filbet, Mouhot, Pareschi, 2006)

- H-theorem: $\frac{d}{dt}|u|^2 = -\,2\,u_2^2$
- macroscopic limit: $\frac{du_1}{dt} = -\,k^2\,u_1$
- generalized entropy: $\mathsf{H}(u) = |u|^2 - \frac{\delta\,k}{1+k^2}\,u_1\,u_2$

$$\begin{aligned} \frac{d\mathsf{H}}{dt} &= -\left(2 - \frac{\delta\,k^2}{1+k^2}\right) u_2^2 - \frac{\delta\,k^2}{1+k^2}\,u_1^2 + \frac{\delta\,k}{1+k^2}\,u_1\,u_2 \\ &\le -(2-\delta)\,u_2^2 - \frac{\delta\Lambda}{1+\Lambda}\,u_1^2 + \frac{\delta}{2}\,u_1 u_2 \end{aligned}$$

Plots for the toy problem



H^1 hypocoercivity

Definition of the φ-entropies

$$\mathcal{E}[w] := \int_{\mathbb{R}^d} \varphi(w)\,d\mu$$

φ is a nonnegative convex continuous function on $\mathbb{R}^+$ such that $\varphi(1) = 0$ and $1/\varphi''$ is concave on $(0, +\infty)$:

$$\varphi'' \geq 0\,, \quad \varphi \geq \varphi(1) = 0 \quad \text{and} \quad (1/\varphi'')'' \leq 0$$

Classical examples

$$\varphi_p(w) := \tfrac{1}{p-1}\left(w^p - 1 - p\,(w-1)\right) \quad p \in (1,2]$$

$$\varphi_1(w) := w\,\log w - (w-1)$$

To a *potential* ψ such that $e^{-\psi} \in \mathrm{L}^1(\mathbb{R}^d, dx)$, we associate the probability measure

$$d\mu = e^{-\psi}\,dx$$

Diffusions

Ornstein-Uhlenbeck equation or *backward Kolmogorov equation*

$$\frac{\partial w}{\partial t} = \mathsf{L}\, w := \Delta w - \nabla\psi \cdot \nabla w$$

- $\displaystyle -\int_{\mathbb{R}^d} (\mathsf{L}\, w_1)\, w_2\, d\mu = \int_{\mathbb{R}^d} \nabla w_1 \cdot \nabla w_2\, d\mu \quad \forall\, w_1,\, w_2 \in \mathrm{H}^1(\mathbb{R}^d, d\mu)$
- $\displaystyle 1 = \int_{\mathbb{R}^d} w_0\, d\mu = \int_{\mathbb{R}^d} w(t,\cdot)\, d\mu$ and $\lim_{t\to+\infty} w(t,\cdot) = 1$
- $\displaystyle \frac{d}{dt}\mathcal{E}[w] = -\int_{\mathbb{R}^d} \varphi''(w)\, |\nabla_x w|^2\, d\mu =: -\mathcal{I}[w]$ *(Fisher information)*

If for some $\Lambda > 0$: *entropy – entropy production* inequality

$$\mathcal{I}[w] \geq \Lambda\, \mathcal{E}[w] \quad \forall\, w \in \mathrm{H}^1(\mathbb{R}^d, d\mu)$$

$$\mathcal{E}[w(t,\cdot)] \leq \mathcal{E}[w_0]\, e^{-\Lambda\, t} \quad \forall\, t \geq 0$$

Fokker-Planck equation : $u = w\, \mu$ converges to $u_\star = \mu$

$$\frac{\partial u}{\partial t} = \Delta u + \nabla_x \cdot (u\, \nabla_x \psi)$$

Properties of the φ-entropies

- *Generalized Csiszár-Kullback-Pinsker inequality* (Pinsker), (Csiszár 1967), (Kullback 1967), (Cáceres, Carrillo, JD, 2002)

Proposition

Let $p \in [1,2]$, $w \in \mathrm{L}^1_+ \cap \mathrm{L}^p(\mathbb{R}^d, d\gamma)$, $\varphi \in C^2(0,+\infty)$ *such that* $\varphi(1) = \varphi'(1) = 0$. *If* $A := \inf_{s\in(0,\infty)} s^{2-p}\,\varphi''(s) > 0$, *then*

$$\mathcal{E}[w] \geq 2^{-\frac{2}{p}}\, A \,\min\left\{1, \|w\|^{p-2}_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}\right\} \|w-1\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}$$

- *Convexity, tensorization and sub-additivity*
- *Stability :* Holley-Stroock perturbation results

Entropy – entropy production, *carré du champ*

On a smooth convex bounded domain Ω, consider

$$\frac{\partial w}{\partial t} = \mathsf{L}\, w := \Delta w - \nabla\psi \cdot \nabla w\,, \quad \nabla w \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

$$\frac{d}{dt} \int_\Omega \frac{w^p - 1}{p-1}\, d\mu = -\frac{4}{p} \int_\Omega |\nabla z|^2\, d\mu \quad \text{and} \quad z = w^{p/2}$$

$$\frac{d}{dt} \int_\Omega |\nabla z|^2\, d\mu \le -\,2\,\Lambda(p) \int_\Omega |\nabla z|^2\, d\mu$$

where $\Lambda(p) > 0$ is the best constant in the inequality

$$\frac{2}{p}\,(p-1) \int_\Omega |\nabla X|^2\, d\mu + \int_\Omega \mathrm{Hess}\,\psi : X \otimes X\, d\mu \ge \Lambda(p) \int_\Omega |X|^2\, d\mu$$

