Relative entropy methods for nonlinear diffusion models

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Outline

- Introduction to the notion of entropy
- A particularly simple case: the **heat equation**
- A case with homogeneity: **fast diffusion** (and porous media) equations
  - Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities
- More about **homogeneity** and the entropy – entropy production approach
  - Algebraic rates *vs.* exponential decay
  - The Bakry-Emery method revisited
  - The gradient flow interpretation
- From kinetic to diffusive models
  - Diffusion limit
  - Hypocoercivity
- Other applications, other models

About *entropy* in physics

- Entropy has been introduced as a state function in thermodynamics by R. Clausius in 1865, in the framework of the second law of thermodynamics, in order to interpret the the results of S. Carnot
- A statistical physics approach: Boltzmann's formula (1877) defines the entropy of a systems in terms of a counting of the micro-states of a physical system
- Boltzmann's equation: $\partial_t f + v \cdot \nabla_x f = Q(f,f)$

  describes the evolution of a gas of particles having binary collisions at the kinetic level f is a time dependent distribution function (probability density) defined on the phase space $\mathbb{R}^d \times \mathbb{R}^d$, thus a function of time t, position x and velocity v. The entropy $H[f] := \iint_{\mathbb{R}^d \times \mathbb{R}^d} f \, \log f \, dx \, dv$ measures the irreversibility: **H-Theorem** (1872)

$$\frac{d}{dt} H[f] = \iint_{\mathbb{R}^d \times \mathbb{R}^d} Q(f,f) \, \log f \, dx \, dv \leq 0$$

- Other notions of entropy:
  - Shannon entropy in information theory, entropy in probability theory (with reference to an arbitrary measure)
  - Other approaches: Carathéodory (1908), Lieb-Yngvason (1997)

About *entropy* in partial differential equations

- In kinetic theory, entropy is one of the few *a priori* estimates available: it has been used for producing existence results [DiPerna, Lions], compactness results with application to hydrodynamic limits [Bardos, Golse, Levermore, Saint-Raymond], convergence of numerical schemes, etc.
- Nash, Lax, DiPerna: regularity for parabolic equations, hydodynamics, compensated-compactness, geometry, etc.
- Modeling issues: entropy estimates are compatible with other physical estimates. Exponential convergence is an issue in physics (time-scales), for numerics, for multi-scale analysis
- For the last 10 years, it has motivated a very large number of studies in the area of nonlinear diffusions, systems of PDEs, in connection with probability, gradient flow and mass transportation techniques
- It can be used to obtain rates of decay or intermediate aymptotics, in connection with functional inequalities

Entropy (a loose definition): a special kind of Lyapunov functional that combines well with other *a priori* estimates and can be used to investigate the large time behaviour

Heat equation and entropy

Consider the **heat equation** on the euclidean space $\mathbb{R}^d$

$$\frac{\partial u}{\partial t} = \Delta u \;, \quad u_{|t=0} = u_0$$

As $t \to \infty$, we know that $u(t,x) \sim G(t,x) := (4\,\pi\,t)^{-d/2}\,e^{-\frac{|x|^2}{4\,t}}$. This is easy to quantify in L^∞ or in L^2. How to give (sharp) estimates in L^1 ? Assume that $u_0 \geq 0$, $u_0\,(1+|x|^2)$, $u_0\,\log u_0 \in L^1(\mathbb{R}^d)$ and consider the **entropy**

$$\mathcal{S}[u] := \int_{\mathbb{R}^d} u\,\log u\,dx$$

- Time-dependent rescaling and Fokker-Planck equation
- Relative entropy (free energy), entropy – entropy production (relative Fisher information) and logarithmic Sobolev inequality
- Csiszár-Kullback inequality and intermediate asymptotics
- The Bakry-Emery method for proving the logarithmic Sobolev inequality

The *entropy* approach (1/2)

Time-dependent rescaling: the change of variables $u(\tau,y)=R^{-d}\,v(t,y/R)$ with $t=\log R$, $R=R(\tau)=\sqrt{1+2\,\tau}$ changes the heat equation $u_\tau=\Delta u$ into the Fokker-Planck equation:

$$v_t=\Delta v+\nabla\cdot(x\,v)\;,\quad v_{|t=0}=u_0$$

with stationary solutions $v_\infty(x):=(2\,\pi)^{-d/2}\,M\,e^{-|x|^2/2}$

Relative entropy (free energy): choose $M=\int_{\mathbb{R}^d}u_0\,dx$ and define

$$\Sigma[v]:=\int_{\mathbb{R}^d}v\,\log\left(\frac{v}{v_\infty}\right)\,dx=\int_{\mathbb{R}^d}v\,\log v\,dx+\frac{1}{2}\int_{\mathbb{R}^d}|x|^2\,v\,dx+Const$$

If v is a solution of the Fokker-Planck equation, then

$$\frac{d}{dt}\,\Sigma[v]=-I[v]$$

where $I[v]=\int_{\mathbb{R}^d}v\left|\frac{\nabla v}{v}+x\right|^2\,dx$ is the relative Fisher information
Observe that exponential decay holds by the **logarithmic Sobolev inequality** [Gross 75]

$$\Sigma[v]\le\frac{1}{2}\,I[v]$$

The *entropy* approach (2/2)

Large time behaviour is controlled by

$$\Sigma[v(t,\cdot)] = \int_{\mathbb{R}^d} v\,\log\left(\frac{v}{v_\infty}\right)\,dx \le \Sigma[u_0]\,e^{-2\,t}$$

Using the **Csiszár-Kullback inequality**

$$\|v(t,\cdot) - v_\infty\|^2_{L^1(\mathbb{R}^d)} \le 4\,M\,\Sigma[v] \le 4\,M\,\Sigma[u_0]\,e^{-2\,t}$$

we get **intermediate asymptotics** for the heat equation, namely

$$\|u(\tau,\cdot) - u_\infty(\tau,\cdot)\|^2_{L^1(\mathbb{R}^d)} \le 4\,M\,\Sigma[v] \le \frac{4\,M\,\Sigma[u_0]}{1+2\,\tau}$$

with $u_\infty(\tau,y) := R^{-d}\,v_\infty(\log R, y/R),\ R = R(\tau) = \sqrt{1+2\,\tau}$

Remark: The **Bakry-Emery** method gives a proof of the logarithmic Sobolev inequality based on the heat equation:

$$\frac{d}{dt}\Big(I[v] - 2\,\Sigma[v]\Big) \le 0$$

Sharp rates of decay of solutions to the nonlinear fast diffusion equation

Fast diffusion equations: outline

- Introduction
  - Fast diffusion equations: entropy methods and Gagliardo-Nirenberg inequalities [del Pino, J.D.]
  - Fast diffusion equations: the finite mass regime
  - Fast diffusion equations: the infinite mass regime
- Relative entropy methods and linearization
  - the linearization of the functionals approach: [Blanchet, Bonforte, J.D., Grillo, Vázquez]
  - sharp rates: [Bonforte, J.D., Grillo,Vázquez]
  - An improvement based on the center of mass: [Bonforte, J.D., Grillo, Vázquez]
- An improvement based on the variance: [J.D., Toscani]

Some references

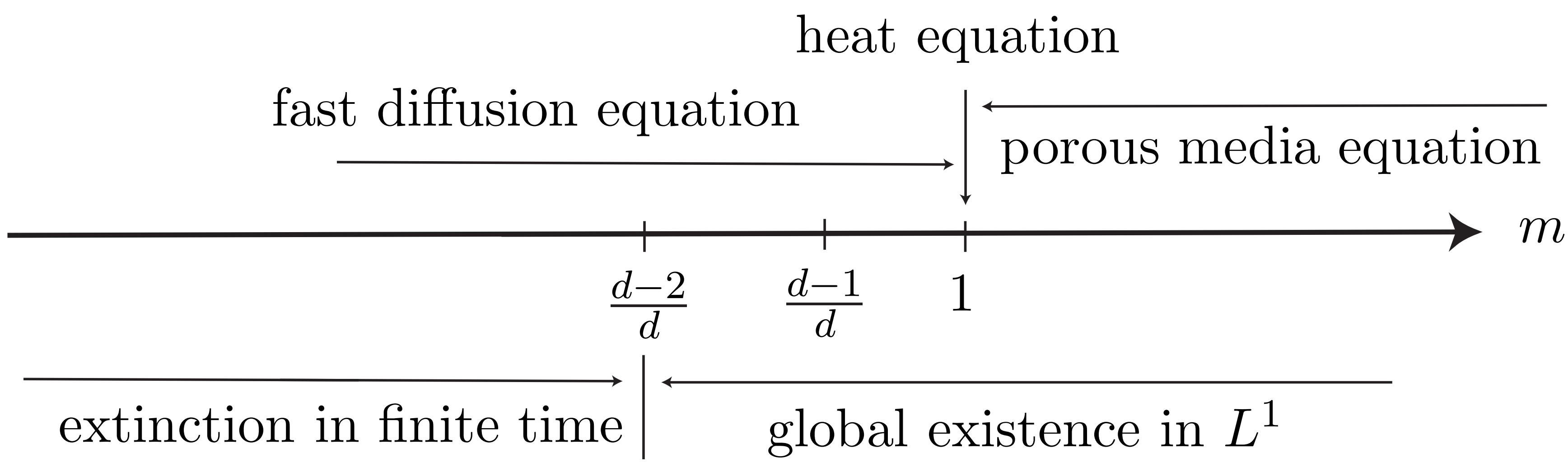
- J.D. and G. Toscani, Fast diffusion equations: matching large time asymptotics by relative entropy methods, Preprint
- Matteo Bonforte, J.D., Gabriele Grillo, and Juan-Luis Vázquez. Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities, submitted to Proc. Nat. Acad. Sciences
- A. Blanchet, M. Bonforte, J.D., G. Grillo, and J.-L. Vázquez. Asymptotics of the fast diffusion equation via entropy estimates. Archive for Rational Mechanics and Analysis, 191 (2): 347-385, 02, 2009
- A. Blanchet, M. Bonforte, J.D., G. Grillo, and J.-L. Vázquez. Hardy-Poincaré inequalities and applications to nonlinear diffusions. C. R. Math. Acad. Sci. Paris, 344(7): 431-436, 2007
- M. Del Pino and J.D., Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. J. Math. Pures Appl. (9), 81 (9): 847-875, 2002

Fast diffusion equations: entropy methods

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$ as $t \to +\infty$

[Friedmann, Kamin, 1980] $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$



Existence theory, critical values of the parameter m

Some references

Generalized entropies and nonlinear diffusions (EDP, uncomplete):
[Del Pino, J.D.], [Carrillo, Toscani], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vázquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub],... [del Pino, Sáez], [Daskalopulos, Sesum]...

