Méthodes d'entropie et résultats de stabilité dans les inégalités de Gagliardo-Nirenberg-Sobolev

Jean Dolbeault

http://www.ceremade.dauphine.fr/~dolbeaul

Ceremade, Université Paris-Dauphine

21 septembre 2021

Groupe de travail Analyse non-linéaire et EDP

Joint work on ***Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method*** arXiv:2007.03674 (Apr. 29, 2021) with

Matteo Bonforte
▷ *Universidad Autónoma de Madrid and ICMAT*

Bruno Nazaret
▷ *Université Paris 1 Panthéon-Sorbonne and Mokaplan team*

Nikita Simonov
▷ *Ceremade, Université Paris-Dauphine (PSL)*

Outline

A brief introduction to some stability issues in Sobolev and related inequalities

The stability result of G. Bianchi and H. Egnell

In Sobolev's inequality (with optimal constant S_d),

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq 0$$

is there a natural way to bound the l.h.s. from below in terms of a "distance" to the set of optimal [Aubin-Talenti] functions when $d \geq 3$?
A question raised in [Brezis, Lieb (1985)]

▷ [Bianchi, Egnell (1991)] There is a positive constant α such that

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \alpha \inf_{\varphi\in\mathfrak{M}} \|\nabla f - \nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ Various improvements, *e.g.*, [Cianchi, Fusco, Maggi, Pratelli (2009)] there are constants α and κ and $f \mapsto \lambda(f)$ such that

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \big(1+\kappa\,\lambda(f)^\alpha\big)\,\mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

*However, the question of **constructive** estimates is still widely open*

Gagliardo-Nirenberg-Sobolev inequalities

We consider the inequalities

$$\boxed{\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\rm GNS}(p)\,\|f\|_{2p}}\qquad\text{(GNS)}$$

$\theta=\frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}$, $\quad p\in(1,+\infty)$ if $d=1$ or 2, $\quad p\in(1,p^*]$ if $d\ge3$, $\quad p^*=\frac{d}{d-2}$

Theorem (del Pino, JD)

Equality case in (GNS) is achieved if and only if

$$f\in\mathfrak{M}:=\left\{g_{\lambda,\mu,y}\,:\,(\lambda,\mu,y)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\right\}$$

Aubin-Talenti functions: $g_{\lambda,\mu,y}(x):=\mu\,g\left((x-y)/\lambda\right)$, $g(x)=\left(1+|x|^2\right)^{-\frac{1}{p-1}}$

[del Pino, JD, 2002], [Gunson, 1987, 1991]

Related inequalities

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p}\tag{GNS}$$

▷ *Sobolev's inequality*: $d\ge 3$, $p=p^*=d/(d-2)$

$$\|\nabla f\|_2^2\ge\mathsf{S}_d\,\|f\|_{2p^*}^2$$

▷ *Euclidean Onofri inequality*

$$\int_{\mathbb{R}^2}e^{h-\overline{h}}\,\frac{dx}{\pi\,(1+|x|^2)^2}\le e^{\frac{1}{16\pi}\int_{\mathbb{R}^2}|\nabla h|^2\,dx}$$

$d=2$, $p\to+\infty$ with $f_p(x):=\mathrm{g}(x)\left(1+\frac{1}{2p}\,(h(x)-\overline{h})\right)$, $\overline{h}=\int_{\mathbb{R}^2}h(x)\,\frac{dx}{\pi\,(1+|x|^2)^2}$

▷ *Euclidean logarithmic Sobolev inequality in scale invariant form*

$$\frac{d}{2}\log\left(\frac{2}{\pi\,d\,e}\int_{\mathbb{R}^d}|\nabla f|^2\,dx\right)\ge\int_{\mathbb{R}^d}|f|^2\log|f|^2\,dx$$

$\|f\|_2=1$, or $\int_{\mathbb{R}^d}|\nabla f|^2\,dx\ge\frac{1}{2}\int_{\mathbb{R}^d}|f|^2\log\left(\frac{|f|^2}{\|f\|_2^2}\right)dx+\frac{d}{4}\log\left(2\pi\,e^2\right)\int_{\mathbb{R}^d}|f|^2\,dx$

A variational point of view on stability

Optimality by concentration-compactness

Deficit functional, scale invariance, weak stability

Deficit functional

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GNS}\,\|f\|_{2p}^{2p\gamma}$$

Lemma

(GNS) *is equivalent to* $\delta[f] \ge 0$ *if and only if*

$$\mathscr{K}_{\rm GNS} = C(p,d)\,\mathscr{C}_{\rm GNS}^{2p\gamma}$$

where $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$ *and* $C(p,d)$ *is an explicit positive constant*

Take $f_\lambda(x) = \lambda^{\frac{d}{2p}}\,f(\lambda\,x)$ and optimize on $\lambda > 0$ to get (*weak stability*)

$$\delta[f] \ge \delta[f] - \inf_{\lambda>0}\delta[f_\lambda] =: \delta_\star[f] \ge 0$$

A simplification: $\delta[f] = \delta[|f|]$ so we shall assume that $f \ge 0$ a.e.