Proposition

$$\int_\Omega \frac{w^p - 1}{p-1}\, d\mu \le \frac{4}{p\,\Lambda} \int_\Omega |\nabla w^{p/2}|^2\, d\mu \quad \textit{for any } w \textit{ s.t.} \quad \int_\Omega w\, d\mu = 1$$

(Bakry, Emery 1985)

φ-hypocoercivity (H^1 framework)

▷ Adapt φ-entropies to kinetic equations

▷ Villani's strategy: derive H^1 estimates (using a twisted Fisher information) and then use standard interpolation inequalities to establish decay rates for the entropy

The *kinetic Fokker-Planck equation*, or *Vlasov-Fokker-Planck equation*

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v\, f)$$

with $\psi(x) = |x|^2/2$ and $\|f\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$ has a unique nonnegative stationary solution

$$\mathscr{M}(x,v) = (2\,\pi)^{-d}\, e^{-\frac{1}{2}(|x|^2+|v|^2)}$$

and $g = f/\mathscr{M}$ solves the *kinetic Ornstein-Uhlenbeck equation*

$$\frac{\partial g}{\partial t} + \mathsf{T} g = \mathsf{L}\, g$$

with transport operator T and Ornstein-Uhlenbeck operator L

$$\mathsf{T} g := v \cdot \nabla_x g - x \cdot \nabla_v g \quad \text{and} \quad \mathsf{L}\, g := \Delta_v g - v \cdot \nabla_v g$$

The function $h = g^{p/2}$ solves $\frac{\partial h}{\partial t} + \mathsf{T} h = \mathsf{L}\, h + \frac{2-p}{p}\, \frac{|\nabla_v h|^2}{h}$

Sharp rates for the kinetic Fokker-Planck equation

Let $\psi(x) = |x|^2/2$, $d\mu := \mathscr{M}\, dx\, dv$, $\mathcal{E}[g] := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \varphi_p(g)\, d\mu$

Proposition

Let $p \in [1,2]$ and consider a nonnegative solution of the kinetic Fokker-Planck equation. There is a constant $\mathcal{C} > 0$ such that

$$\mathcal{E}[g(t,\cdot,\cdot)] \le \mathcal{C}\, e^{-t} \quad \forall\, t \ge 0$$

and the rate e^{-t} is sharp as $t \to +\infty$

(Villani), (Arnold, Erb): a twisted *Fisher information* functional

$$\mathcal{J}_\lambda[h] = (1-\lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2\, d\mu + (1-\lambda) \int_{\mathbb{R}^d} |\nabla_x h|^2\, d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2\, d\mu$$

(Arnold, Erb) relies on $\lambda = 1/2$ and $\frac{d}{dt} \mathcal{J}_{1/2}[h(t,\cdot)] \le -\mathcal{J}_{1/2}[h(t,\cdot)]$

Improved rates (in the large entropy regime)

Rewrite the decay of the *Fisher information* functional as

$$-\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^d} X^\perp\cdot\mathfrak{M}_0\,X\,d\mu=\int_{\mathbb{R}^d} X^\perp\cdot\mathfrak{M}_1\,X\,d\mu+\int_{\mathbb{R}^d} Y^\perp\cdot\mathfrak{M}_2\,Y\,d\mu$$

where $X=(\nabla_v h,\nabla_x h)\;,\quad Y=(\mathsf{H}_{vv},\mathsf{H}_{xv},Fvv,Fxv)$

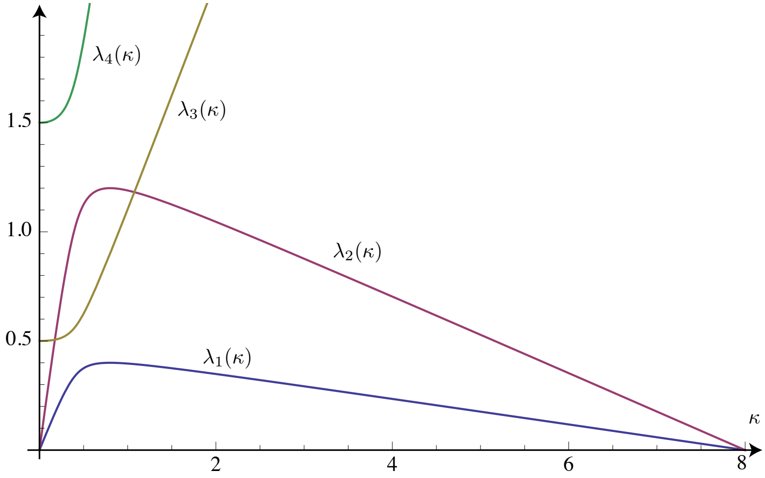
$$\mathfrak{M}_0=\begin{pmatrix}1&\lambda\\ \lambda&\nu\end{pmatrix}\otimes\mathrm{Id}_{\mathbb{R}^d}\,,\quad \mathfrak{M}_1=\begin{pmatrix}1-\lambda&\frac{1+\lambda-\nu}{2}\\ \frac{1+\lambda-\nu}{2}&\lambda\end{pmatrix}\otimes\mathrm{Id}_{\mathbb{R}^d}$$

$$\mathfrak{M}_2=\begin{pmatrix}1&\lambda&-\frac{\kappa}{2}&-\frac{\kappa\,\lambda}{2}\\ \lambda&\nu&-\frac{\kappa\,\lambda}{2}&-\frac{\kappa\,\nu}{2}\\ -\frac{\kappa}{2}&-\frac{\kappa\,\lambda}{2}&2\,\kappa&2\,\kappa\,\lambda\\ -\frac{\kappa\,\lambda}{2}&-\frac{\kappa\,\nu}{2}&2\,\kappa\,\lambda&2\,\kappa\nu\end{pmatrix}\otimes\mathrm{Id}_{\mathbb{R}^d\times\mathbb{R}^d}$$

