The two-dimensional Keller-Segel model after blow-up

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Outline

- Write a generalized weak formulation of the Keller-Segel model allowing for measure valued cell densities
- the main mathematical problem is an appropriate definition of the only nonlinear term, the convective flux (the spatial dimension is two, is of essential importance)

1. The model, a brief review of the literature
2. A priori estimates for solutions, Poupaud's theory
3. Limiting problem, measure valued solutions, strong formulation (formal)
4. Local density profiles of the regularized problem approximating point aggregates

1. The model, a brief review of the literature

The Keller-Segel model

Patlak-Keller-Segel model

$$\partial_t \varrho + \nabla \cdot (\varrho \nabla S - \nabla \varrho) = 0$$
$$-\Delta S = \varrho$$

... a nondimensionalized form

- dynamics of the chemical are much faster than those for the cell dynamics: a parabolic-elliptic system
- the chemotactic sensitivity, the cell diffusivity, the chemical diffusivity, and the reaction rate are constant

... initial data

$$\varrho_I \in L^1_+(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \qquad \int_{\mathbb{R}^2} |x|^2 \varrho_I(x)\,dx < \infty$$

The Keller-Segel model

$$\partial_t \varrho + \nabla \cdot (\varrho \nabla S - \nabla \varrho) = 0$$
$$-\Delta S = \varrho$$

preserves the mass

$$\frac{d}{dt} \int_{\mathbb{R}^2} \varrho \, dx = 0$$

and has a Lyapunov functional: the free energy functional

$$\frac{d}{dt} \left[\int_{\mathbb{R}^2} \varrho \log \varrho \, dx + \frac{1}{2} \int_{\mathbb{R}^2} S[\varrho] \, \varrho \, dx \right] = - \int_{\mathbb{R}^2} \varrho \left| \frac{\nabla \varrho}{\varrho} - \nabla S[\varrho] \right|^2 dx$$

$S[\varrho](x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y] \, \varrho(y) \, dy$

N.B. $\int_{\mathbb{R}^2} S[\varrho] \, \varrho \, dx = - \int_{\mathbb{R}^2} |\nabla S[\varrho]|^2 \, dx$ only if $\int_{\mathbb{R}^2} \varrho \, dx = 0$

A standard computation with moments

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\,\varrho(x,t)\,dx = \int_{\mathbb{R}^2} |x|^2\,\Delta\varrho(x,t)\,dx + \frac{\chi}{2\,\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} 2x\cdot\frac{x-y}{|x-y|^2}\,\varrho(x,t)\,\varrho(y,t)\,dx\,dy$$

Two integrations by parts for the first term, one integration by parts for the second one and symmetrization w.r.t. x and y variables give

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\,\varrho(x,t)\,dx = 4M\left(1-\frac{M}{8\pi}\right)$$

$M = 8\pi$ is a threshold between two regimes

Finite time blow-up

Existence of smooth solutions depends on the mass

$$M = \int_{\mathbb{R}^2} \varrho_I \, dx$$

- $M < 8\pi$, a global bounded solution exists, vanishing occurs as $t \to \infty$, [Jäger, Luckhaus], [Dolbeault, Perthame], [Blanchet, Dolbeault, Perthame]
- $M > 8\pi$, blow-up in finite time occurs
- $M = 8\pi$, a global solution exists in this case, which possibly becomes unbounded as $t \to \infty$, [Biler, Karch, Laurençot, Nadzieja], [Blanchet, Carrillo, Masmoudi]

Aggregation: at the blow-up time, mass concentrates in a point, and then new dynamics has to be established, [Herrero, Velázquez], [Velázquez]

Regularized models

- volume filling effects taken into account by a density dependent chemotactic sensitivity [Hillen, Painter], [Velázquez], [Dolak, Schmeiser],
- finite sampling radius, which results in a regularization of the chemical concentration [Hillen, Painter, Schmeiser]
- kinetic transport models, whose macroscopic limit is the Keller-Segel model [Chalub, Markowich, Perthame]
- density dependent chemotactic sensitivities, [Velázquez]