Minimization and concentration-compactness

$$I_M = \inf\left\{(p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} : f \in \mathscr{H}_p(\mathbb{R}^d)\,,\quad \|f\|_{2p}^{2p} = M\right\}$$

$$I_1 = \mathscr{K}_{\mathrm{GNS}} \text{ and } I_M = I_1\,M^\gamma \text{ for any } M>0$$

Lemma

If $d \ge 1$ and p is an admissible exponent with $p<d/(d-2)$, then

$$I_{M_1+M_2} < I_{M_1} + I_{M_2} \quad \forall\,M_1, M_2 > 0$$

Lemma

Let $d \ge 1$ and p be an admissible exponent with $p<d/(d-2)$ if $d\ge3$. If $(f_n)_n$ is minimizing and if $\limsup_{n\to+\infty}\sup_{y\in\mathbb{R}^d}\int_{B(y)}|f_n|^{p+1}\,dx = 0$, then

$$\lim_{n\to\infty}\|f_n\|_{2p} = 0$$

... *Existence*

Existence of a minimizer, further properties

Proposition

Assume that $d\ge 1$ is an integer and let p be an admissible exponent with $p<d/(d-2)$ if $d\ge 3$. Then there is a radial minimizer of δ

- *Pólya-Szegö principle* : there is a radial minimizer solving

$$-2\,(p-1)^2\,\Delta f+4\,\big(d-p\,(d-2)\big)\,f^p-C\,f^{2\,p-1}=0$$

If $f=\mathbf{g}$, then $C=8\,p$

- *A rigidity result* : if f is a (smooth) minimizer and $\mathrm{P}=-\frac{p+1}{p-1}\,f^{1-p}$, then

$$\big(d-p\,(d-2)\big)\int_{\mathbb{R}^d}f^{p+1}\left|\Delta\mathrm{P}+(p+1)^2\,\frac{\int_{\mathbb{R}^d}|\nabla f|^2\,dx}{\int_{\mathbb{R}^d}f^{p+1}\,dx}\right|^2dx$$
$$+2\,d\,p\int_{\mathbb{R}^d}f^{p+1}\,\left\|\mathrm{D}^2\mathrm{P}-\tfrac{1}{d}\,\Delta\mathrm{P}\,\mathrm{Id}\right\|^2dx=0$$

$\triangleright$ $\mathbf{g}(x)=\big(1+|x|^2\big)^{-\frac{1}{p-1}}$ is a minimizer and $\delta[\mathbf{g}]=0$

Non-constructive stability results

Relative entropy and Fisher information

- *Free energy* or *relative entropy functional*

$$\mathcal{E}[f|g] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(f^{p+1} - g^{p+1} - \tfrac{1+p}{2p}\, g^{1-p} \left(f^{2p} - g^{2p} \right) \right) dx \geq 0$$

Lemma (Csiszár-Kullback inequality)

Let $d \geq 1$ and $p > 1$. There exists a constant $C_p > 0$ such that

$$\left\| f^{2p} - g^{2p} \right\|^2_{\mathrm{L}^1(\mathbb{R}^d)} \leq C_p\, \mathcal{E}[f|g] \quad \text{if} \quad \|f\|_{2p} = \|g\|_{2p}$$

- *Relative Fisher information*

$$\mathcal{J}[f|g] := \frac{p+1}{p-1} \int_{\mathbb{R}^d} \left| (p-1) \nabla f + f^p \nabla g^{1-p} \right|^2 dx$$

Best matching profile

- Nonlinear extension of the *Heisenberg uncertainty principle*

$$\left(\frac{d}{p+1}\int_{\mathbb{R}^d} f^{p+1}\,dx\right)^2 \le \int_{\mathbb{R}^d} |\nabla f|^2\,dx \int_{\mathbb{R}^d} |x|^2 f^{2p}\,dx$$

▷ Take $g = \mathbf{g}$ in $\mathscr{J}[f|g]$ and expand the square

- If $\mathrm{g}_f := g \in \mathfrak{M}$ is such that $\int_{\mathbb{R}^d} f^{2p}\,(1, x, |x|^2)\,dx = \int_{\mathbb{R}^d} g^{2p}\,(1, x, |x|^2)\,dx$

$$\text{then} \quad \mathscr{E}[f|g] = \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(f^{p+1} - g^{p+1}\right)dx$$