With constant coefficients

$$\lambda_\star(\lambda,\nu)=\max\left\{\min_X\frac{X^\perp\cdot\mathfrak{M}_1\,X}{X^\perp\cdot\mathfrak{M}_0\,X}\;:\;(\lambda,\nu)\in\mathbb{R}^2\text{ s.t. }\mathfrak{M}_2\geq 0\right\}$$

For $(\lambda, \nu) = (1/2)$, $\lambda_\star = 1/2$ and the eigenvalues of $\mathfrak{M}_2(\frac{1}{2}, 1)$ are given as a function of $\kappa = 8\,(2-p)/p \in [0,8]$ are all nonnegative



Theorem

Let $p \in (1,2)$ and h be a solution of the kinetic Ornstein-Uhlenbeck equation. Then there exists a function $\lambda : \mathbb{R}^+ \to [1/2, 1)$ such that $\lambda(0) = \lim_{t\to+\infty} \lambda(t) = 1/2$ and a function $\rho > 1$ s.t.

$$\frac{d}{dt}\mathcal{J}_{\lambda(t)}[h(t,\cdot)] \leq -\,\rho(t)\,\mathcal{J}_{\lambda(t)}[h(t,\cdot)]$$

As a consequence, for any $t \geq 0$ we have the global estimate

$$\mathcal{J}_{\lambda(t)}[h(t,\cdot)] \leq \mathcal{J}_{1/2}[h_0]\,\exp\left(-\int_0^t \rho(s)\,ds\right)$$

L^2 Hypocoercivity

- ▷ Abstract statement, diffusion limit
- ▷ Mode-by-mode analysis in Fourier variables
- ▷ Refined decay rates in the whole space

Collaboration with C. Mouhot and C. Schmeiser
+ E. Bouin, S. Mischler

An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$
* denotes the adjoint with respect to $\langle\cdot,\cdot\rangle$

Π is the orthogonal projection onto the null space of L

The estimate

$$\frac{1}{2}\,\frac{d}{dt}\|F\|^2 = \langle\mathsf{L}F, F\rangle \le -\,\lambda_m\,\|(1-\Pi)F\|^2$$

is not enough to conclude that $\|F(t,\cdot)\|^2$ decays exponentially
$\Leftarrow$ *microscopic coercivity*

Formal macroscopic / diffusion limit

$F = F(t,x,v)$, $\mathsf{T} = v\cdot\nabla_x$, L good collision operator. Scaled evolution equation

$$\varepsilon\,\frac{dF}{dt} + \mathsf{T}F = \frac{1}{\varepsilon}\,\mathsf{L}F$$

on the Hilbert space $\mathcal{H}$. $F_\varepsilon = F_0 + \varepsilon\,F_1 + \varepsilon^2\,F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$

$$\begin{aligned}
\varepsilon^{-1}: &\qquad \mathsf{L}F_0 = 0\,,\\
\varepsilon^{0}: &\qquad \mathsf{T}F_0 = \mathsf{L}F_1\,,\\
\varepsilon^{1}: &\qquad \tfrac{dF_0}{dt} + \mathsf{T}F_1 = \mathsf{L}F_2
\end{aligned}$$

The first equation reads as $u = F_0 = \Pi F_0$
The second equation is simply solved by $F_1 = -\,(\mathsf{T}\Pi)\,F_0$
After projection, the third equation is

$$\tfrac{d}{dt}\,(\Pi F_0) - \,\Pi\mathsf{T}\,(\mathsf{T}\Pi)\,F_0 = \Pi\mathsf{L}F_2 = 0$$

$$\partial_t u + (\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)\,u = 0$$

is such that $\frac{d}{dt}\|u\|^2 = -\,2\,\|(\mathsf{T}\Pi)\,u\|^2 \le -\,2\,\lambda_M\,\|u\|^2$
$\Leftarrow$ *Macroscopic coercivity*

The macro part and the Poincaré inequality

▷ Free transport operator: $\mathsf{T}F = v\cdot\nabla_x F$
If $F_0(x,v) = u(x)\,\mathcal{M}(v)$ with $\mathcal{M}(v) = (2\pi)^{-d/2}\,e^{-|v|^2/2}$ then

$$(\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)F_0 = (-\Delta_x u)\,\mathcal{M}$$

and we obtain the heat equation (*e.g.* on $\mathbb{T}^d$)

$$\partial_t u = \Delta\, u$$

▷ With an external potential ψ so that $\mathsf{T}F = v\cdot\nabla_x F - \nabla_x\psi\cdot\nabla_v F$
we obtain the Fokker-Planck equation

$$\partial_t u = \Delta\, u + \nabla\cdot(u\,\nabla\psi)$$

The operator $\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$ is such that

$$\langle \mathsf{A}\mathsf{T}\Pi F, F\rangle \ge \frac{\lambda_M}{1+\lambda_M}\,\|\Pi F\|^2$$

if the Poincaré inequality $\int_{\mathbb{R}^d}|\nabla u|^2\,d\mu \ge \lambda_M\int_{\mathbb{R}^d}|u-\bar u|^2\,d\mu$ holds

The assumptions in the compact case

λ_m, λ_M, and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity:*

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \, \|(1-\Pi)F\|^2 \tag{H1}$$

▷ *macroscopic coercivity:*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \, \|\Pi F\|^2 \tag{H2}$$