All these models have global solutions

The Keller-Segel model appears as a formal limit

In Velázquez' approach (formal asymptotics), the limit the cell density is the sum of a smooth part and of a finite number of point aggregates: locally in time existence

Our regularization

Replace the Poisson equation $\Delta S = -\varrho$ by

$$S_\varepsilon[\varrho](x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|+\varepsilon)\varrho(y)\,dy$$

to get the regularized model

$$\partial_t \varrho^\varepsilon + \nabla\cdot(\varrho^\varepsilon \nabla S_\varepsilon[\varrho^\varepsilon] - \nabla \varrho^\varepsilon) = 0$$

Similar to the finite-sampling-radius model [Hillen, Painter, Schmeiser]

$$\varrho^\varepsilon(t=0) = \varrho_I \in L^1_+(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)\,.$$

[F. Poupaud, Diagonal defect measures, adhesion dynamics and Euler equations, Meth. Appl. Anal. **9** (2002), pp. 533–561] (two-dimensional incompressible Euler equations and Keller-Segel model without diffusion of cells)

2. A priori estimates for solutions, Poupaud's theory

A priori estimates and diagonal defect measures

Theorem 1 *For every $\varepsilon > 0$, the regularized problem has a global solution satisfying*

$$\|\varrho^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^2)} = \|\varrho_I\|_{L^1(\mathbb{R}^2)} := M$$
$$\|\varrho^\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le c\left(1+\frac{1}{\varepsilon^2}\right)$$

with an ε-independent constant c

Existence of a local solution, mass conservation are known, and by [Hillen, Painter, Schmeiser]

$$|\nabla S_\varepsilon[\varrho^\varepsilon](x,t)| \le \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{\varrho^\varepsilon(y,t)dy}{|x-y|+\varepsilon} \le \frac{M}{2\pi\varepsilon}$$

The nonlinear term

For $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \varphi \varrho \nabla S_\varepsilon[\varrho]\, dx = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\varphi(x) - \varphi(y))(x-y)}{|x-y|(|x-y|+\varepsilon)} \varrho(x)\varrho(y)\, dx\, dy$$

holds, implying the uniform-in-ε estimate

$$\left| \int_{\mathbb{R}^2} \varphi \varrho^\varepsilon \nabla S_\varepsilon[\varrho^\varepsilon] dx \right| \leq \frac{M^2}{4\pi} |\varphi|_{1,\infty}$$

Let

$$m^\varepsilon(t,x) := \int_{\mathbb{R}^2} \mathcal{K}^\varepsilon(x-y) \varrho^\varepsilon(t,x) \varrho^\varepsilon(t,y) dy \quad \text{with } \mathcal{K}^\varepsilon(x) = \frac{x^{\otimes 2}}{|x|(|x|+\varepsilon)}$$

of matrix valued functions. Following [Poupaud], we consider $\varrho^\varepsilon(t,\cdot)$ and $m^\varepsilon(t,\cdot)$ as time dependent measures $\varrho^\varepsilon(t)$ and $m^\varepsilon(t)$.

Tight equicontinuity

Lemma 1 *[Poupaud] The families $\{\varrho^\varepsilon(t)\}_{\varepsilon>0}$ and $\{m^\varepsilon(t)\}_{\varepsilon>0}$ are tightly bounded locally uniformly in t, and $\{\varrho^\varepsilon(t)\}_{\varepsilon>0}$ is tightly equicontinuous in t.*

$$\left| \frac{d}{dt} \int_{\mathbb{R}^2} \varphi \varrho^\varepsilon \, dx \right| \leq c \, |\varphi|_{2,\infty}$$

with c independent of ε and t. This implies equicontinuity in $W^{2,\infty}(\mathbb{R}^2)'$:

$$\left| \int_{\mathbb{R}^2} \varphi \varrho^\varepsilon(t,x) \, dx - \int_{\mathbb{R}^2} \varphi \varrho^\varepsilon(s,x) \, dx \right| \leq C(\varphi)|t-s|$$