▷ A smaller space: $\mathscr{W}_p(\mathbb{R}^d) := \left\{f \in \mathscr{H}_p(\mathbb{R}^d)\,:\,|x|\,|f|^p \in \mathrm{L}^2(\mathbb{R}^d)\right\}$

Lemma

For any $f \in \mathscr{W}_p(\mathbb{R}^d)$, $\mathrm{g}_f \in \mathfrak{M}$ is uniquely defined and

$$\mathscr{E}[f|\mathrm{g}_f] = \inf_{g \in \mathfrak{M}} \mathscr{E}[f|g]$$

A first (weak) stability result

Lemma (A weak stability result)

If $g_f = \mathbf{g}$, then

$$\delta[f] \ge \delta_\star[f] \approx \mathscr{E}[f|\mathbf{g}]^2$$

▷ Up to the invariances, $\mathbf{g}$ is the unique minimizer for $f \mapsto \delta[f]$

Lemma (Entropy - entropy production inequality)

If $\|f\|_{2p} = \|g\|_{2p}$ with $\delta[g] = 0$, then

$$\frac{p+1}{p-1}\,\delta[f] = \mathscr{J}[f|g] - 4\,\mathscr{E}[f|g] \ge 0$$

▷ From now on, we will assume that $g_f = \mathbf{g}$, *i.e.*

$$\int_{\mathbb{R}^d} f^{2p}\,(1, x, |x|^2)\,dx = \int_{\mathbb{R}^d} \mathbf{g}^{2p}\,(1, x, |x|^2)\,dx$$

Stability in (GNS)

- [Bianchi, Egnell (1991)] There is a positive constant α such that

$$\mathsf{S}_d\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \alpha \inf_{\varphi\in\mathfrak{M}} \|\nabla f - \nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- Various extensions

$\triangleright$ L^q norm of the gradient by [Chianchi, Fusco, Maggi, Pratelli (2009)], [Figalli, Neumayer (2018)], [Neumayer (2020)], [Figalli, Zhang (2020)]
$\triangleright$ (GNS) inequalities by [Carlen, Figalli (2013)], [Seuffert (2017)], [Nguyen (2019)]

Theorem

There exists a constant $C>0$ such that

$$\delta[f] \ge C\,\mathcal{E}[f|\mathbf{g}]$$

for any $f\in\mathcal{W}_p(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} f^{2p}\,(1,x,|x|^2)\,dx = \int_{\mathbb{R}^d} \mathbf{g}^{2p}\,(1,x,|x|^2)\,dx$$

Towards constructive stability results

A strategy based on a spectral gap

- The spectral gap inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2\, \mathbf{g}^{2p}\, dx \geq \frac{4p}{p-1} \int_{\mathbb{R}^d} |u|^2\, \mathbf{g}^{3p-1}\, dx$$

valid for any function u such that $\int_{\mathbb{R}^d} u\, \mathbf{g}^{3p-1}\, dx = 0$, can be improved with a constant $\Lambda_\star > 4p/(p-1)$ under the constraint that

$$\int_{\mathbb{R}^d} \left(1, x, |x|^2\right) u\, \mathbf{g}^{3p-1}\, dx = 0$$

- A Taylor expansion with $f = \mathbf{g} + \eta\, h$ gives

$$\lim_{\eta \to 0} \frac{\delta[f_\eta]}{\mathscr{E}[f_\eta|\mathrm{g}]} \geq \frac{(p-1)^2}{p\,(p+1)} \left[\Lambda_\star - \frac{4\,p}{p-1}\right]$$

▷Analysis along a minimizing sequence...

How can we make this strategy constructive ?

From the carré du champ method to stability results

Carré du champ method (adapted from D. Bakry and M. Emery)

$$\frac{\partial u}{\partial t} = \Delta u^m \,, \quad \frac{d\mathscr{F}}{dt} = -\mathscr{I} \,, \quad \frac{d\mathscr{I}}{dt} \le -\Lambda \mathscr{I}$$

deduce that $\mathscr{I} - \Lambda \mathscr{F}$ is monotone non-increasing with limit 0

$$\boxed{\mathscr{I}[u] \ge \Lambda \mathscr{F}[u]}$$

▷ An ***improved entropy – entropy production inequality*** (weak form)

$$\mathscr{I} \ge \Lambda \psi(\mathscr{F})$$

for some ψ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi'' > 0$

$$\mathscr{I} - \Lambda \mathscr{F} \ge \Lambda \big(\psi(\mathscr{F}) - \mathscr{F}\big) \ge 0$$