▷ *parabolic macroscopic dynamics:*

$$\Pi \mathsf{T} \Pi \, F = 0 \tag{H3}$$

▷ *bounded auxiliary operators:*

$$\|\mathsf{AT}(1-\Pi)F\| + \|\mathsf{AL}F\| \leq C_M \, \|(1-\Pi)F\| \tag{H4}$$

Equivalence and entropy decay

For some $\delta > 0$ to be chosen, the L^2 entropy / Lyapunov functional is defined by

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

$\triangleright$ *norm equivalence* of $\mathsf{H}[F]$ and $\|F\|^2$

$$\tfrac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \tfrac{2+\delta}{4}\,\|F\|^2$$

Entropy decay: $\frac{d}{dt}\mathsf{H}[F] = -\,\mathsf{D}[F]$

$\triangleright$ *entropy decay rate:* for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$\mathsf{D}[F] \ge \lambda\,\mathsf{H}[F]$$

Theorem

Under (H1)–(H4), *for any* $t \ge 0$,

$$\mathsf{H}[F(t,\cdot)] \le \mathsf{H}[F_0]\,e^{-\lambda\,t}$$
$$\|F(t,\cdot)\|^2 \le \mathcal{C}\,\|F_0\|^2\,e^{-\lambda\,t} \quad \textit{with} \quad \mathcal{C} = \frac{2+\delta}{2-\delta}$$

Basic example: with confinement

Vlasov-Fokker-Planck equation with harmonic potential

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v\, f)$$

▷ *microscopic coercivity:* Gaussian Poincaré inequality in v

$$\int_{\mathbb{R}^d} |F(v) - \rho\, \mathcal{M}(v)|^2 \, \frac{dv}{\mathcal{M}(v)} \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla_v F(v)^2 \, \frac{dv}{\mathcal{M}(v)} \tag{H1}$$

with $\rho = \int_{\mathbb{R}^d} F(v)\, dv$

▷ *macroscopic coercivity:* Gaussian Poincaré inequality in x

$$\int_{\mathbb{R}^d} \left| \rho(x) - \frac{M\, e^{-|x|^2/2}}{(2\pi)^{d/2}} \right|^2 e^{\frac{|x|^2}{2}} \, dx \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla_x \rho(x)|^2 \, e^{\frac{|x|^2}{2}} \, dx \tag{H2}$$

▷ *parabolic macroscopic dynamics* (H3) and *bounded auxiliary operators* (H4) are consequences of elliptic estimates

Basic examples without confinement

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad f(0,x,v) = f_0(x,v)$$

L is the *Fokker-Planck operator* L_1 or the *linear BGK operator* L_2

$$\mathsf{L}_1 f := \Delta_v f + \nabla_v \cdot (v\, f) \quad \text{and} \quad \mathsf{L}_2 f := \rho_f\, \mu - f$$

$\mu(v) = \frac{e^{-\frac12\,|v|^2}}{(2\,\pi)^{d/2}}$ is the normalized Gaussian function

$\rho_f := \int_{\mathbb{R}^d} f\, dv$ is the spatial density

$$d\gamma := \gamma(v)\, dv \quad \text{where} \quad \gamma := \frac{1}{\mu}$$

$$\|f\|^2_{\mathrm{L}^2(dx\, d\gamma)} := \iint_{\mathbb{R}^d\times\mathbb{R}^d} |f(x,v)|^2\, dx\, d\gamma$$

Fourier variables: mode-by-mode hypocoercivity

Let us consider the Fourier transform in x, denote by $\xi \in \mathbb{R}^d$ the Fourier variable, so that $F = \hat{f}$ solves

$$\partial_t F + \mathsf{T} F = \mathsf{L} F\,, \quad F(0,\xi,v) = \hat{f}_0(\xi,v)\,, \quad \mathsf{T} F = i\,(v\cdot\xi) F$$

Goal: apply the abstract method with ξ considered as a parameter

$$\mathcal{H} = \mathrm{L}^2\,(d\gamma)\;, \quad \|F\|^2 = \int_{\mathbb{R}^d} |F|^2\,d\gamma\,, \quad \Pi F = \mu \int_{\mathbb{R}^d} F\,dv = \mu\,\rho_F$$

The operator A is now defined as

$$(\mathsf{A} F)(v) = \frac{-\,i\,\xi}{1+|\xi|^2} \cdot \int_{\mathbb{R}^d} w\,F(w)\,dw\,\mu(v)$$

and, with $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$, we have that

$$|\mathrm{Re}\langle \mathsf{A} F, F\rangle| \le \frac{|\xi|}{1+|\xi|^2}\,X\,Y\,, \quad \|F\|^2 = X^2 + Y^2$$

$$\frac{1}{2}\left(1-\frac{\delta\,|\xi|}{1+|\xi|^2}\right)(X^2+Y^2) \le \mathsf{H}[F] \le \frac{1}{2}\left(1+\frac{\delta\,|\xi|}{1+|\xi|^2}\right)(X^2+Y^2)$$

Entropy – entropy production inequality

$$-\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT}\Pi F, F\rangle \ge X^2 + \frac{\delta\,|\xi|^2}{1+|\xi|^2}\,Y^2$$

$$\begin{aligned}\mathsf{D}[F] &= -\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT}\Pi F, F\rangle + \delta\,(\ldots)\\ &\ge (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y\end{aligned}$$

with $\lambda_m = 1\,,\quad \Lambda_M = |\xi|^2 =: s^2\,,\quad C_M = \dfrac{s\,(1+\sqrt{3}\,s)}{1+s^2}$

$$\begin{aligned}&\mathsf{D}[F] - \lambda\,\mathsf{H}[F]\\ &\quad\ge \left(1 - \tfrac{\delta\,s^2}{1+s^2} - \tfrac{\lambda}{2}\right) X^2 - \tfrac{\delta\,s}{1+s^2}\left(1+\sqrt{3}\,s+\lambda\right) X\,Y + \left(\tfrac{\delta\,s^2}{1+s^2} - \tfrac{\lambda}{2}\right) Y^2\end{aligned}$$

is (for any $s = |\xi| > 0$) a nonnegative quadratic form of X and Y iff...