Some methods

1) [J.D., del Pino] relate entropy and **Gagliardo-Nirenberg** inequalities
2) *entropy – entropy-production method* the **Bakry-Emery** point of view
3) mass transport techniques
4) hypercontractivity for appropriate semi-groups
5) the approach by **linearization** of the entropy

... Fast diffusion equations and Gagliardo-Nirenberg inequalities

We follow the same scheme as for the heat equation

Time-dependent rescaling, Free energy

- **Time-dependent rescaling**: Take $u(\tau,y) = R^{-d}(t)\, v\,(t, y/R(\tau))$ where

$$\frac{\partial R}{\partial \tau} = R^{d(1-m)-1}\,, \quad R(0) = 1\,, \quad t = \log R$$

The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla\cdot(x\, v)\,, \quad v_{|\tau=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional: **Generalized entropy** or **Free energy**

$$\Sigma[v] := \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}\,|x|^2 v \right) dx - \Sigma_0$$

Entropy production is measured by the **Generalized Fisher information**

$$\frac{d}{dt}\Sigma[v] = -I[v]\,, \quad I[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$

Relative entropy and entropy production

- **Stationary solution:** choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) := \left(C + \frac{1-m}{2\,m}\,|x|^2 \right)_+^{-1/(1-m)}$$

Relative entropy: Fix Σ_0 so that $\Sigma[v_\infty] = 0$. The entropy can be put in an m-homogeneous form: for $m \neq 1$,

$$\Sigma[v] = \int_{\mathbb{R}^d} \psi\!\left(\tfrac{v}{v_\infty}\right) v_\infty^m \, dx \quad with\ \psi(t) = \frac{t^m - 1 - m\,(t-1)}{m-1}$$

- **Entropy – entropy production inequality**

Theorem 1. $d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2\,\Sigma[v]$$

Corollary 2. *A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies*

$$\Sigma[v(t,\cdot)] \leq \Sigma[u_0]\,e^{-\,2\,t}$$

An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\Sigma[v] = \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v\right) dx - \Sigma_0 \leq \frac{1}{2}\int_{\mathbb{R}^d} v \left|\frac{\nabla v^m}{v} + x\right|^2 dx = \frac{1}{2} I[v]$$

Rewrite it with $p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx + K \geq 0$$

- $1 < p = \frac{1}{2m-1} \leq \frac{d}{d-2} \iff$ Fast diffusion case: $\frac{d-1}{d} \leq m < 1$; $K < 0$
- $0 < p < 1 \iff$ Porous medium case: $m > 1$, $K > 0$
- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} v\, dx = \int_{\mathbb{R}^d} w^{2p}\, dx\right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal
- $m = m_1 := \frac{d-1}{d}$: Sobolev, $m \to 1$: logarithmic Sobolev

Theorem 3. [Del Pino, J.D.] *Assume that* $1 < p \leq \frac{d}{d-2}$ *(fast diffusion case) and* $d \geq 3$

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \leq A\, \|\nabla w\|^\theta_{L^2(\mathbb{R}^d)}\, \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}, \quad \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \quad y = \frac{p+1}{p-1}$$

Intermediate asymptotics

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau}$+ Csiszár-Kullback inequalities
Undo the change of variables, with

$$u_\infty(t,x) = R^{-d}(t)\, v_\infty\left(x/R(t)\right)$$

Theorem 4. [Del Pino, J.D.] *Consider a solution of $u_t = \Delta u^m$ with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\, u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$*

- **Fast diffusion case:** $\frac{d-1}{d} < m < 1$ *if* $d \geq 3$

$$\limsup_{t\to+\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}}\, \|u^m - u_\infty^m\|_{L^1} < +\infty$$

- **Porous medium case:** $1 < m < 2$

$$\limsup_{t\to+\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}}\, \|\,[u - u_\infty]\, u_\infty^{m-1}\,\|_{L^1} < +\infty$$

Fast diffusion equations: the finite mass regime

Can we consider $m < m_1$?

- If $m \geq 1$: porous medium regime or $m_1 := \frac{d-1}{d} \leq m < 1$, the decay of the entropy is governed by Gagliardo-Nirenberg inequalities, and to the limiting case $m = 1$ corresponds the logarithmic Sobolev inequality
- Displacement convexity holds in the same range of exponents, $m \in (m_1, 1)$, as for the Gagliardo-Nirenberg inequalities
- The fast diffusion equation can be seen as the gradient flow of the generalized entropy with respect to the Wasserstein distance if $m > \widetilde{m}_1 := \frac{d}{d+2}$
- If $m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in L^1 and the Barenblatt self-similar solution has finite mass

...the Bakry-Emery method

We follow the same scheme as for the heat equation

Consider the generalized Fisher information

$$I[v] := \int_{\mathbb{R}^d} v\,|Z|^2\,dx \quad \text{with} \quad Z := \frac{\nabla v^m}{v} + x$$

and compute

$$\frac{d}{dt}\,I[v(t,\cdot)] + 2\,I[v(t,\cdot)] = -2\,(m-1)\int_{\mathbb{R}^d} u^m\,(\mathrm{div} Z)^2\,dx - 2\sum_{i,\,j=1}^{d}\int_{\mathbb{R}^d} u^m\,(\partial_i Z^j)^2\,dx$$

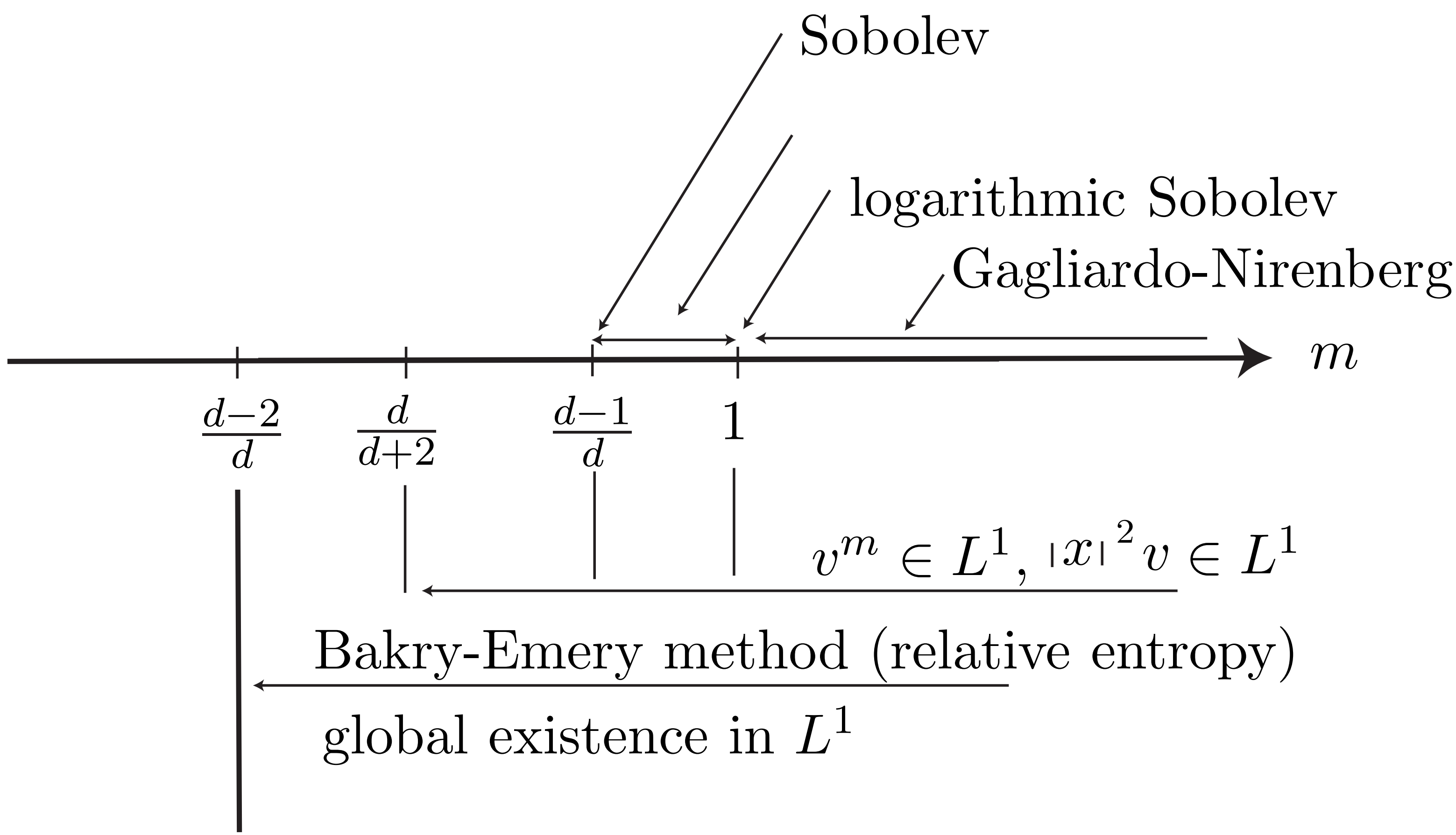
- the Fisher information decays exponentially: $I[v(t,\cdot)] \le I[u_0]\,e^{-\,2\,t}$
- $\lim_{t\to\infty} I[v(t,\cdot)] = 0$ and $\lim_{t\to\infty} \Sigma[v(t,\cdot)] = 0$
- $\frac{d}{dt}\Big(I[v(t,\cdot)] - 2\,\Sigma[v(t,\cdot)]\Big) \le 0$ means $I[v] \ge 2\,\Sigma[v]$