▷ An ***improved constant*** means ***stability***
Under some restrictions on the functions, there is some $\Lambda_\star \ge \Lambda$ such that

$$\mathscr{I} - \Lambda \mathscr{F} \ge (\Lambda_\star - \Lambda) \mathscr{F}$$

Fast diffusion equation and entropy methods

Fast diffusion equation and entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{FDE}$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy – entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)
- Initial time layer: improved entropy – entropy production estimates

The fast diffusion equation in original variables

Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \geq 1$, $m \in (0,1)$

$$\frac{\partial u}{\partial t} = \Delta u^m$$

with initial datum $u(t=0,x) = u_0(x) \geq 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, dx = \mathcal{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \, dx < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$B(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathscr{B}\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$ and $\mathscr{B}$ is the Barenblatt profile with $\int_{\mathbb{R}^d} \mathscr{B} \, dx = \mathcal{M}$

$$\mathscr{B}(x) := \left(1 + |x|^2\right)^{-\frac{1}{1-m}}$$

Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With $m \ge m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\,dx = 0$$

(ii) *Second moment.* With $m > d/(d+2)$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d,(1+|x|^2)\,dx)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2\,u(t,x)\,dx = 2\,d\int_{\mathbb{R}^d} u^m(t,x)\,dx$$

(iii) *Entropy estimate.* With $m \ge m_1 := (d-1)/d$, $u_0^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d,(1+|x|^2)\,dx)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u^m(t,x)\,dx = \frac{m^2}{1-m}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\,dx \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy growth rate

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p}\tag{GNS}$$

$$p=\frac{1}{2m-1}\quad\Longleftrightarrow\quad m=\frac{p+1}{2p}\in[m_1,1)$$

$u=f^{2p}$ so that $u^m=f^{p+1}$ and $u\,|\nabla u^{m-1}|^2=(p-1)^2\,|\nabla f|^2$

$$\mathscr{M}=\|f\|_{2p}^{2p}\,,\quad \mathsf{E}[u]=\|f\|_{p+1}^{p+1}\,,\quad \mathsf{I}[u]=(p+1)^2\,\|\nabla f\|_2^2$$

If u solves (FDE) $\frac{\partial u}{\partial t}=\Delta u^m$

$$\mathsf{E}'\ge\frac{p-1}{2p}\,(p+1)^2\,\Big(\mathscr{C}_{\mathrm{GNS}(p)}\Big)^{\frac{2}{\theta}}\,\|f\|_{2p}^{\frac{2}{\theta}}\,\|f\|_{p+1}^{-\frac{2(1-\theta)}{\theta}}=C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d}u^m(t,x)\,dx\ge\left(\int_{\mathbb{R}^d}u_0^m\,dx+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\quad\forall\,t\ge0$$

Equality case: $u(t,x)=\frac{c_1}{R(t)^d}\,\mathscr{B}\left(\frac{c_2\,x}{R(t)}\right)\,,\ \mathscr{B}(x):=\left(1+|x|^2\right)^{\frac{1}{m-1}}$

Pressure variable and decay of the Fisher information

The t-derivative of the *Rényi entropy power* $\mathsf{E}^{\frac{2}{d}\,\frac{1}{1-m}-1}$ is proportional to

$$\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

▷ *Pressure variable*

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

▷ *Fisher information*

$$\mathsf{I}[u] = \int_{\mathbb{R}^d} u\,|\nabla\mathsf{P}|^2\,dx$$

If u solves (FDE), then

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla\mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla\mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta\mathsf{P} + |\nabla\mathsf{P}|^2\Big)\,dx$$

$$= -2\int_{\mathbb{R}^d} u^m\left(\|\mathrm{D}^2\mathsf{P}\|^2 - (1-m)\,(\Delta\mathsf{P})^2\right)dx$$

Rényi entropy powers and interpolation inequalities

▷ Integrations by parts and completion of squares

$$-\frac{\mathsf{I}}{2\theta}\,\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)$$
$$=\int_{\mathbb{R}^d}u^m\left\|\mathrm{D}^2\mathsf{P}-\frac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right\|^2dx+(m-m_1)\int_{\mathbb{R}^d}u^m\left|\Delta\mathsf{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2dx$$

▷ Analysis of the asymptotic regime as $t\to+\infty$

$$\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^{\theta}\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{\mathscr{M}^{\frac{2\theta}{p}}}=\frac{\mathsf{I}[\mathscr{B}]^{\theta}\,\mathsf{E}[\mathscr{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathscr{B}\|_1^{\frac{2\theta}{p}}}=(p+1)^{2\theta}\,\left(\mathscr{C}_{\mathrm{GNS}}(p)\right)^{2\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$\mathsf{I}[u]^{\theta}\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\ge(p+1)^{2\theta}\,\left(\mathscr{C}_{\mathrm{GNS}}(p)\right)^{2\theta}\,\mathscr{M}^{\frac{2\theta}{p}}$$