Now let $\varphi \in C_b(\mathbb{R}^2)$: $\forall \, \delta > 0$, $\exists \varphi_\delta \in W^{2,\infty}(\mathbb{R}^2)$ s.t. $\|\varphi - \varphi_\delta\|_{L^\infty(\mathbb{R}^2)} \leq \delta$

$$\left| \int_{\mathbb{R}^2} \varphi \varrho^\varepsilon(t,x) \, dx - \int_{\mathbb{R}^2} \varphi \varrho^\varepsilon(s,x) \, dx \right| \leq 2\delta M + C(\varphi_\delta)|t-s|$$

$\Longrightarrow$ tight equicontinuity of $\varrho^\varepsilon(t)$

Limits

Use Prokhorov's criterium: tight boundedness and equicontinuity of $\varrho^\varepsilon(t)$ provides compactness

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi(x,y)\varrho^\varepsilon(t,x)\varrho^\varepsilon(t,y)\,dx\,dy \to \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varphi(x,y)\varrho(t,x)\varrho(t,y)\,dx\,dy$$

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^2}\varphi(t,x)m^\varepsilon(t,x)\,dx\,dt \to \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\varphi(t,x)m(t,x)\,dx\,dt$$

for all $\varphi \in C_b([t_1,t_2]\times\mathbb{R}^2)$ in the second case. Because of the discontinuity of the limiting kernel in $\mathcal{K}$, define the defect measure

$$\nu(t,x) = m(t,x) - \int_{\mathbb{R}^2}\mathcal{K}(x-y)\varrho(t,x)\varrho(t,y)\,dy$$

with $\mathcal{K}(x) = \lim_{\varepsilon\to 0}\mathcal{K}^\varepsilon(x) = \frac{x^{\otimes 2}}{|x|^2}$ for $x \neq 0$ and $\mathcal{K}(0) = 0$

Atomic support

The atomic support (an at most countable set) of $\varrho(t)$ is

$$S_{at}(\varrho(t)) := \{a \in \mathbb{R}^2 : \ \varrho(t)(\{a\}) > 0\}$$

It is an at most countable set

Lemma 2 [Poupaud] ν *is symmetric, nonnegative, and satisfies*

$$\mathrm{tr}(\nu(t,x)) \leq \sum_{a \in S_{at}(\varrho(t))} (\varrho(t)(\{a\}))^2 \delta(x-a)$$

The limit of ϱ^ε is thus characterized by the pair (ϱ, ν)

$\mathcal{M}$: spaces of Radon measures

$\mathcal{M}_1^+$: subset of nonnegative bounded measures

For an interval $I \subset \mathbb{R}$, the set of time dependent admissible measures with diagonal defects is defined by

$$\begin{aligned} \mathcal{DM}^+(I;\mathbb{R}^2) \quad = \quad & \Big\{(\varrho,\nu): \ \varrho(t) \in \mathcal{M}_1^+(\mathbb{R}^2) \ \forall t \in I, \ \nu \in \mathcal{M}(I \times \mathbb{R}^2)^{2\times 2}, \\ & \varrho \text{ is tightly continuous with respect to } t, \\ & \nu \text{ is a nonnegative, symmetric, matrix valued measure,} \\ & \operatorname{tr}(\nu(t,x)) \leq \sum_{a \in S_{at}(\varrho(t))} (\varrho(t)(\{a\}))^2 \delta(x-a)\Big\} \end{aligned}$$