[Carrillo, Toscani], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]

$I[v] \ge 2\,\Sigma[v]$ holds for any $m > m_c$

Fast diffusion: finite mass regime

Inequalities...



... existence of solutions of $u_t = \Delta u^m$

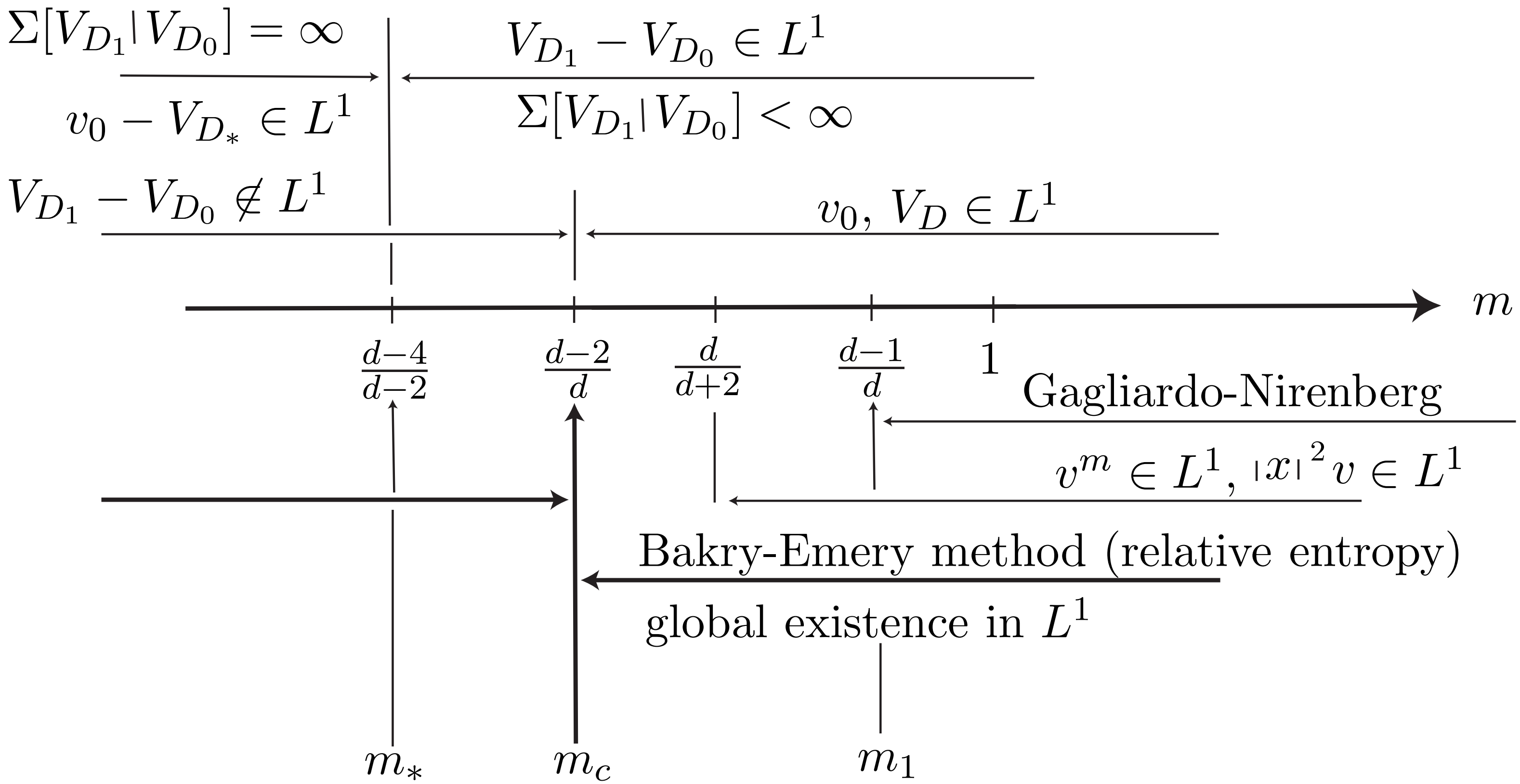
More references: Extensions and related results

- Mass transport methods: inequalities / rates [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]
- General nonlinearities [Biler, J.D., Esteban], [Carrillo-DiFrancesco], [Carrillo-Juengel-Markowich-Toscani-Unterreiter] and gradient flows [Jordan-Kinderlehrer-Otto], [Ambrosio-Savaré-Gigli], [Otto-Westdickenberg] [J.D.-Nazaret-Savaré], etc
- Non-homogeneous nonlinear diffusion equations [Biler, J.D., Esteban], [Carrillo, DiFrancesco]
- Extension to systems and connection with Lieb-Thirring inequalities [J.D.-Felmer-Loss-Paturel, 2006], [J.D.-Felmer-Mayorga]
- Drift-diffusion problems with mean-field terms. An example: the Keller-Segel model [J.D-Perthame, 2004], [Blanchet-J.D-Perthame, 2006], [Biler-Karch-Laurençot-Nadzieja, 2006], [Blanchet-Carrillo-Masmoudi, 2007], etc
- ... connection with linearized problems [Markowich-Lederman], [Carrillo-Vázquez], [Denzler-McCann], [McCann, Slepčev], [Kim, McCann], [Koch, McCann, Slepčev]

Fast diffusion equations: the infinite mass regime – Linearization of the entropy

- If $m > m_c := \frac{d-2}{d} \le m < m_1$, solutions globally exist in $L^1(\mathbb{R}^d)$ and the Barenblatt self-similar solution has finite mass.
- For $m \le m_c$, the Barenblatt self-similar solution has infinite mass

Extension to $m \le m_c$? Work in relative variables !



Entropy methods and linearization: intermediate asymptotics, vanishing

[A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez], [J.D., Toscani]

- work in relative variables
- use the properties of the flow
- write everything as relative quantities (to the Barenblatt profile)
- compare the functionals (entropy, Fisher information) to their linearized counterparts

$\Longrightarrow$ *Extend the domain of validity of the method to the price of a restriction of the set of admissible solutions*

Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where $m_c := \frac{d-2}{d}$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$
- $0 < m < m_c$, $T < +\infty$: vanishing in finite time $\lim_{\tau \nearrow T} u(\tau, y) = 0$

Alternative approach by comparison techniques: [Daskalopoulos, Sesum] (without rates)

Fast diffusion equation and Barenblatt solutions

$$\frac{\partial u}{\partial \tau} = -\nabla \cdot (u\,\nabla u^{m-1}) = \frac{1-m}{m}\,\Delta u^m \tag{1}$$

with $m < 1$. We look for positive solutions $u(\tau, y)$ for $\tau \geq 0$ and $y \in \mathbb{R}^d$, $d \geq 1$, corresponding to nonnegative initial-value data $u_0 \in L^1_{\rm loc}(dx)$
In the limit case $m = 0$, u^m/m has to be replaced by $\log u$
Barenblatt type solutions are given by

$$U_{D,T}(\tau, y) := \frac{1}{R(\tau)^d}\left(D + \tfrac{1-m}{2\,d\,|m-m_c|}\left|\tfrac{y}{R(\tau)}\right|^2\right)_+^{-\frac{1}{1-m}}$$

- If $m > m_c := (d-2)/d$, $U_{D,T}$ with $R(\tau) := (T+\tau)^{\frac{1}{d\,(m-m_c)}}$ describes the large time asymptotics of the solutions of equation (1) as $\tau \to \infty$ (mass is conserved)
- If $m < m_c$ the parameter T now denotes the *extinction time* and $R(\tau) := (T-\tau)^{-\frac{1}{d\,(m_c-m)}}$
- If $m = m_c$ take $R(\tau) = e^\tau$, $U_{D,T}(\tau, y) = e^{-d\,\tau}\left(D + e^{-2\tau}\,|y|^2/2\right)^{-d/2}$

Two crucial values of m: $m_* := \frac{d-4}{d-2} < m_c := \frac{d-2}{d} < 1$

Rescaling

A time-dependent change of variables

$$t := \tfrac{1-m}{2} \log\left(\tfrac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\tfrac{1}{2\,d\,|m-m_c|}}\,\frac{y}{R(\tau)}$$