The fast diffusion equation in self-similar variables

- ▷ Rescaling and self-similar variables
- ▷ Relative entropy and the entropy – entropy production inequality
- ▷ Large time asymptotics and spectral gaps

Entropy – entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x)=\frac{1}{\kappa^d\,R^d}\,v\left(\tau,\frac{x}{\kappa\,R}\right)\quad\text{where}\quad\frac{dR}{dt}=R^{1-\mu}\,,\quad\tau(t):=\tfrac12\log R(t)$$

$\frac{\partial u}{\partial t}=\Delta u^m$ is changed into *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial\tau}+\nabla\cdot\left[v\left(\nabla v^{m-1}-2\,x\right)\right]=0 \qquad (r\,\mathrm{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v]:=-\frac{1}{m}\int_{\mathbb{R}^d}\left(v^m-\mathcal{B}^m-m\,\mathcal{B}^{m-1}\,(v-\mathcal{B})\right)dx$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}+2\,x\right|^2dx$$

are such that $\mathcal{I}[v]\ge4\,\mathcal{F}[v]$ by (GNS) [del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)]\le\mathcal{F}[v_0]\,e^{-4\,t}$$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \tag{H}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,,\quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,dx$, for $\alpha<0$

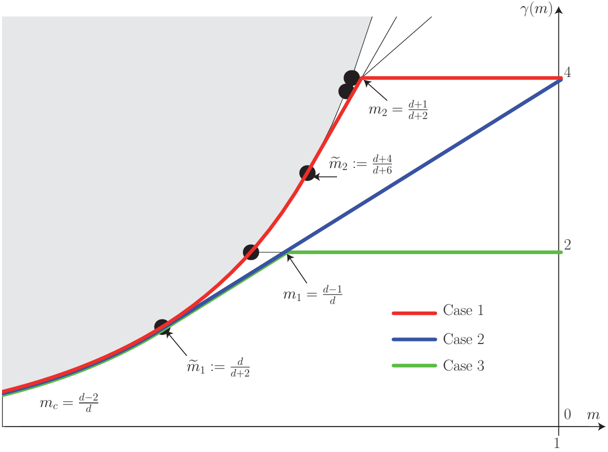
Lemma

Under assumption (H),

$$\mathscr{F}[v(t,\cdot)] \le C\,e^{-2\gamma(m)\,t} \quad \forall\, t\ge 0\,,\quad \gamma(m):=(1-m)\,\Lambda_{1/(m-1),d}$$

Moreover $\gamma(m):=2$ *if* $1-1/d\le m<1$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Initial and asymptotic time layers

▷ Asymptotic time layer: constraint, spectral gap and improved entropy – entropy production inequality

▷ Initial time layer: the carré du champ inequality and a backward estimate

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathscr{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,dx$$

Hardy-Poincaré inequality. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}^{2-m}\,dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,dx = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,dx = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2\,(d\,m - d + 1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\,dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ and

$$(1-\varepsilon)\,\mathscr{B} \le v \le (1+\varepsilon)\,\mathscr{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then

$$\mathscr{I}[v] \ge (4+\eta)\,\mathscr{F}[v]$$

The initial time layer improvement: backward estimate

Hint: for some strictly convex function ψ with $\psi(0)=\psi'(0)=0$, we have

$$\mathscr{I}-4\mathscr{F}\ge 4\big(\psi(\mathscr{F})-\mathscr{F}\big)\ge 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions) for (FDE), we expect some improvement

Rephrasing the *carré du champ* method, $\mathscr{Q}[v]:=\frac{\mathscr{I}[v]}{\mathscr{F}[v]}$ is such that

$$\frac{d\mathscr{Q}}{dt}\le\mathscr{Q}\,(\mathscr{Q}-4)$$

Lemma

Assume that $m>m_1$ and v is a solution to $(r\,\mathrm{FDE})$ with nonnegative initial datum v_0. If for some $\eta>0$ and $t_\star>0$, we have $\mathscr{Q}[v(t_\star,\cdot)]\ge 4+\eta$, then

$$\mathscr{Q}[v(t,\cdot)]\ge 4+\frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}}\quad\forall\,t\in[0,t_\star]$$