3. Limiting problem, measure valued solutions, strong formulation (formal)

Regularized measures

$$\frac{(\varphi(x)-\varphi(y))(x-y)}{|x-y|(|x-y|+\varepsilon)} = \mathcal{K}^\varepsilon(x-y)\nabla\varphi(x) + L^\varepsilon(\varphi)(x,y)$$

with

$$L^\varepsilon(\varphi)(x,y) = \frac{(\varphi(x)-\varphi(y)-(x-y)\cdot\nabla\varphi(x))(x-y)}{|x-y|(|x-y|+\varepsilon)}$$

converges uniformly to the continuous $L^0(\varphi)(x,y)$ for any test function $\varphi \in C_b^1(\mathbb{R}^2)$

$$\int_0^T \int_{\mathbb{R}^2} \varphi(t,x)\varrho^\varepsilon(t,x)\nabla S^\varepsilon[\varrho^\varepsilon](t,x)dx\,dt = -\frac{1}{4\pi}\int_0^T\int_{\mathbb{R}^2} m^\varepsilon(t,x)\nabla\varphi(t,x)\,dx\,dt$$
$$-\frac{1}{4\pi}\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varrho^\varepsilon(t,x)\varrho^\varepsilon(t,y)L^\varepsilon(\varphi)(t,x,y)\,dx\,dy\,dt$$

Limiting problem

$$\int_0^T \int_{\mathbb{R}^2} \varphi(t,x) j[\varrho,\nu](t,x) dx\, dt$$

$$= \quad -\frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^4} (\varphi(t,x) - \varphi(t,y)) K(x-y) \varrho(t,x) \varrho(t,y) dx\, dy\, dt$$

$$-\frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2} \nu(t,x) \nabla\varphi(t,x) dx\, dt$$

for $\varphi \in C_b^1((0,T) \times \mathbb{R}^2)$

Theorem 2 *For every $T > 0$, ϱ^ε converges tightly and uniformly in time to $\varrho(t)$ and there exists $\nu(t)$ such that $(\varrho,\nu) \in \mathcal{DM}^+((0,T);\mathbb{R}^2)$ is a generalized solution of*

$$\partial_t \varrho + \nabla \cdot (j[\varrho,\nu] - \nabla\varrho) = 0$$

$\varrho(t=0) = \varrho_I$ *holds in the sense of tight continuity*

Strong formulation (formal)

Ansatz. $\varrho = \overline{\varrho} + \hat{\varrho}$, $\hat{\varrho}(t,x) = \sum_{n\in N} M_n(t)\delta_n(t,x)$, $\delta_n(t,x) = \delta(x - x_n(t))$
$(\varrho,\nu) \in \mathcal{DM}^+((0,T);\mathbb{R}^2) \implies \nu(t,x) = \sum_{n\in N} \nu_n(t)\delta_n(t,x)$, $\mathrm{tr}(\nu_n) \leq M_n^2$

$$j[\varrho,\nu] = \overline{\varrho}\nabla S_0[\overline{\varrho} + \hat{\varrho}] + \sum_n M_n\delta_n \nabla S_0\left[\overline{\varrho} + \sum_{m\neq n} M_m\delta_m\right] + \frac{1}{4\pi}\sum_n M_n\nu_n\nabla\delta_n$$

$$\begin{aligned}
\partial_t\overline{\varrho} \quad &+ \quad \nabla\cdot(\overline{\varrho}\nabla S_0[\overline{\varrho}] - \nabla\overline{\varrho}) + \nabla\overline{\varrho}\cdot\nabla S_0[\hat{\varrho}] \\
&+ \quad \sum_n \delta_n(\dot{M}_n - \overline{\varrho}M_n) \\
&- \quad \sum_n M_n\nabla\delta_n\left(\dot{x}_n - \nabla S_0\left[\overline{\varrho} + \textstyle\sum_{m\neq n} M_m\delta_m\right]\right) \\
&+ \quad \sum_n \left(\tfrac{1}{4\pi}\nu_n : \nabla^2\delta_n - M_n\Delta\delta_n\right) = 0
\end{aligned}$$