If $m = m_c$, we take $t = \tau/d$ and $x = e^{-\tau}\,y/\sqrt{2}$

The generalized Barenblatt functions $U_{D,T}(\tau, y)$ are transformed into stationary *generalized Barenblatt profiles* $V_D(x)$

$$V_D(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}} \quad x \in \mathbb{R}^d$$

If u is a solution to (1), the function $v(t,x) := R(\tau)^d\,u(\tau,y)$ solves

$$\frac{\partial v}{\partial t} = -\nabla \cdot \left[v\,\nabla\left(v^{m-1} - V_D^{m-1}\right)\right] \quad t > 0\,, \quad x \in \mathbb{R}^d \qquad (2)$$

with initial condition $v(t=0,x) = v_0(x) := R(0)^{-d}\,u_0(y)$

Goal

We are concerned with the *sharp rate* of convergence of a solution v of the rescaled equation to the *generalized Barenblatt profile* V_D in the whole range $m < 1$. Convergence is measured in terms of the **relative entropy**

$$\mathcal{E}[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - V_D^m - m\, V_D^{m-1} (v - V_D) \right] dx$$

for all $m \neq 0$, $m < 1$

Assumptions on the initial datum v_0

(H1) *$V_{D_0} \leq v_0 \leq V_{D_1}$ for some $D_0 > D_1 > 0$*

(H2) *if $d \geq 3$ and $m \leq m_*$, $(v_0 - V_D)$ is integrable for a suitable $D \in [D_1, D_0]$*

- The case $m = m_* = \frac{d-4}{d-2}$ will be discussed later
- If $m > m_*$, we define D as the unique value in $[D_1, D_0]$ such that $\int_{\mathbb{R}^d} (v_0 - V_D)\, dx = 0$

Our goal is to find the best possible rate of decay of $\mathcal{E}[v]$ if v solves (2)

Sharp rates of convergence

Theorem 5. [Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if $m < 1$ and $m \neq m_*$, the entropy decays according to*

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,(1-m)\,\Lambda\, t} \quad \forall\, t \geq 0$$

The sharp decay rate Λ is equal to the best constant $\Lambda_{\alpha,d} > 0$ in the Hardy–Poincaré inequality of Theorem 16 *with $\alpha := 1/(m-1) < 0$*
The constant $C > 0$ depends only on m, d, D_0, D_1, D and $\mathcal{E}[v_0]$

- Notion of *sharp rate* has to be discussed
- Rates of convergence in more standard norms: $L^q(dx)$ for $q \geq \max\{1, d\,(1-m)/\,[2\,(2-m)+d\,(1-m)]\}$, or C^k by interpolation
- By undoing the time-dependent change of variables, we deduce results on the *intermediate asymptotics* of (1), i.e. rates of decay of $u(\tau,y) - U_{D,T}(\tau,y)$ as $\tau \to +\infty$ if $m \in [m_c, 1)$, or as $\tau \to T$ if $m \in (-\infty, m_c)$

Strategy of proof

Assume that $D = 1$ and consider $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$, with $\alpha = 1/(m-1) < 0$, and $\mathcal{L}_{\alpha,d} := -h_{1-\alpha} \operatorname{div}\left[\, h_\alpha\, \nabla \cdot \,\right]$ on $L^2(d\mu_\alpha)$:

$$\int_{\mathbb{R}^d} f\, (\mathcal{L}_{\alpha,d}\, f)\, d\mu_{\alpha-1} = \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha$$

A first order expansion of $v(t,x) = h_\alpha(x)\left[1 + \varepsilon\, f(t,x)\, h_\alpha^{1-m}(x)\right]$ solves

$$\frac{\partial f}{\partial t} + \mathcal{L}_{\alpha,d}\, f = 0$$

Theorem 6. *Let $d \geq 3$. For any $\alpha \in (-\infty, 0) \setminus \{\alpha_*\}$, there is a positive constant $\Lambda_{\alpha,d}$ such that*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$

under the additional condition $\int_{\mathbb{R}^d} f\, d\mu_{\alpha-1} = 0$ if $\alpha < \alpha_*$

$$\Lambda_{\alpha,d} = \begin{cases} \frac{1}{4}\,(d-2+2\,\alpha)^2 & \text{if } \alpha \in \left[-\frac{d+2}{2}, \alpha_*\right) \cup (\alpha_*, 0) \\ -4\,\alpha - 2\,d & \text{if } \alpha \in \left[-d, -\frac{d+2}{2}\right) \\ -2\,\alpha & \text{if } \alpha \in (-\infty, -d) \end{cases}$$

[Denzler, McCann], [Blanchet, Bonforte, J.D., Grillo, Vázquez]

Proof: Relative entropy and relative Fisher information and interpolation

For $m \neq 0, 1$, the *relative entropy* of J. Ralston and W.I. Newmann and the *generalized relative Fisher information* are given by

$$\mathcal{F}[w] := \tfrac{m}{1-m} \int_{\mathbb{R}^d} \left[w - 1 - \tfrac{1}{m}\left(w^m - 1\right)\right] V_D^m \, dx$$

$$\mathcal{I}[w] := \int_{\mathbb{R}^d} \left| \tfrac{1}{m-1} \nabla \left[\left(w^{m-1} - 1\right) V_D^{m-1}\right] \right|^2 v \, dx$$

where $w = \frac{v}{V_D}$. If v is a solution of (2), then $\frac{d}{dt}\mathcal{F}[w(t,\cdot)] = -\mathcal{I}[w(t,\cdot)]$

- **Linearization:** $f := (w-1)\, V_D^{m-1}$, $h_1(t) := \inf_{\mathbb{R}^d} w(t,\cdot)$, $h_2(t) := \sup_{\mathbb{R}^d} w(t,\cdot)$ and $h := \max\{h_2, 1/h_1\}$. We notice that $h(t) \to 1$ as $t \to +\infty$

$$h^{m-2} \int_{\mathbb{R}^d} |f|^2 \, V_D^{2-m} \, dx \leq \frac{2}{m}\, \mathcal{F}[w] \leq h^{2-m} \int_{\mathbb{R}^d} |f|^2 \, V_D^{2-m} \, dx$$

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, V_D \, dx \leq [1 + X(h)]\, \mathcal{I}[w] + Y(h) \int_{\mathbb{R}^d} |f|^2 \, V_D^{2-m} \, dx$$

where X and Y are functions such that $\lim_{h\to1} X(h) = \lim_{h\to1} Y(h) = 0$

$$h_2^{2(2-m)}/h_1 \leq h^{5-2m} =: 1 + X(h)$$

$$\left[\,(h_2/h_1)^{2(2-m)} - 1\right] \leq d\,(1-m)\,\left[h^{4(2-m)} - 1\right] =: Y(h)$$

Proof (continued)

- A new **interpolation** inequality: for $h > 0$ small enough

$$\mathcal{F}[w] \leq \frac{h^{2-m}\,[1 + X(h)]}{2\,\big[\Lambda_{\alpha,d} - m\,Y(h)\big]}\, m\,\mathcal{I}[w]$$

- Another **interpolation** allows to close the system of estimates: for some C, t large enough,

$$0 \leq h - 1 \leq \mathsf{C}\,\mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$$

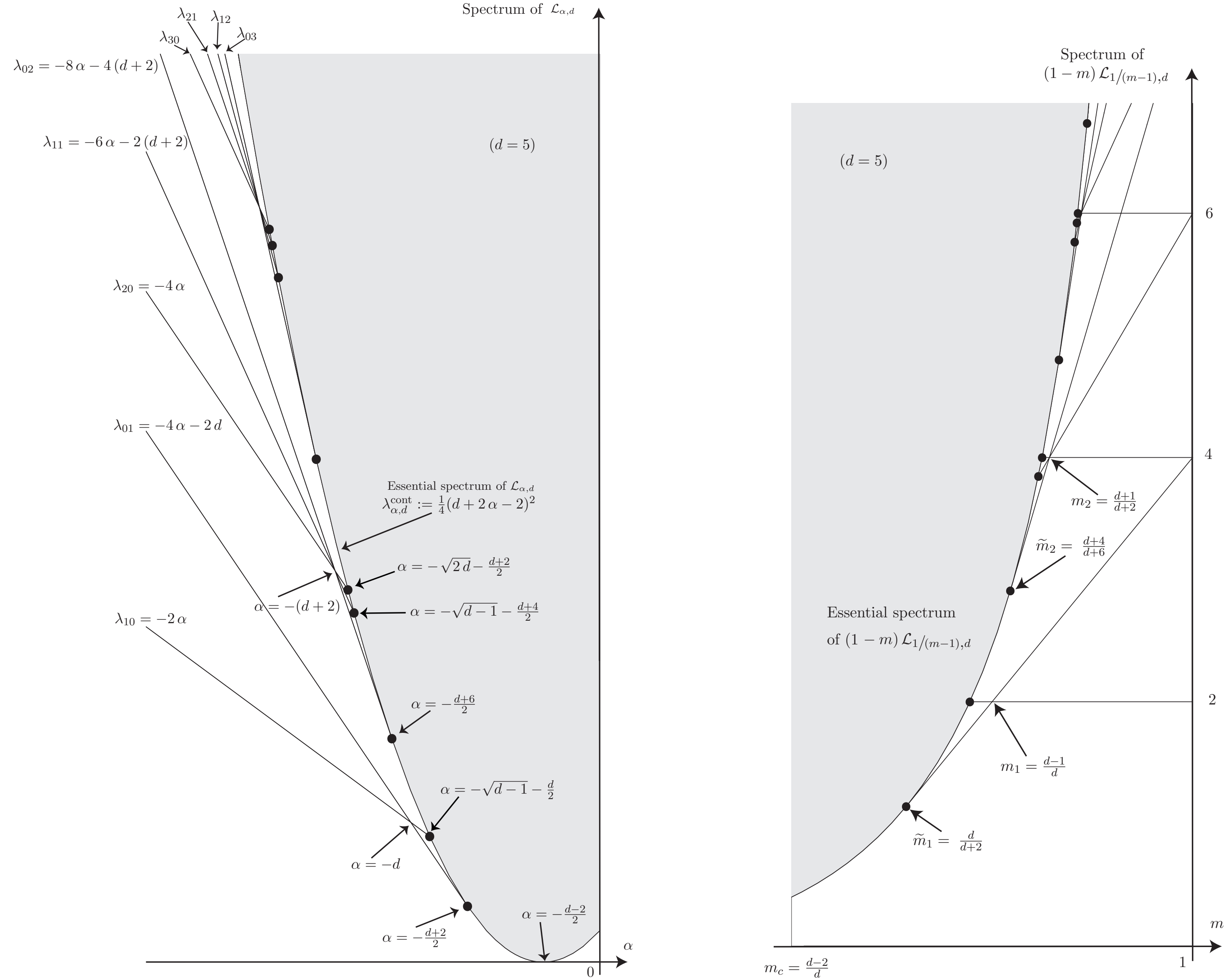
Hence we have a nonlinear differential inequality