Stability in Gagliardo-Nirenberg-Sobolev inequalities

Our strategy

Choose $\varepsilon > 0$*, small enough*

Get a threshold time $t_\star(\varepsilon)$

0 $\qquad$ $t_\star(\varepsilon)$ $\qquad$ t

Backward estimate $\qquad$ *Forward estimate*

The threshold time and the uniform convergence in relative error

▷ The regularity results allow us to glue the initial time layer estimates with the asymptotic time layer estimates

The improved entropy – entropy production inequality holds for any time along the evolution along (r FDE)

(and in particular for the initial datum)

If v is a solves $(r\,\mathrm{FDE})$ for some nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} v_0\,dx \le A < \infty \tag{H_A}$$

then

$$(1-\varepsilon)\,\mathscr{B} \le v(t,\cdot) \le (1+\varepsilon)\,\mathscr{B} \quad \forall\, t \ge t_\star$$

for some ***explicit*** $t_\star$ depending only on ε and A

Large time asymptotics and Barenblatt solutions

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad \text{(FDE)}$$

admits the self-similar *Barenblatt* solution

$$B(t,x) = \frac{t^{1/(1-m)}}{\left[b_0\,\frac{t^{2/\mu}}{\mathcal{M}^{2/\mu(1-m)}} + b_1\,|x|^2\right]^{1/(1-m)}}$$

where $\mu = 2 - d\,(1-m) > 0$, such that

$$\lim_{t\to+\infty} \|u(t) - B(t)\|_{\mathrm{L}^1(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t\to+\infty} t^{d/\mu}\,\|u(t) - B(t)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0$$

The uniform convergence in relative error is a matter of tails

We are interested in the convergence in *relative error*, *i.e.*, the convergence of

$$\left|\frac{u(t,x)-B(t,x)}{B(t,x)}\right|$$

with $\mathscr{M}=\int_{\mathbb{R}^d} u_0\,dx$. If the initial data is $u_0(x)=\left(1+|x|^2\right)^{-m/(1-m)}$, then the solution of (FDE) satisfies

$$\frac{1}{\left[(c\,t+1)^{1/(1-m)}+|x|^2\right]^{\frac{m}{1-m}}}\le u(t,x)\le\frac{(1+t)^{\frac{m}{1-m}}}{\left(1+t+|x|^2\right)^{\frac{m}{1-m}}}$$

Global Harnack Principle

The *Global Harnack Principle* holds if for some $t>0$ large enough

$$\mathscr{B}_{M_1}(t-\tau_1,x)\le u(t,x)\le \mathscr{B}_{M_2}(t+\tau_2,x) \tag{GHP}$$

[Vázquez, 2003], [Bonforte, Vázquez, 2006]: (GHP) holds if $u_0 \lesssim |x|^{-\frac{2}{1-m}}$
[Vázquez, 2003], [Bonforte, Simonov, 2020]: (GHP) holds if

$$\mathsf{A}[u_0]:=\sup_{R>0} R^{\frac{2}{1-m}-d}\int_{\mathbb{R}^d\setminus B_R(0)}|u_0|\,dx<\infty$$

Theorem

[Bonforte, Simonov, 2020] If $M+\mathsf{A}[u_0]<\infty$, then

$$\lim_{t\to\infty}\left\|\frac{u(t)-B(t)}{B(t)}\right\|_\infty=0$$

Uniform convergence in relative error

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m\in(m_1,1)$ if $d\ge2$, $m\in(1/3,1)$ if $d=1$ and let $\varepsilon\in(0,1/2)$, small enough, $A>0$, and $G>0$ be given. There exists an explicit threshold time $T\ge0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t}=\Delta u^m \tag{FDE}$$

with nonnegative initial datum $u_0\in\mathrm L^1(\mathbb R^d)$ satisfying

$$\mathsf A[u_0]=\sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}}\int_{|x|>r} u_0\,dx\le A<\infty \tag{$\mathrm H_A$}$$

$\int_{\mathbb R^d} u_0\,dx=\int_{\mathbb R^d} B\,dx=\mathcal M$ and $\mathcal F[u_0]\le G$, then

$$\sup_{x\in\mathbb R^d}\left|\frac{u(t,x)}{B(t,x)}-1\right|\le\varepsilon\quad\forall\,t\ge T$$

The threshold time

Proposition

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\varepsilon \in (0, \varepsilon_{m,d})$, $A > 0$ and $G > 0$

$$T = \mathsf{c}_\star \frac{1 + A^{1-m} + G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a} = \frac{\alpha}{\vartheta} \frac{2-m}{1-m}$, $\alpha = d\,(m - m_c)$ *and* $\vartheta = \nu/(d+\nu)$

$$\mathsf{c}_\star = \mathsf{c}_\star(m,d) = \sup_{\varepsilon \in (0, \varepsilon_{m,d})} \max\left\{\varepsilon\,\kappa_1(\varepsilon,m), \varepsilon^{\mathsf{a}} \kappa_2(\varepsilon,m), \varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m) := \max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m} - 1}, \frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m) := \frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}} \quad \text{and} \quad \kappa_3(\varepsilon,m) := \frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