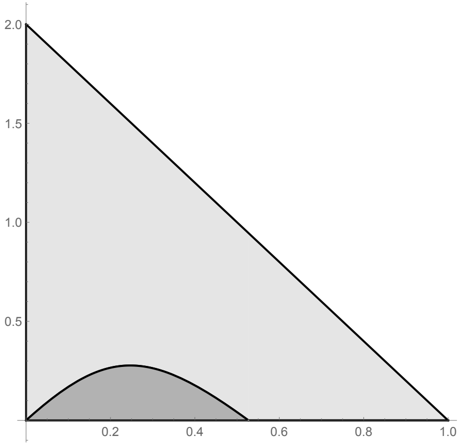


Figure: Horizontal axis: $\delta/2$, vertical axis: λ. Admissible region: grey triangle. Negative discriminant: dark grey area, shown here for $s = 5$

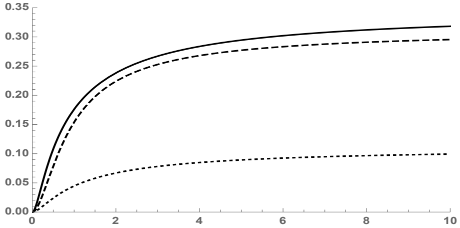


Figure: $s \mapsto \lambda(s)$

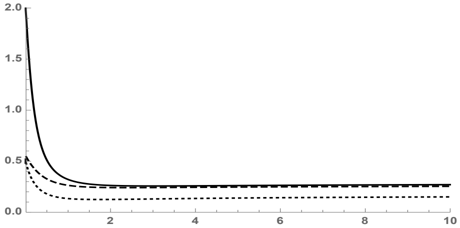


Figure: $s \mapsto \delta(s)$

Results (whole space, no external potential)

On the whole Euclidean space, we can define the entropy

$$\mathsf{H}[f] := \frac{1}{2}\,\|f\|^2_{\mathrm{L}^2(dx\,d\gamma)} + \delta\,\langle \mathsf{A} f, f\rangle_{dx\,d\gamma}$$

Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|^2_{\mathrm{L}^2(dx)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(dx)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(dx)}$$

proves that

$$\mathsf{H}[f] \le C\left(\mathsf{H}[f_0] + \|f_0\|^2_{\mathrm{L}^1(dx\,dv)}\right)(1+t)^{-\frac{d}{2}}$$

(Bouin, JD, Mischler, Mouhot, Schmeiser)

Theorem

There exists a constant $C>0$ such that, for any $t\ge 0$

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma)} \le C\left(\|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma)} + \|f_0\|^2_{\mathrm{L}^2(d\gamma;\,\mathrm{L}^1(dx))}\right)(1+t)^{-\frac{d}{2}}$$

Comments

▷ Use of the *enlargement of the space* method or *factorization method* of (Gualdani, Mischler, Mouhot)

▷ Not limited to Maxwellian local equilibria

▷ Can be compared with spectral methods based on *Lyapunov matrix inequalities* and *twisted Euclidean norms* (Arnold, JD, Schmeiser, Wöhrer, 2021)

▷ Sharper but in most cases still suboptimal estimates can be given with A defined as a pseudo-differential operator (Arnold, JD, Schmeiser, Wöhrer, 2021)

Functional inequalities and hypocoercivity

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In collaboration with Lanoir Addala, Emeric Bouin, Kleber Carrapatoso, Frédéric Hérau, Laurent Lafleche, Xingyu Li, Stéphane Mischler, Clément Mouhot, Christian Schmeiser, Lazhar Tayeb

The global picture: from diffusive to kinetic

- Depending on the local equilibria and on the external potential (H1) and (H2) (which are Poincaré type inequalities) can be replaced by other functional inequalities:

▷ *microscopic coercivity* (H1)

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \|(1-\Pi)F\|^2$$

$\Longrightarrow$ ***weak Poincaré inequalities*** or ***Hardy-Poincaré inequalities***

▷ *macroscopic coercivity* (H2)

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2$$

$\Longrightarrow$ ***Nash inequality***, ***weighted Nash*** or ***Caffarelli-Kohn-Nirenberg inequalities***

- This can be done at the level of the *diffusion equation* (homogeneous case) or at the level of the *kinetic equation* (non-homogeneous case)

Diffusion (Fokker-Planck) equations

Potential	$V = 0$	$V(x) = \gamma \log\|x\|$ $\gamma < d$	$V(x) = \|x\|^\alpha$ $\alpha \in (0,1)$	$V(x) = \|x\|^\alpha$ $\alpha \geq 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 1: $\partial_t u = \Delta u + \nabla \cdot (u \nabla V)$

Kinetic Fokker-Planck equations

B = Bouin, L = Lafleche, M = Mouhot, MM = Mischler, Mouhot
S = Schmeiser

Potential	$V = 0$	$V(x) = \gamma \log\|x\|$ $\gamma < d$	$V(x) = \|x\|^\alpha$ $\alpha \in (0,1)$	$V(x) = \|x\|^\alpha$ $\alpha \geq 1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v) = e^{-\langle v \rangle^\beta}$, $\beta \geq 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b < 1$, $\beta = 2$ convergence	DMS, Mischler-Mouhot $e^{-\lambda t}$ convergence
$F(v) = e^{-\langle v \rangle^\beta}$, $\beta \in (0,1)$	BDLS: $t^{-\zeta}$, $\zeta =$ $\min\left\{\frac{d}{2}, \frac{k}{\beta}\right\}$ decay			
$F(v) = \langle v \rangle^{-d-\beta}$	BDLS, fractional			