$$\frac{d}{dt}\mathcal{F}[w(t,\cdot)] \leq -2\,\frac{\Lambda_{\alpha,d} - m\,Y(h)}{\big[1 + X(h)\big]\,h^{2-m}}\,\mathcal{F}[w(t,\cdot)]$$

- A **Gronwall** lemma (take $h = 1 + \mathsf{C}\,\mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$) then shows that

$$\limsup_{t\to\infty} e^{\,2\,\Lambda_{\alpha,d}\,t}\,\mathcal{F}[w(t,\cdot)] < +\infty$$

Plots $(d = 5)$



Spectrum of $\mathcal{L}_{\alpha,d}$
λ_{21}
λ_{12}
λ_{30}
λ_{03}
$\lambda_{02} = -8\,\alpha - 4\,(d+2)$
$\lambda_{11} = -6\,\alpha - 2\,(d+2)$
$(d = 5)$
$\lambda_{20} = -4\,\alpha$
$\lambda_{01} = -4\,\alpha - 2\,d$
Essential spectrum of $\mathcal{L}_{\alpha,d}$
$\lambda^{\rm cont}_{\alpha,d} := \frac{1}{4}(d + 2\,\alpha - 2)^2$
$\alpha = -\sqrt{2\,d} - \frac{d+2}{2}$
$\alpha = -(d+2)$
$\alpha = -\sqrt{d-1} - \frac{d+4}{2}$
$\lambda_{10} = -2\,\alpha$
$\alpha = -\frac{d+6}{2}$
$\alpha = -\sqrt{d-1} - \frac{d}{2}$
$\alpha = -d$
$\alpha = -\frac{d-2}{2}$
$\alpha = -\frac{d+2}{2}$
α
0
Spectrum of
$(1-m)\,\mathcal{L}_{1/(m-1),d}$
$(d = 5)$
6
4
$m_2 = \frac{d+1}{d+2}$
$\widetilde{m}_2 = \frac{d+4}{d+6}$
Essential spectrum
of $(1-m)\,\mathcal{L}_{1/(m-1),d}$
2
$m_1 = \frac{d-1}{d}$
$\widetilde{m}_1 = \frac{d}{d+2}$
m
1
$m_c = \frac{d-2}{d}$

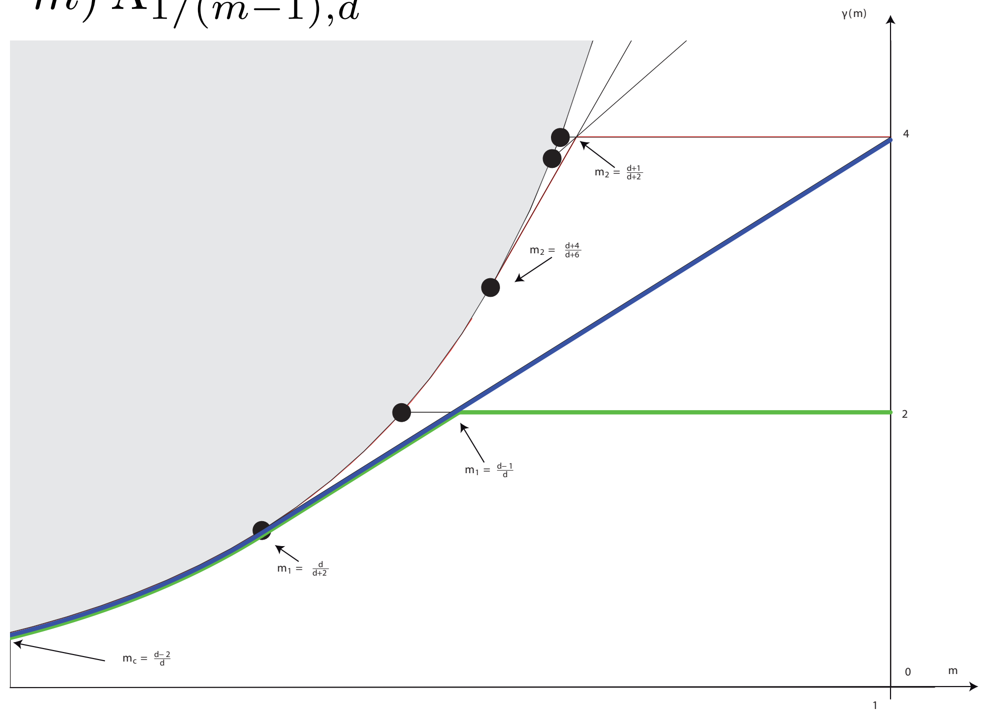
Remarks, improvements

- Optimal constants in interpolation inequalities does not mean optimal *asymptotic* rates
- The critical case ($m = m_*$, $d \geq 3$): **Slow asymptotics** [Bonforte, Grillo, Vázquez] If $|v_0 - V_D|$ is bounded a.e. by a radial $L^1(dx)$ function, then there exists a positive constant C^* such that $\mathcal{E}[v(t,\cdot)] \leq C^*\, t^{-1/2}$ for any $t \geq 0$
- Can we improve the rates of convergence by imposing restrictions on the initial data ?
  - [Carrillo, Lederman, Markowich, Toscani (2002)] Poincaré inequalities for linearizations of very fast diffusion equations (radially symmetric solutions)
  - Formal or partial results: [Denzler, McCann (2005)], [McCann, Slepčev (2006)], [Denzler, Koch, McCann (announcement)],
- Faster convergence ?
  - Improved Hardy-Poincaré inequality: under the conditions $\int_{\mathbb{R}^d} f\, d\mu_{\alpha-1} = 0$ and $\int_{\mathbb{R}^d} x\, f\, d\mu_{\alpha-1} = 0$ (center of mas),
    $$\widetilde{\Lambda}_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha$$
  - Next ? Can we kill other linear modes ?

[Bonforte, J.D., Grillo, Vázquez] Assume that $m \in (m_1, 1)$, $d \geq 3$. Under Assumption (H1), if v is a solution of (2) with initial datum v_0 such that $\int_{\mathbb{R}^d} x\, v_0\, dx = 0$ and if D is chosen so that $\int_{\mathbb{R}^d} (v_0 - V_D)\, dx = 0$, then

$$\mathcal{E}[v(t,\cdot)] \leq \widetilde{C}\, \mathrm{e}^{-\gamma(m)\, t} \quad \forall\, t \geq 0$$

with $\gamma(m) = (1-m)\, \widetilde{\Lambda}_{1/(m-1),d}$



Higher order matching asymptotics

For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1} \right) = 0$$

The strategy is easy to understand using a time-dependent rescaling and the relative entropy formalism. Define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t, x) \;, \quad R = R(\tau) \;, \quad t = \tfrac{1}{2} \log R \;, \quad x = \frac{y}{R}$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \, \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2\, x \right) \right] = 0 \quad t > 0 \;, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on R)

$$2\, \sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \tfrac{1}{\sigma}\,|x|^2 \right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d \tag{3}$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1}\,(v - B_\sigma) \right] dx$$

Let us briefly sketch the strategy of our method before giving all details

The time derivative of this relative entropy is

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt} \left(\frac{d}{d\sigma}\,\mathcal{F}_\sigma[v] \right)_{|\sigma=\sigma(t)}}_{\text{choose it } = 0} + \frac{m}{m-1} \int_{\mathbb{R}^d} \left(v^{m-1} - B_{\sigma(t)}^{m-1} \right) \frac{\partial v}{\partial t}\, dx$$

$$\Longleftrightarrow$$

$$\text{Minimize } \mathcal{F}_\sigma[v] \text{ w.r.t. } \sigma \iff \int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx = \int_{\mathbb{R}^d} |x|^2\, v\, dx \tag{4}$$

Second step: the entropy / entropy production estimate

According to the definition of B_σ, we know that $2\,x = \sigma^{\frac{d}{2}(m-m_c)}\,\nabla B_\sigma^{m-1}$
Using the new change of variables, we know that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} v\left|\nabla\left[v^{m-1} - B_{\sigma(t)}^{m-1}\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[v] = \frac{m}{1-m}\int_{\mathbb{R}^d}\left[w - 1 - \frac{1}{m}\left(w^m - 1\right)\right] B_\sigma^m\,dx$$