The last row this gives

$$\nu_n = 4\pi M_n \,\mathrm{id}$$

As a consequence of $\mathrm{tr}(\nu_n) = 8\pi M_n \leq M_n^2$, point masses have to be at least 8π (there is only a finite number of them)

$$\partial_t \overline{\varrho} + \nabla\cdot(\overline{\varrho}\nabla S_0[\overline{\varrho}] - \nabla\overline{\varrho}) - \frac{1}{2\pi}\nabla\overline{\varrho}\cdot\sum_n M_n \frac{x-x_n}{|x-x_n|^2} = 0$$

$$\dot{M}_n = \overline{\varrho}(x = x_n) M_n$$

$$\dot{x}_n = \nabla S_0[\overline{\varrho}](x = x_n) - \frac{1}{2\pi}\sum_{m\neq n} M_m \frac{x_n - x_m}{|x_n - x_m|^2}$$

Note that $\frac{d}{dt}\left(\int_{\mathbb{R}^2}\overline{\varrho}\,dx + \sum_n M_n\right) = 0$

Comparison with Velázquez' results

Formal limit of [Velázquez]: there is a factor (≤ 1) in front of the right hand side of the force term for x_n, which depends on the details of the regularization

A local-in-time existence result can be found in [Velázquez]

In general, one has to expect blow-up events in the smooth part $\overline{\varrho}$ and/or collisions of point aggregates in finite time. At such points in time, a restart is required with either an additional point aggregate after a blow-up event or with a smaller number of point aggregates after a collision. A rigorous theory producing global solutions by such a procedure is still missing

Long time behaviour

Assume again

$$\nu(t,x) = 4\pi\,\mathrm{id} \sum_{a \in S_{at}(\varrho(t))} \varrho(t)(\{a\})\delta(x-a)$$

and

$$\int_{\mathbb{R}^2} |x|^2 \varrho_I\, dx < \infty$$

With $\hat{M} = \sum_{a \in S_{at}(\varrho(t))} \varrho(t)(\{a\})$ and $\overline{M} = M - \hat{M}$

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 \varrho\, dx &= 4M - \frac{1}{2\pi}\int_{\mathbb{R}^4} (1-\chi_D)\varrho \otimes \varrho\, dy\, dx - \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{tr}(\nu)\, dx \\ &= \overline{M}\left(4 - \frac{M}{2\pi} - \frac{\hat{M}}{2\pi}\right) - \frac{1}{2\pi}\sum_{a \neq b,\ a,b \in S_{at}(\varrho(t))} \varrho(t)(\{a\})\varrho(t)(\{b\}) \end{aligned}$$

For $M > 8\pi$, the r.h.s is the sum of two nonpositive terms: the second order moment is a Lyapunov function.

The dissipation term only vanishes when $\overline{M} = 0$, and when the atomic support of $\varrho(t)$ consists of only one point located at

$$X_I = \frac{1}{M} \int_{\mathbb{R}^2} x \varrho_I dx$$

Example 1. $\overline{\varrho} = 0$, two aggregates:
$\varrho(t, x) = M_1 \delta(x - x_1(t)) + M_2 \delta(x - x_2(t))$. The asymptotic state is reached in finite time

$$t = \frac{\pi |x_1(0) - x_2(0)|^2}{M_1 + M_2}$$

by collision of the aggregates

Example 2. Initial mass $\overline{M}(0) = \int_{\mathbb{R}^2} \overline{\varrho}(t=0)dx < M_{crit} = \sup_{q>1} \frac{4}{q\,C_q}$ of the smooth part, one aggregate. The corresponding solution may exist only on a finite time interval

$$\partial_t \overline{\varrho} + \nabla \cdot (\overline{\varrho} \nabla S_0[\overline{\varrho}] - \nabla \overline{\varrho}) - \tfrac{M_1}{2\pi} \nabla \overline{\varrho} \cdot \tfrac{x - x_1}{|x - x_1|^2} = 0$$