Table 1: $\partial_t f + v \cdot \nabla_x f = F\,\nabla_v\big(F^{-1}\,\nabla_v f\big)$. Notation: $\langle v \rangle = \sqrt{1+|v|^2}$

Two additional examples

Linearized Vlasov-Poisson-Fokker-Planck system

The *Vlasov-Poisson-Fokker-Planck system* in presence of an external potential ψ is

$$\begin{aligned}\partial_t f + v\cdot\nabla_x f - (\nabla_x\psi + \nabla_x\phi)\cdot\nabla_v f &= \Delta_v f + \nabla_v\cdot(v\,f)\\ -\Delta_x\phi = \rho_f &= \int_{\mathbb{R}^d} f\,dv\end{aligned} \qquad \text{(VPFP)}$$

Linearized problem around $f_\star$: $f = f_\star\,(1+\eta\,h)$, $\iint_{\mathbb{R}^d\times\mathbb{R}^d} h\,f_\star\,dx\,dv = 0$

$$\partial_t h + v\cdot\nabla_x h - (\nabla_x\psi + \nabla_x\phi_\star)\cdot\nabla_v h + v\cdot\nabla_x\psi_h - \Delta_v h + v\cdot\nabla_v h = \eta\,\nabla_x\psi_h\cdot\nabla_v h$$

$$-\Delta_x\psi_h = \int_{\mathbb{R}^d} h\,f_\star\,dv$$

Drop the $\mathcal{O}(\eta)$ term : *linearized Vlasov-Poisson-Fokker-Planck / Ornstein-Uhlenbeck* system

$$\partial_t h + v\cdot\nabla_x h - (\nabla_x\psi + \nabla_x\phi_\star)\cdot\nabla_v h + v\cdot\nabla_x\psi_h - \Delta_v h + v\cdot\nabla_v h = 0$$

$$-\Delta_x\psi_h = \int_{\mathbb{R}^d} h\,f_\star\,dv\,,\quad \iint_{\mathbb{R}^d\times\mathbb{R}^d} h\,f_\star\,dx\,dv = 0 \qquad \text{(VPFPlin)}$$

Hypocoercivity

Let us define the norm

$$\|h\|^2 := \iint_{\mathbb{R}^d\times\mathbb{R}^d} h^2\, f_\star\, dx\, dv + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2\, dx$$

(Addala, JD, Li, Tayeb)

Theorem

Let us assume that $d \ge 1$, $\psi(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. Then there exist two positive constants $\mathcal{C}$ and λ such that any solution h of (VPFPlin) *with an initial datum h_0 of zero average with $\|h_0\|^2 < \infty$ is such that*

$$\|h(t,\cdot,\cdot)\|^2 \le \mathcal{C}\, \|h_0\|^2\, e^{-\lambda\, t} \quad \forall\, t \ge 0$$

Fractional diffusion limits and hypocoercivity

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f$$

with *fat tail local equilibrium* μ

$$\forall\, v \in \mathbb{R}^d, \quad \mu(v) = \frac{c_\gamma}{\langle v\rangle^{d+\gamma}} \quad \text{where} \quad \langle v\rangle := \sqrt{1+|v|^2}\,.$$

▷ *Fokker-Planck* type operator ($\beta = 2$)

$$\mathsf{L}_1 f := \nabla_v \cdot \big(\mu\, \nabla_v \big(\mu^{-1} f\big)\big)$$

▷ *linear Boltzmann* operator, or *scattering* collision operator

$$\mathsf{L}_2 f := \int_{\mathbb{R}^d} \mathrm{b}(\cdot, v') \Big(f(v')\, \mu(\cdot) - f(\cdot)\, \mu(v')\Big)\, \mathrm{d}v'$$

with *collision frequency* $\nu(v) := \int_{\mathbb{R}^d} \mathrm{b}(v, v')\, \mu(v')\, \mathrm{d}v' \underset{|v| \to +\infty}{\sim} |v|^{-\beta}$

▷ the *fractional Fokker-Planck* operator ($0 < \sigma < 2$, $\beta = \sigma - \gamma$)

$$\mathsf{L}_3 f := \Delta_v^{\sigma/2} f + \nabla_v \cdot (E\, f)$$

\+ technical conditions

(Bouin, JD, Lafleche, 2022)

Theorem

Let $d \geq 2$, $\beta \in \mathbb{R}$, $\gamma > \max\{0, -\beta\}$ *and* $k \in [0, \gamma)$ *such that* $\gamma \neq 2 + \beta$ *or if* $\gamma = 2 + \beta$ *and* $\frac{k}{\beta_+} > \frac{d}{2}$. *If* f *is a solution with initial condition* $f^{\rm in} \in \mathrm{L}^1(\,\mathrm{d}x\,\mathrm{d}v) \cap \mathrm{L}^2\big(\langle v\rangle^k\,\mathrm{d}x\,\mu^{-1}\,\mathrm{d}v\big)$, *then for any* $t \geq 0$,