(Repeating) define the *relative Fisher information* by

$$\mathcal{I}_\sigma[v] := \int_{\mathbb{R}^d}\left|\frac{1}{m-1}\nabla\left[\left(w^{m-1}-1\right)B_\sigma^{m-1}\right]\right|^2 B_\sigma\,w\,dx$$

so that $\quad \frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\,m\,(1-m)\,\sigma(t)\,\mathcal{I}_{\sigma(t)}[v(t,\cdot)] \quad \forall\, t>0$

When linearizing, one more mode is killed and $\sigma(t)$ scales out

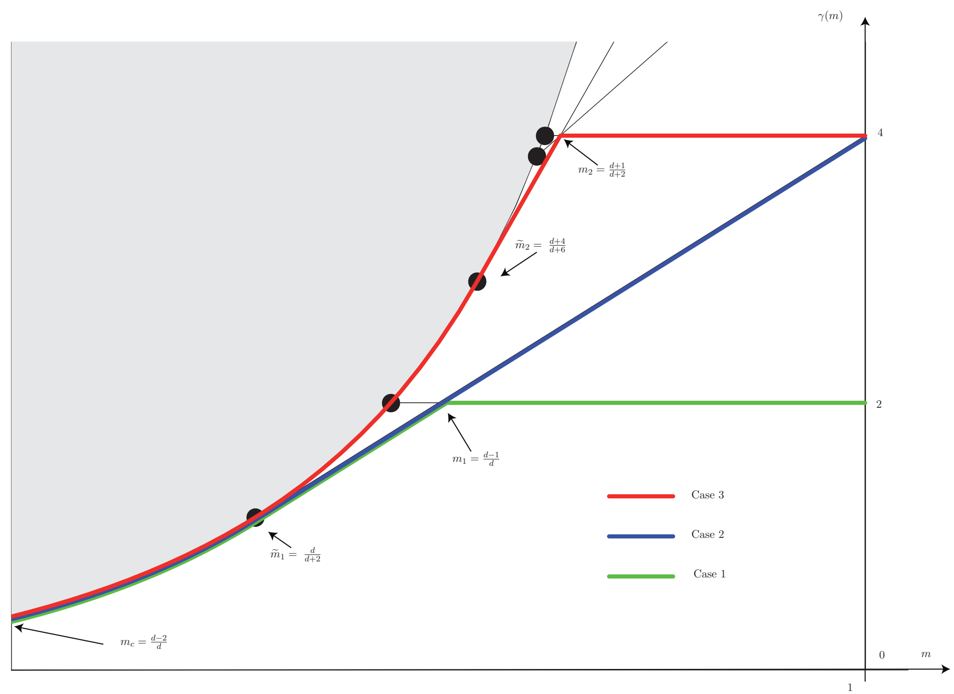
Improved rates of convergence

Theorem 7. *Let* $m \in (\widetilde{m}_1, 1)$, $d \geq 2$, $v_0 \in L^1_+(\mathbb{R}^d)$ *such that* v_0^m, $|y|^2\, v_0 \in L^1(\mathbb{R}^d)$

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,\gamma(m)\,t} \quad \forall\, t \geq 0$$

where

$$\gamma(m) = \begin{cases} \frac{((d-2)\,m-(d-4))^2}{4\,(1-m)} & \text{if } m \in (\widetilde{m}_1, \widetilde{m}_2] \\ 4\,(d+2)\,m - 4\,d & \text{if } m \in [\widetilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$



[Denzler, Koch, McCann], in progress

More about homogeneity

Algebraic rates vs. exponential decay

[J. Carrillo, J.D. , I. Gentil, A. Jüngel]

Consider the one dimensional porous medium/fast diffusion equation

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \,, \quad x \in S^1 \,, \quad t > 0 \quad \text{with } u(\cdot, t=0) = u_0 \geq 0$$

- The method also applies to the thin film equation $u_t = -(u^m \, u_{xxx})_x$ the Derrida-Lebowitz-Speer-Spohn (DLSS) equation $u_t = -(u\,(\log u)_{xx})_{xx}$
- Some references: [Cáceres, Carrillo, Toscani], [Gualdani, Jüngel, Toscani], [Jüngel, Matthes], [Laugesen]

More entropies ? $p \in (0, +\infty)$, $q \in \mathbb{R}$, $v \in H^1_+(S^1)$, $\mu_p[v] := \left(\int_{S^1} v^{1/p} \, dx\right)^p$

$$\Sigma_{p,q}[v] := \frac{1}{p\,q\,(p\,q-1)} \left[\, \int_{S^1} v^q \, dx - (\mu_p[v])^q \,\right] \quad \text{if } p\,q \neq 1 \text{ and } q \neq 0$$

$$\Sigma_{1/q,q}[v] := \int_{S^1} v^q \, \log\left(\frac{v^q}{\int_{S^1} v^q \, dx}\right) dx \quad \text{if } p\,q = 1 \text{ and } q \neq 0$$

$$\Sigma_{p,0}[v] := -\frac{1}{p} \int_{S^1} \log\left(\frac{v}{\mu_p[v]}\right) dx \quad \text{if } q = 0$$

Functional inequalities

- $\Sigma_{p,q}[v]$ is non-negative by convexity of $u \mapsto \frac{u^{p\,q}-1-p\,q\,(u-1)}{p\,q\,(p\,q-1)}$

Proposition 8. *Global functional inequalities: For all $p \in (0,+\infty)$ and $q \in (0,2)$, there exists a positive constant $\kappa_{p,q}$ such that, for any $v \in H^1_+(S^1)$,*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{\kappa_{p,q}} \int_{S^1} |v'|^2 \, dx$$

Small entropies regime: For any $p > 0$, $q \in \mathbb{R}$ and $\varepsilon_0 > 0$, there exists a positive constant C such that, for any $\varepsilon \in (0, \varepsilon_0]$, if $v \in H^1_+(S^1)$ is such that $\Sigma_{p,q}[v] \leq \varepsilon$ and $\mu_p[v] = 1$

$$\Sigma_{p,q}[v] \leq \frac{1 + C\sqrt{\varepsilon}}{8\,p^2\,\pi^2} \int_{S^1} |v'|^2 \, dx$$

Application to porous media: convergence rates

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \quad x \in S^1, \; t > 0$$

With $v := u^p$, $p := \frac{m+k}{2}$, $q := \frac{k+1}{p} = 2\,\frac{k+1}{m+k}$, let $\mathcal{E}[u] := \Sigma_{p,q}[v]$

$$\mathcal{E}[u] = \begin{cases} \frac{1}{k\,(k+1)} \int_{S^1} \left(u^{k+1} - \bar{u}^{k+1}\right)\, dx & \text{if} \quad k \in \mathbb{R} \setminus \{-1, 0\} \\ \int_{S^1} u \log\left(\frac{u}{\bar{u}}\right)\, dx & \text{if} \quad k = 0 \\ -\int_{S^1} \log\left(\frac{u}{\bar{u}}\right)\, dx & \text{if} \quad k = -1 \end{cases}$$

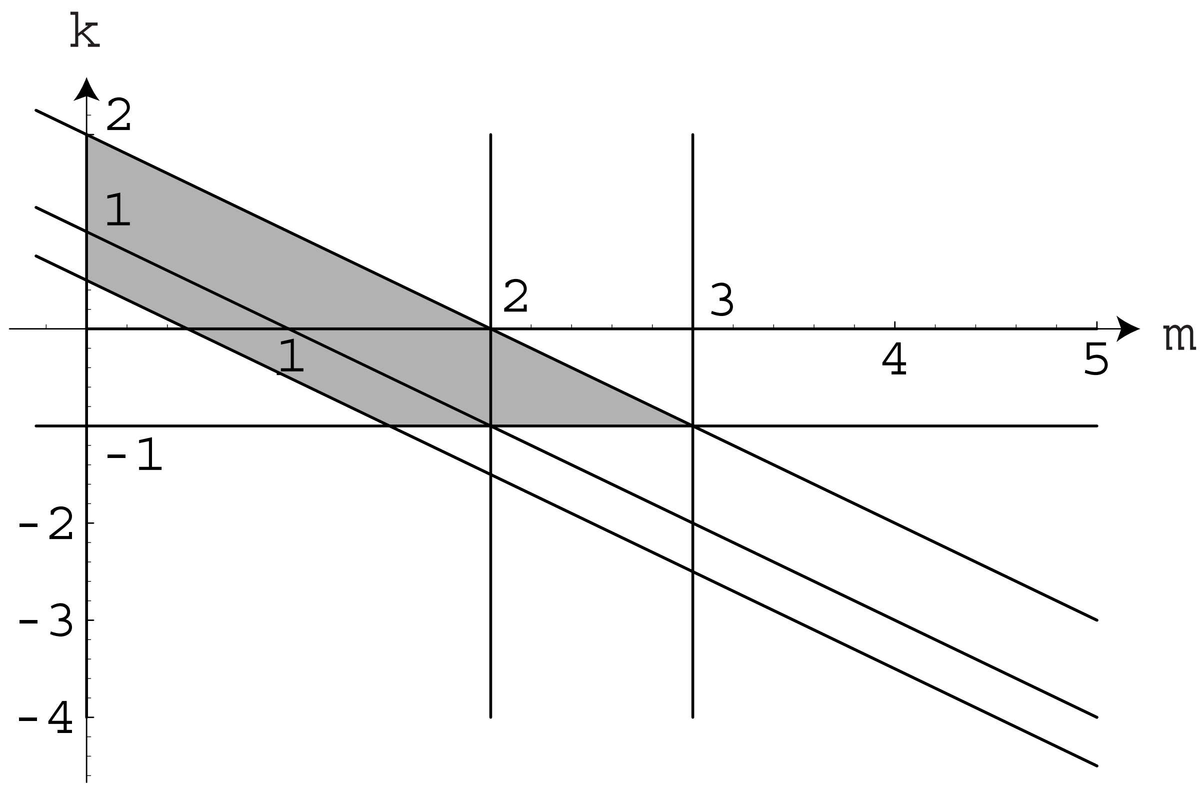
Proposition 9. *Let $m \in (0, +\infty)$, $k \in \mathbb{R} \setminus \{-m\}$, $q = 2\,(k+1)/(m+k)$, $p = (m+k)/2$ and u be a smooth positive solution*