$$\dot{M}_1 = \overline{\varrho}(x = x_1) M_1$$

$$\dot{x}_1 = \nabla S_0[\overline{\varrho}](x = x_1)$$

$\frac{d}{dt} \int_{\mathbb{R}^2} \overline{\varrho}^q dx = (q-1)\left(\int_{\mathbb{R}^2} \overline{\varrho}^{q+1} dx - \frac{4}{q} \int_{\mathbb{R}^2} |\nabla \overline{\varrho}^{q/2}|^2 dx \right) - \overline{\varrho}(x = x_1)^q M_1$

Gagliardo-Nirenberg: $\int_{\mathbb{R}^2} |u|^{2(1+1/q)} dx \le C_q \int_{\mathbb{R}^2} |u|^{2/q} dx \int_{\mathbb{R}^2} |\nabla u|^2 dx$

$$\frac{d}{dt} \int_{\mathbb{R}^2} \overline{\varrho}^q dx \le (q-1)\left(C_q \overline{M} - \frac{4}{q} \right) \int_{\mathbb{R}^2} |\nabla \overline{\varrho}^{q/2}|^2 dx - \overline{\varrho}(x = x_1)^q M_1 \le 0$$

$M_{crit} = 8\,\pi$?

4. Local density profiles of the regularized problem approximating point aggregates

Local density profiles

For fixed t and $a \in S_{at}(\varrho(t))$, let $\varepsilon\xi = x - a$ and $\varepsilon^2\varrho^\varepsilon = R^\varepsilon$

$$\varepsilon^2\partial_t R^\varepsilon + \nabla_\xi \cdot (R^\varepsilon \nabla_\xi S_1[R^\varepsilon] - \nabla_\xi R^\varepsilon) = 0$$

R^ε is uniformly bounded, implying compactness of $\nabla_\xi S_1[R^\varepsilon]$. The L^∞-weak* limit R of R^ε (take subsequences, formal) satisfies

$$\nabla_\xi \cdot (R\nabla_\xi S_1[R] - \nabla_\xi R) = 0$$

Observe that

$$\int_{\mathbb{R}^2} R(\xi)d\xi = \frac{1}{8\pi}\int_{\mathbb{R}^4} \frac{|\xi - \eta|}{|\xi - \eta| + 1} R(\xi)R(\eta)d\eta d\xi \le \frac{1}{8\pi}\left(\int_{\mathbb{R}^2} R(\xi)d\xi\right)^2$$

This shows that either R vanishes or its mass is not smaller than 8π

Free energy

$$F_\varepsilon[\varrho] := \int_{\mathbb{R}^2} \left(\varrho \log \varrho - \frac{1}{2} \varrho S_\varepsilon[\varrho] \right) dx$$

$$= \int_{\mathbb{R}^2} \varrho \log \varrho \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^4} \log(|x-y| + \varepsilon) \varrho(x) \varrho(y) dy \, dx$$

and

$$\frac{d}{dt} F_\varepsilon[\varrho^\varepsilon] = - \int_{\mathbb{R}^2} \varrho^\varepsilon |\nabla (\log \varrho^\varepsilon - S_\varepsilon[\varrho^\varepsilon])|^2 dx$$

With an arbitrary $a \in \mathbb{R}^2$ and $R(\xi) = \varepsilon^2 \varrho(a + \varepsilon \xi)$ we have

$$F_\varepsilon[\varrho] = \left(2M - \frac{M^2}{4\pi} \right) \log \frac{1}{\varepsilon} + F_1[R]$$

Free energy

Lemma 3 *Let* $R \in L^1_+(\mathbb{R}^2)$ *be radial,* $\int_{\mathbb{R}^2} \log(1+|x|)\, R(x)\, dx < \infty$*,*
$M = \int_{\mathbb{R}^2} R\, dx$