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\,\mathrm{d}x\,\mu^{-1}\,\mathrm{d}v)} \lesssim \frac{\big\|f^{\rm in}\big\|^2_{\mathrm{L}^1(\,\mathrm{d}x\,\mathrm{d}v)} + \big\|f^{\rm in}\big\|^2_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mu^{-1}\,\mathrm{d}v)}}{(1+t)^\tau}$$

with $\tau = \min\left\{\frac{d}{\alpha}, \frac{k}{\beta_+}\right\}$ *and* $\alpha = \min\left\{\frac{\gamma+\beta}{1+\beta}, 2\right\}$

The exponent α arises from the *fractional diffusion limit*

$$\partial_t u + (-\Delta)^{\alpha/2} u = 0$$

cf. (Mellet, Mischler, Mouhot, 2011), (Jara, Komorowski, Olla), (Bouin, Mouhot)

Special macroscopic modes and hypocoercivity

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Joint work with with Kleber Carrapatoso, Frédéric Hérau, Stéphane Mischler, Clément Mouhot, Christian Schmeiser

The equation

Consider the kinetic equation

$$\partial_t f = \mathscr{L} f := \mathscr{T} f + \mathscr{C} f \,, \quad f_{|t=0} = f_0$$

with *transport operator* $\mathscr{T}$ given by

$$\mathscr{T} f := -\, v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f$$

where $\phi \in C^2(\mathbb{R}^d, \mathbb{R})$. Let $\rho(x) := e^{-\phi(x)}$ and $\langle \varphi \rangle := \int_{\mathbb{R}^d} \varphi \, \rho \, \mathrm{d}x$

Linear collision operator

$\mathscr{C}$ acts only on $v \in \mathbb{R}^d$, is self-adjoint in $\mathrm{L}^2(\mu^{-1})$, with $\mu(v) := \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}$ and has the $(d+2)$-dimensional kernel of *collision invariants* given by

$$\operatorname{Ker}\mathscr{C} = \operatorname{Span}\left\{\mu, v_1\,\mu, \ldots, v_d\,\mu, |v|^2\,\mu\right\}$$

▷ *Spectral gap property*

$$-\int_{\mathbb{R}^d} f(v)\,\mathscr{C}f(v)\,\frac{dv}{\mu(v)} \geq \mathrm{c}_{\mathscr{C}}\,\|f - \Pi f\|^2_{\mathrm{L}^2(\mu^{-1})}$$

where Π denotes the $\mathrm{L}^2(\mu^{-1})$-orthogonal projection onto $\operatorname{Ker}\mathscr{C}$

▷ For any polynomial function $p(v) : \mathbb{R}^d \to \mathbb{R}$ of degree at most 4, the function $p\,\mu$ is in the domain of $\mathscr{C}$ and

$$C(p) := \|\mathscr{C}(p\,\mu)\|_{\mathrm{L}^2(\mu^{-1})} < \infty$$

Other assumptions (1/2)

▷ *Normalization conditions*:

$$\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1\,, \quad \int_{\mathbb{R}^d} x\,\rho(x)\,\mathrm{d}x = 0\,, \quad \langle \nabla_x^2 \phi \rangle = \int_{\mathbb{R}^d} \nabla_x^2 \phi\,\rho\,\mathrm{d}x = \mathrm{Id}_{d\times d}$$

▷ *Growth/regularity assumption*

$$|\nabla_x^2 \phi| \le \varepsilon\,|\nabla_x \phi|^2 + C_\varepsilon$$

▷ *Poincaré inequality*

$$c_{\mathrm{P}} \int_{\mathbb{R}^d} |u - \langle u \rangle|^2\,\rho\,\mathrm{d}x \le \int_{\mathbb{R}^d} |\nabla_x u|^2\,\rho\,\mathrm{d}x$$

▷ *Moment bounds on* ρ

$$\int_{\mathbb{R}^d} \left(|x|^4 + |\phi|^2 + |\nabla_x \phi|^4 \right) \rho\,\mathrm{d}x \le \mathrm{C}_\phi$$

Other assumptions (2/2)

▷ *Semi-group property*

$$t \mapsto e^{t\mathscr{L}} \text{ is a strongly continuous semi-group on } \mathrm{L}^2(\mathscr{M}^{-1})$$

where $\mathscr{M}$ is the *global Maxwellian equilibrium*

$$\mathscr{M}(x,v) := \rho(x)\,\mu(v) = \frac{e^{-\frac{1}{2}|v|^2-\phi(x)}}{(2\,\pi)^{d/2}}$$

Special macroscopic modes (1/2)

Special macroscopic modes $\quad \mathscr{C} F = 0\,, \quad \partial_t F = \mathscr{T} F$

$$F = \big(r(t,x) + m(t,x)\cdot v + e(t,x)\,\mathfrak{E}(v)\big)\,\mathscr{M}\,, \quad \mathfrak{E}(v) := \frac{|v|^2 - d}{\sqrt{2\,d}}$$

▷ *Energy mode* $F = \mathcal{H}\,\mathscr{M}$ with

$$\mathcal{H}(x,v) := \tfrac{1}{2}\left(|v|^2 - d\right) + \phi(x) - \langle\phi\rangle$$

▷ The set of *infinitesimal rotations compatible with* ϕ defined as

$$\mathcal{R}_\phi := \left\{x \mapsto A\,x \;:\; A \in \mathfrak{M}^{\rm skew}_{d\times d}(\mathbb{R}) \text{ s.t. } \forall\,x \in \mathbb{R}^d\,,\; \nabla_x\phi(x)\cdot A\,x = 0\right\}$$

gives rise *rotation modes compatible with* ϕ

$$(A\,x\cdot v)\,\mathscr{M}(x,v)\,, \quad A \in \mathcal{R}_\phi$$

Special macroscopic modes (2/2): harmonic modes

Harmonic directions $E_\phi := \underset{\mathbb{R}^d}{\mathrm{Span}}\{\nabla_x\phi(x) - x\}_{x\in\mathbb{R}^d}$, $d_\phi := \dim E_\phi$