i) Short-time Algebraic Decay: If $m > 1$ and $k > -1$, then

$$\mathcal{E}[u(\cdot,t)] \leq \left[\mathcal{E}[u_0]^{-(2-q)/q} + \frac{2-q}{q}\,\lambda\,\kappa_{p,q}\,t\right]^{-q/(2-q)}$$

ii) Asymptotically Exponential Decay: If $m > 0$ and $m + k > 0$, there exists $C > 0$ and $t_1 > 0$ such that for $t \geq t_1$,

$$\mathcal{E}[u(\cdot,t)] \leq \mathcal{E}[u(\cdot,t_1)]\,\exp\left(-\frac{8\,p^2\,\pi^2\,\lambda\,\bar{u}^{p(2-q)}\,(t-t_1)}{1 + C\sqrt{\mathcal{E}[u(\cdot,t_1)]}}\right)$$


k
2
1
2
3
m
1
4
5
-1
-2
-3
-4

The Bakry-Emery method revisited

[J.D., B. Nazaret, G. Savaré]

Consider a domain $\Omega \subset \mathbb{R}^d$, $d\gamma = g\,dx$, $g = e^{-F}$ and a generalized *Ornstein-Uhlenbeck operator:* $\Delta_g v := \Delta v - \mathrm{D}F \cdot \mathrm{D}v$

$$v_t = \Delta_g v \quad x \in \Omega\,,\ t \in \mathbb{R}^+$$
$$\nabla v \cdot n = 0 \quad x \in \partial\Omega\,,\ t \in \mathbb{R}^+$$

With $s := v^{p/2}$ and $\alpha := (2-p)/p$, $p \in (1,2]$

$$\mathcal{E}_p(t) := \frac{1}{p-1} \int_\Omega \Big[v^p - 1 - p\,(v-1) \Big]\, d\gamma$$
$$\mathcal{I}_p(t) := \tfrac{4}{p} \int_\Omega |\mathrm{D}s|^2\, d\gamma$$
$$\mathcal{K}_p(t) := \int_\Omega |\Delta_g s|^2\, d\gamma + \alpha \int_\Omega \Delta_g s\, \tfrac{|\mathrm{D}s|^2}{s}\, d\gamma$$

A simple computation shows that

$$\frac{d}{dt}\mathcal{E}_p(t) = -\mathcal{I}_p(t) \quad \text{and} \quad \frac{d}{dt}\mathcal{I}_p(t) = -\frac{8}{p}\,\mathcal{K}_p(t)$$

An extension of the criterion of Bakry-Emery

Using the commutation relation $[\mathrm{D}, \Delta_g]\, s = -\mathrm{D}^2 F\, \mathrm{D}s$, we get

$$\int_\Omega (\Delta_g s)^2\, d\gamma = \int_\Omega |\mathrm{D}^2 s|^2\, d\gamma + \int_\Omega \mathrm{D}^2 F\, \mathrm{D}s \cdot \mathrm{D}s\, d\gamma - \underbrace{\sum_{i,j=1}^{d} \int_{\partial\Omega} \partial^2_{ij} s\, \partial_i s\, n_j\, g\, d\mathcal{H}^{d-1}}_{\geq 0 \text{ if } \Omega \text{ is convex}}$$

$$\mathcal{K}_p = \int_\Omega |\Delta_g s|^2\, d\gamma + 4\,\alpha \int_\Omega \Delta_g s\, |\mathrm{D}z|^2\, d\gamma \geq (1-\alpha) \int_\Omega |\mathrm{D}^2 s|^2\, d\gamma + \int_\Omega V\, |\mathrm{D}s|^2\, d\gamma$$

$$\text{with } V(x) := \inf_{\xi \in S^{d-1}} \left(\mathrm{D}^2 F(x)\, \xi, \xi\right)$$

Theorem 10. *Let $F \in C^2(\Omega)$, $\gamma = e^{-F} \in L^1(\Omega)$, and Ω be a convex domain in $\mathbb{R}^d$.*

If $\lambda_1(p) := \inf \dfrac{\int_\Omega \left(2\, \frac{p-1}{p}\, |\mathrm{D}w|^2 + V\, |w|^2\right) d\gamma}{\int_\Omega |w|^2\, d\gamma}$ *is positive, then*

$$\mathcal{I}_p(t) \leq \mathcal{I}_p(0)\, e^{-2\,\lambda_1(p)\, t}$$

$$\mathcal{E}_p(t) \leq \mathcal{E}_p(0)\, e^{-2\,\lambda_1(p)\, t}$$

Generalized entropies

Consider the weighted porous media equation

$$v_t = \Delta_g v^m$$

$d\gamma$ is a probability measure, $p \in (1,2)$

$$\begin{aligned}
\mathcal{E}_{m,p}(t) &:= \frac{1}{m+p-2} \int_\Omega \left[v^{m+p-1} - 1 \right] d\gamma \\
\mathcal{I}_{m,p}(t) &:= c(m,p) \int_\Omega |\mathrm{D}s|^2 \, d\gamma \\
\mathcal{K}_{m,p}(t) &:= \int_\Omega s^{\beta(m-1)} \, |\Delta_g s|^2 \, d\gamma + \alpha \int_\Omega s^{\beta(m-1)} \, \Delta_g s \, \frac{|\mathrm{D}s|^2}{s} \, d\gamma
\end{aligned}$$

with $v =: s^\beta$, $\beta := \frac{1}{p/2+m-1}$, $\alpha := \frac{2-p}{p+2(m-1)}$ and $c(m,p) = \frac{4\,m\,(m+p-1)}{(2m+p-2)^2}$

Adapting the Bakry-Emery method...

Written in terms of $s = v^{1/\beta}$, the evolution is governed by

$$\frac{1}{m}\, s_t = s^{\beta(m-1)} \left[\Delta_g s + \alpha\, \frac{|\mathrm{D}s|^2}{s} \right]$$

A computation shows that

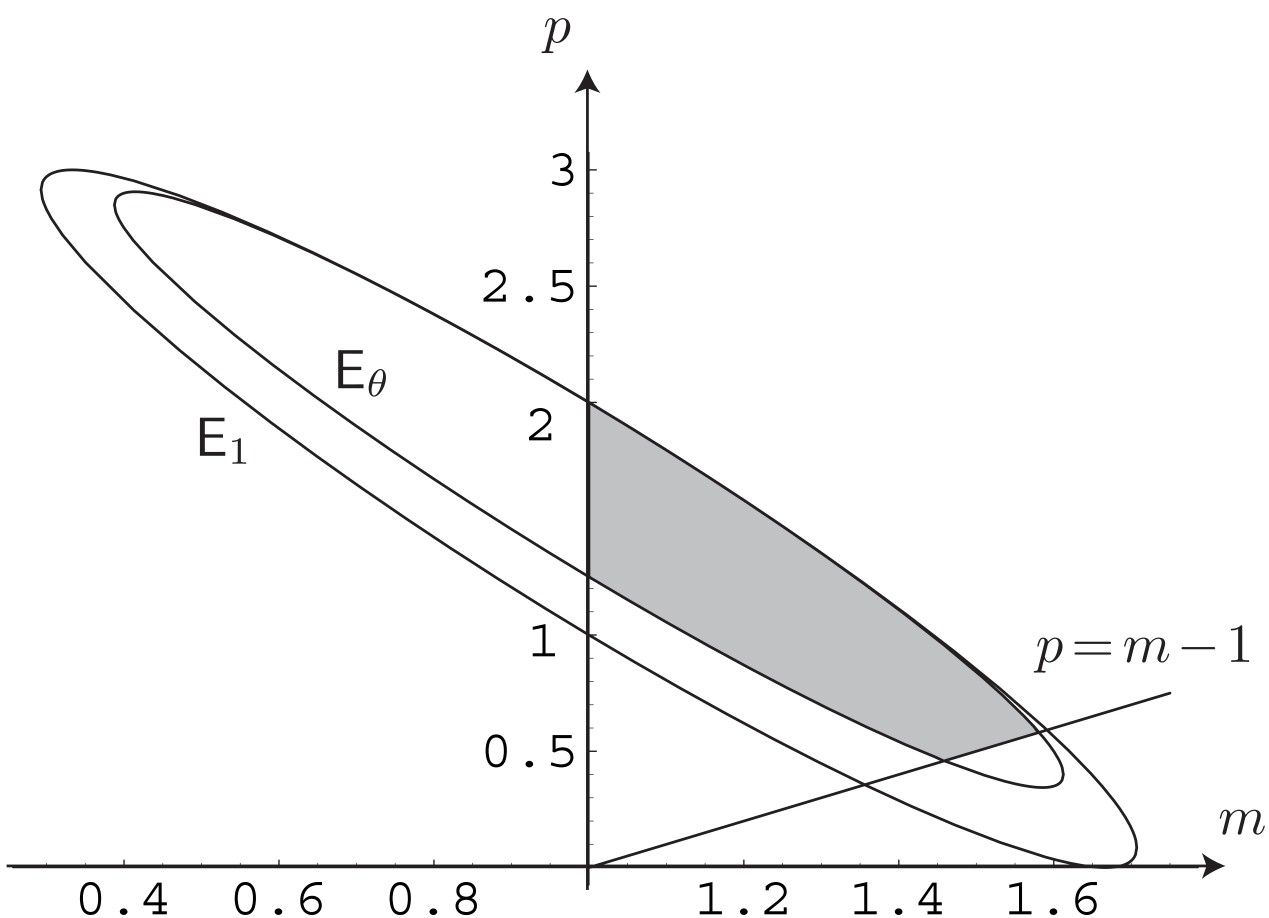
$$\begin{aligned} \frac{d}{dt}\mathcal{E}_{m,p}(t) &:= -\mathcal{I}_{m,p}(t) \\ \frac{1}{m}\,\frac{d}{dt}\mathcal{I}_{m,p}(t) &:= -2\,c(m,p)\,\mathcal{K}_{m,p}(t) \end{aligned}$$