Improved entropy – entropy production inequality (subcritical case)

Theorem

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. Then there is a positive number ζ such that

$$\mathscr{I}[v]\ge(4+\zeta)\,\mathscr{F}[v]$$

for any nonnegative function $v\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v]=G$, $\int_{\mathbb{R}^d}v\,dx=\mathscr{M}$, $\int_{\mathbb{R}^d}x\,v\,dx=0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon\in(0,2\,\varepsilon_\star)\,,\quad \varepsilon_\star:=\frac12\min\{\varepsilon_{m,d},\chi\,\eta\}\quad\text{with}\quad t_\star=t_\star(\varepsilon)=\frac12\log R(T)$$

$$(1-\varepsilon)\,\mathscr{B}\le v(t,\cdot)\le(1+\varepsilon)\,\mathscr{B}\quad\forall\,t\ge t_\star$$

and, as a consequence, the *initial time layer estimate*

$$\mathscr{I}[v(t,.)]\ge(4+\zeta)\,\mathscr{F}[v(t,.)]\quad\forall\,t\in[0,t_\star]\quad\text{where}\quad\zeta=\frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}}$$

Two consequences

$$\zeta = \mathsf{Z}\big(A, \mathscr{F}[u_0]\big)\,,\quad \mathsf{Z}(A,G) := \frac{\zeta_\star}{1 + A^{(1-m)\frac{2}{\alpha}} + G}\,,\quad \zeta_\star := \frac{4\,\eta\,c_\alpha}{4+\eta}\left(\frac{\varepsilon_\star^a}{2\,\alpha\,c_\star}\right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (r FDE) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ and v_0 satisfies (H_A), then

$$\mathscr{F}[v(t,.)] \le \mathscr{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\, t \ge 0$$

▷ The ***stability in the entropy - entropy production estimate*** $\mathscr{I}[v] - 4\,\mathscr{F}[v] \ge \zeta\,\mathscr{F}[v]$ also holds in a stronger sense

$$\mathscr{I}[v] - 4\,\mathscr{F}[v] \ge \frac{\zeta}{4+\zeta}\,\mathscr{I}[v]$$

Stability results (subcritical case)

▷ We rephrase the results obtained by entropy methods in the language of stability *à la* Bianchi-Egnell

Subcritical range

$$p^* = +\infty \text{ if } d = 1 \text{ or } 2, \qquad p^* = \frac{d}{d-2} \text{ if } d \geq 3$$

$$\lambda[f] := \left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2p}{d-p(d-4)}}\,,\quad \kappa[f] := \frac{\mathcal{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

$$\mathsf{A}[f] := \frac{\mathcal{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}}\,\|f\|_{2p}^{2p}}\,\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f(x+x_f)|^{2p}\,dx$$

$$\mathsf{E}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}}\,f^{p+1} - \mathsf{g}^{p+1} - \frac{1+p}{2p}\,\mathsf{g}^{1-p}\left(\frac{\kappa[f]^{2p}}{\lambda[f]^2}\,f^{2p} - \mathsf{g}^{2p}\right)\right)dx$$

$$\mathfrak{S}[f] := \frac{\mathcal{M}^{\frac{p-1}{2p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf{Z}(\mathsf{A}[f],\mathsf{E}[f])$$

Theorem

Let $d\ge1$, $p\in(1,p^*)$

$$\text{If } f\in\mathcal{W}_p(\mathbb{R}^d) := \left\{f\in\mathrm{L}^{2p}(\mathbb{R}^d)\,:\,\nabla f\in\mathrm{L}^2(\mathbb{R}^d),\ |x|\,f^p\in\mathrm{L}^2(\mathbb{R}^d)\right\},$$

$$\left(\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\right)^{2p\gamma} - \left(\mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p}\right)^{2p\gamma} \ge \mathfrak{S}[f]\,\|f\|_{2p}^{2p\gamma}\,\mathsf{E}[f]$$

With $\mathscr{K}_{\rm GNS} = C(p,d)\,\mathscr{C}_{\rm GNS}^{2p\gamma}$, $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GNS}\,\|f\|_{2p}^{2p\gamma}$$