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \log(1+|x-y|)\, R(y)\, dy \geq \frac{M}{4\pi} \log|x| \quad \forall\, x \in \mathbb{R}^2$$

$L^1_{+,M} := \{R \in L^1_+(\mathbb{R}^2) : \int_{\mathbb{R}^2} R\, d\xi = M\}$, $J_M := \inf_{R \in L^1_{+,M}} F_1[R] \geq -\infty$

Theorem 3 $J_M = -\infty$ *for* $M < 8\pi$*, and* $J_M > -\infty$ *for* $M \geq 8\pi$*. If* $M > 8\pi$*, there exists a radial nonincreasing minimizer*

Logarithmic HLS inequality, [Carlen, Loss]
Riesz symmetrization inequality
Variational method

References

- P. Biler, G. Karch, P. Laurençot, T. Nadzieja, The 8π-problem for radially symmetric solutions of a chemotaxis model in the plane, *Math. Meth. Appl. Sci.* **29** (2006), pp. 1563–1583.
- A. Blanchet, J. Dolbeault, B. Perthame, Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions, *Electron. J. Differential Equations* No. 44 (2006), 32 pp.
- A. Blanchet, N. Masmoudi, J. Carrillo, Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$, to appear in *Comm. Pure Appl. Math.* (2007).
- E. Carlen, M. Loss, Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on S^n, *Geom. Funct. Anal.* **2** (1992), pp. 90–104.
- F.A.C.C. Chalub, P. Markowich, B. Perthame, C. Schmeiser, Kinetic models for chemotaxis and their drift-diffusion limits, *Monatsh. Math.* **142** (2004), pp. 123–141.
- M. del Pino, J. Dolbeault, Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions, *J. Math. Pures Appl. (9)* **81** (2002), pp. 847–875.
- Y. Dolak, C. Schmeiser, The Keller-Segel model with logistic sensitivity function and small diffusivity. *SIAM J. Appl. Math.* **66** (2005), pp. 286–308.

- J. Dolbeault, B. Perthame, Optimal Critical Mass in the two dimensional Keller-Segel model in $\mathbb{R}^2$, *C. R. Acad. Sci. Paris, Ser I* **339** (2004), pp. 611–616.
- M. A. Herrero and J. J. L. Velázquez, A blow-up mechanism for a chemotaxis model, *Ann. Scuola Norm. Sup. Pisa* **24** (1997), pp. 633–683.
- T. Hillen and K. Painter, Global existence for a parabolic chemotaxis model with prevention of overcrowding, *Adv. in Appl. Math.* **26** (2001), pp. 280–301.
- T. Hillen, K. Painter, C. Schmeiser, Global existence for chemotaxis with finite sampling radius, *DCDS-B* **7** (2007), 125–144.
- W. Jäger, S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, *Trans. Amer. Math. Soc.* **329** (1992), pp. 819–824.
- E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.* **26** (1970), pp. 399–415.
- E. Lieb, M. Loss, *Analysis*, 2nd edition, Grad. Stud. in Math. 14, AMS, Providence, RI, 2001.
- C.S. Patlak, Random walk with persistence and external bias, *Bull. Math. Biophys.* **15** (1953), pp. 311–338.

- F. Poupaud, Diagonal defect measures, adhesion dynamics and Euler equations, *Meth. Appl. Anal.* **9** (2002), pp. 533–561.
- J.J.L. Velázquez, Point dynamics in a singular limit of the Keller-Segel model I: motion of the concentration regions, *SIAM J. Appl. Math.* **64** (2004), pp. 1198–1223.
- J.J.L. Velázquez, Point dynamics in a singular limit of the Keller-Segel model II: formation of the concentration regions, *SIAM J. Appl. Math.* **64** (2004), pp. 1224–1248.
- J.J.L. Velázquez, Well-posedness of a model of point dynamics for a limit of the Keller-Segel system, *J. Diff. Equs* **206** (2004), pp. 315–352.