$\triangleright$ the potential is *partially harmonic* if $1 \le d_\phi \le d-1$
harmonic directional modes are defined by

$(x_i \cos t - v_i \sin t)\,\mathscr{M}$, $\quad (x_i \sin t + v_i \cos t)\,\mathscr{M}$, $\quad i \in I_\phi := \{d_\phi+1, \ldots, d\}$

$\triangleright$ If $d_\phi = 0$, the potential $\phi(x) = \frac{1}{2}\,|x|^2 + \frac{d}{2}\,\log(2\,\pi)$ is *fully harmonic*
In addition to the harmonic directional modes, there are *harmonic pulsating modes*

$$\left(\tfrac{1}{2}\left(|x|^2 - |v|^2\right)\cos(2\,t) - x\cdot v\,\sin(2\,t)\right)\mathscr{M}$$
$$\left(\tfrac{1}{2}\left(|x|^2 - |v|^2\right)\sin(2\,t) + x\cdot v\,\cos(2\,t)\right)\mathscr{M}$$

(Boltzmann, 1876) (Cercignani, 1983) (Uhlenbeck, Ford, 1963)

Theorem (Special macroscopic modes and hypocoercivity)

(1) *All special macroscopic modes are given by*

$$F = \alpha\,\mathscr{M} + \beta\,\mathcal{H}\,\mathscr{M} + A\,x\cdot v\,\mathscr{M} + F_{\mathrm{dir}} + F_{\mathrm{pul}}$$

(2) *There are explicit constants* $C>0$ *and* $\lambda>0$ *such that, for any solution* $f\in\mathcal{C}\big(\mathbb{R}^+;\mathrm{L}^2(\mathscr{M}^{-1})\big)$ *with initial datum* f_0*, there exists a unique special macroscopic mode* F *such that*

$$\forall\,t\ge0\,,\quad \|f(t)-F(t)\|_{\mathrm{L}^2(\mathscr{M}^{-1})}\le C\,e^{-\lambda\,t}\,\|f_0-F(0)\|_{\mathrm{L}^2(\mathscr{M}^{-1})}$$

A micro-macro decomposition

$$\partial_t h = \mathcal{L}\, h := \mathscr{T} h + \mathcal{C} h\,, \quad \mathcal{C} h := \mu^{-1} \mathscr{C}\,(\mu\, h)$$

with $\mathrm{Ker}\,\mathcal{C} = \mathrm{Span}\left\{1, v_1, \ldots, v_d, |v|^2\right\}$ and

$$h := \frac{f - \alpha\,\mathscr{M} - \beta\,\mathcal{H}\,\mathscr{M} - F_{\mathrm{rot}} - F_{\mathrm{dir}} - F_{\mathrm{pul}}}{\mathscr{M}}$$

Micro-macro decomposition

$$h = h^{\|} + h^{\perp}\,, \quad h^{\|} := r + m \cdot v + e\,\mathfrak{E}(v)$$

$$(r, m, e)(t, x) := \int_{\mathbb{R}^d} \big(1, v, \mathfrak{E}(v)\,\big)\, h(t, x, v)\, \mu(v)\, \mathrm{d}v$$

- f is a special macroscopic modes iff $h^{\perp} = 0$
- all steady states are special macroscopic modes: factorization (use entropy-dissipation arguments)

Sketch of the proof

The function $h = h^\parallel + h^\perp = r + m\cdot v + e\,\mathfrak{E}(v) + h^\perp$ is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2 \le -\,2\,\mathrm{c}_{\mathscr{C}}\,\|h^\perp\|^2$$

With the *Witten-Laplace operator* $\Omega := -\,\Delta_x + \nabla_x\phi\cdot\nabla_x + 1$ and

$$E[h] := \int_{\mathbb{R}^d} (v\otimes v - \mathrm{Id}_{d\times d})\,h\,\mu\,\mathrm{d}v\,,\quad \Theta[h] := \int_{\mathbb{R}^d} v\left(\mathfrak{E}(v) - \sqrt{\tfrac{2}{d}}\right) h\,\mu\,\mathrm{d}v$$

we build a *Lyapunov functional*

$$\begin{aligned}\mathcal{F}[h] := \|h\|^2 &+ \varepsilon\,\langle\Omega^{-1}\,\nabla_x e, \Theta[h]\rangle + \varepsilon^{\frac{3}{2}}\,\langle\Omega^{-1}\,\nabla_x^{\mathrm{sym}}\,m_s, E[h] - \sqrt{\tfrac{2}{d}}\,\langle e\rangle\,\mathrm{Id}_{d\times d}\rangle\\ &+ \varepsilon^{\frac{7}{4}}\,\langle\Omega^{-1}\,\nabla_x w_s, m_s\rangle + \varepsilon^{\frac{15}{8}}\,\langle -\Omega^{-1}\,\partial_t w_s, w_s\rangle\\ &- \varepsilon^{\frac{61}{32}}\,\langle(X - Y\cdot\nabla_x\phi), \nabla_x\phi\cdot A\,x\rangle - \varepsilon^{\frac{62}{32}}\,\langle b, b'\rangle - \varepsilon_6\,\langle c', c''\rangle\end{aligned}$$

such that, for some $\lambda \ge 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[h] \le -\,\lambda\,\mathcal{F}[h] \quad\text{and}\quad \|h\|^2 \lesssim \mathcal{F}[h] \lesssim \|h\|^2$$

These slides can be found at

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Thank you for your attention !