Theorem

Let $d\ge1$ and $p\in(1,p^*)$. There is an explicit $\mathscr{C}=\mathscr{C}[f]$ such that, for any $f\in\mathrm{L}^{2p}(\mathbb{R}^d,(1+|x|^2)\,dx)$ such that $\nabla f\in\mathrm{L}^2(\mathbb{R}^d)$ and $\mathsf{A}[f^{2p}]<\infty$,

$$\delta[f]\ge\mathscr{C}[f]\inf_{\varphi\in\mathfrak{M}}\int_{\mathbb{R}^d}\left|(p-1)\nabla f+f^p\,\nabla\varphi^{1-p}\right|^2dx$$

▷ The dependence of $\mathscr{C}[f]$ on $\mathsf{A}[f^{2p}]$ and $\mathscr{F}[f^{2p}]$ is explicit and does not degenerate if $f\in\mathfrak{M}$

▷ Can we remove the condition $\mathsf{A}[f^{2p}]<\infty$?

Stability in Sobolev's inequality (critical case)

- ▷ A constructive stability result
- ▷ The main ingredient of the proof

A constructive stability result

Let $2\,p^\star=2d/(d-2)=2^*$, $d\ge3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d)=\left\{f\in\mathrm{L}^{p^\star+1}(\mathbb{R}^d)\,:\,\nabla f\in\mathrm{L}^2(\mathbb{R}^d)\,,\;|x|\,f^{p^\star}\in\mathrm{L}^2(\mathbb{R}^d)\right\}$$

Theorem

Let $d\ge3$ and $A>0$. Then for any nonnegative $f\in\mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d}(1,x,|x|^2)\,f^{2^*}\,dx=\int_{\mathbb{R}^d}(1,x,|x|^2)\,\mathrm{g}\,dx\quad\text{and}\quad\sup_{r>0}r^d\int_{|x|>r}f^{2^*}\,dx\le A$$

we have

$$\delta[f]:=\|\nabla f\|_2^2-\mathsf{S}_d^2\,\|f\|_{2^*}^2\ge\frac{\mathscr{C}_\star(A)}{4+\mathscr{C}_\star(A)}\int_{\mathbb{R}^d}\left|\nabla f+\frac{d-2}{2}\,f^{\frac{d}{d-2}}\,\nabla\mathrm{g}^{-\frac{2}{d-2}}\right|^2dx$$

$\mathscr{C}_\star(A)=\mathfrak{C}_\star\left(1+A^{1/(2d)}\right)^{-1}$ and $\mathfrak{C}_\star>0$ depends only on d

We can remove the normalization of f, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x + x_f) \quad \text{and} \quad Z[f] := \Big(1 + \mu[f]^{-d}\,\lambda[f]^d\,A[f]\Big)$$

the *Bianchi-Egnell type result*

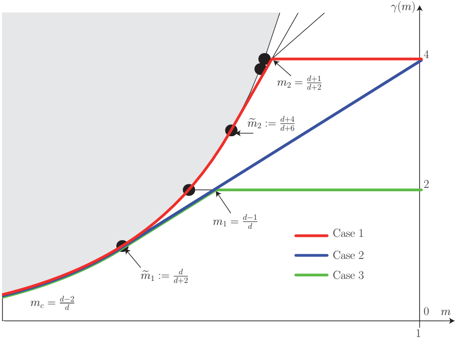
$$\delta[f] \geq \frac{\mathfrak{C}_\star\,Z[f]}{4 + Z[f]} \inf_{g \in \mathfrak{M}} \mathscr{J}[f|g]$$

with x_f, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

Extending the subcritical result in the critical case

To improve the spectral gap for $m=m_1$, we need to adjust the Barenblatt function $\mathscr{B}_\lambda(x)=\lambda^{-d/2}\,\mathscr{B}\Big(x/\sqrt{\lambda}\Big)$ in order to match $\int_{\mathbb{R}^d}|x|^2\,v\,dx$ where the function v solves (r FDE) or to further rescale v according to

$$v(t,x)=\frac{1}{\mathfrak{R}(t)^d}\,w\left(t+\tau(t),\frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{\mathrm{d}\tau}{\mathrm{d}t}=\left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d}|x|^2\,v\,dx\right)^{-\frac{d}{2}(m-m_c)}-1\,,\quad \tau(0)=0\quad\text{and}\quad\mathfrak{R}(t)=e^{2\tau(t)}$$

Lemma

$t\mapsto\lambda(t)$ and $t\mapsto\tau(t)$ are bounded on $\mathbb{R}^+$

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/
▷ Lectures

More related papers can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/list/
▷ Preprints and papers

For final versions, use `Dolbeault` as login and `Jean` as password

E-mail: dolbeault@ceremade.dauphine.fr

Thank you for your attention !