Exactly as in the linear case, define for any $\theta \in (0,1)$

$$\lambda_1(m,\theta) := \inf_{w \in H^1(\Omega, d\gamma)\setminus\{0\}} \frac{\int_\Omega \Big((1-\theta)\,|\mathrm{D}w|^2 + V\,|w|^2 \Big)\, d\gamma}{\int_\Omega |w|^2\, d\gamma}$$

The non-local condition

Assume that for some $\theta \in (0,1)$, $\lambda_1(m,\theta) > 0$. Admissible parameters m and p correspond to $(m,p) \in \mathsf{E}_\theta$, $1 < m < p+1$, where the set E_θ is a portion of an ellipse (grey area)



Results for the porous media equation

Lemma 11. *With the above notations, if* Ω *is convex and* $(m,p) \in \mathsf{E}_\theta$ *are admissible, then*

$$\mathcal{I}_{m,p}^{\frac{4}{3}} \leq \frac{1}{3}\,[4\,c(m,p)]^{\frac{4}{3}}\mathsf{K}^{\frac{1}{3}}\left[(m+p-2)\,\mathcal{E}_{m,p}+1\right]^{\frac{4-3q}{3(2-q)}}\mathcal{K}_{m,p}$$

Theorem 12. *Under the above conditions there exists a positive constant* κ *which depends on* $\mathcal{E}_{m,p}(0)$ *such that any smooth solution* u *of the porous media equation satisfies, for any* $t>0$

$$\mathcal{I}_{m,p}(t) \leq \frac{\mathcal{I}_{m,p}(0)}{\left[1+\frac{\kappa}{3}\,\sqrt[3]{\mathcal{I}_{m,p}(0)}\,t\right]^3}$$

$$\mathcal{E}_{m,p}(t) \leq \frac{3\,\left[\mathcal{I}_{m,p}(0)\right]^{\frac{8}{3}}}{2\,\kappa\left[1+\frac{\kappa}{3}\,\sqrt[3]{\mathcal{I}_{m,p}(0)}\,t\right]^2}$$

The gradient flow interpretation

[J.D., B. Nazaret, G. Savaré]

As in the Bakry-Emery approach (linear case) let $p \in (1,2]$ and define

$$\mathcal{E}_p[v] := \frac{1}{p-1} \int_{\mathbb{R}^d} \Big[v^p - 1 - p\,(v-1) \Big]\, d\gamma \;, \quad \mathcal{E}_1[v] := \int_{\mathbb{R}^d} \Big[v\, \log v - \;(v-1) \Big]\, d\gamma$$

with $\Omega = \mathbb{R}^d$ and $d\gamma(x) = (2\,\pi)^{-d/2}\, e^{-|x|^2/2}\, dx$ (gaussian measure). Along the flow associated to the Ornstein-Uhlenbeck equation

$$v_t = \Delta v - x \cdot \nabla v$$

we have found that $\mathcal{E}_p[v(t,\cdot)] \le \mathcal{E}_p[v_0]\, e^{-2\,t}$, which amounts to the **generalized Poincaré inequality** [Beckner]

$$\mathcal{E}_p[v] \le \frac{p}{2} \int_{\mathbb{R}^d} v^{p-2}\, |\nabla v|^2\, d\gamma \quad \text{or} \quad \frac{\int_{\mathbb{R}^d} f^2\, d\gamma - \left(\int_{\mathbb{R}^d} f^q\, d\gamma \right)^{2/q}}{q-2} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\gamma$$

(take $q = 2/p$, $v^p = f^2$, $q \in [1,2)$)

Why do we have such a choice in the linear case ?

Wasserstein distances

$p > 1$, μ_0 and μ_1 probability measures on $\mathbb{R}^d$

- Transport plans between μ_0 and μ_1 : $\Gamma(\mu_0, \mu_1)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ having μ_0 and μ_1 as marginals.
- Wasserstein distance between μ_0 ans μ_1

$$W^2(\mu_0, \mu_1) = \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\Sigma(x, y) \ : \ \Sigma \in \Gamma(\mu_0, \mu_1) \right\}$$

- The Benamou-Brenier characterization (2000)

$$W^2(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} |\mathbf{v}_t|^2 \rho_t \, dx \, dt \ : \ (\rho_t, \mathbf{v}_t)_{t \in [0,1]} \text{ admissible} \right\}$$

where admissible paths $(\rho_t, \mathbf{v}_t)_{t \in [0,1]}$ are such that

$$\partial_t \rho_t + \nabla \cdot (\rho_t \mathbf{v}_t) = 0 \ , \quad \rho_0 = \mu_0 \ , \quad \rho_1 = \mu_1$$

Gradient flows

- [Jordan, Kinderlehrer, Otto 98] : Formal Riemannian structure on $\mathcal{P}(\mathbb{R}^d)$: the McCann interpolant is a geodesic. For an integral functional such as

$$\mathcal{E}_1[\rho] := \int_{\mathbb{R}^d} F(\rho(x))\, dx$$

the gradient flow of $\mathcal{E}_1$ w.r.t. $W = W_1$ is

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\rho \, \nabla \left(F'(\rho) \right) \right]$$

- [Ambrosio, Gigli, Savaré 05] : Rigorous framework for JKO's calculus in the framework of length spaces (based on the optimal transportation)
- [Otto, Westdickenberg 05] : Use the Brenier-Benamou formulation to prove

$$W^2(\mu_0^t, \mu_1^t) \leq W^2(\mu_0, \mu_1)$$

along the heat flow on a compact Riemannian manifold

A generalization of the Benamou-Brenier approach

Given a function h on $\mathbb{R}^+$, define the admissible paths by

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (h(\rho_t)\, \mathbf{v}_t) = 0 \\ \rho_0 = \mu_0 \,, \quad \rho_1 = \mu_1 \end{cases}$$

and consider the distance

$$W_p^2(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} |\mathbf{v}_t|^2\, h_p(\rho_t)\, dx\, dt \;:\; (\rho_t, \mathbf{v}_t)_{t \in [0,1]} \text{ admissible} \right\}$$

$$h_p(\rho) = \rho^{2-p} \,, \quad 1 \le p \le 2$$

- $p = 1$: Wasserstein case
- $p = 2$: homogeneous Sobolev distance on $\dot{W}^{-1,2}$

$$\|\mu_1 - \mu_0\|_{\dot{W}^{-1,2}} = \sup \left\{ \int_{\mathbb{R}^d} \xi\, d(\mu_1 - \mu_0) \;:\; \xi \in \mathcal{C}_c^1(\mathbb{R}^d) \,, \; \int_{\mathbb{R}^d} |\nabla \xi|^2 \le 1 \right\}$$

The heat equation as a gradient flow w.r.t. W_p

Denote by S_t the semi-group associated to the heat equation. Let $p < \frac{d+2}{d}$ and consider the generalized entropy functional

$$\mathcal{E}_p[\mu] = \frac{1}{p\,(p-1)} \int_{\mathbb{R}^d} \rho^p(x)\,dx \quad \text{if } d\mu = \rho\,dx$$

Theorem 13. *If $\mu \in \mathcal{P}(\mathbb{R}^d)$, $\mathcal{E}_p[\mu] < +\infty$, then $\mathcal{E}_p[S_t\,\mu] < +\infty$ for all $t > 0$ and*

$$\frac{1}{2}\frac{d}{dt} W_p^2(S_t\,\mu, \sigma) + \mathcal{E}_p[S_t\,\mu] \leq \mathcal{E}_p[\sigma]$$

- Beckner inequalities w.r.t. gaussian weight
- The heat equation can be seen as the gradient flow of $\mathcal{E}_p$ w.r.t. W_p

From kinetic to diffusive models

We consider a distribution function $f = f(t, x, v)$ solving a non-homogeneous kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = Q(f)$$

- **Generalized entropies** *i.e.* *Free energies / entropies / energy under Casimir constraints* are useful to characterize special stationary states und prove their nonlinear stability: [Guo], [Rein], [Schaeffer], etc.
- Generalized entropies allow to prove existence, characterize large time attractors, take singular limits or quantify the rate of convergence towards an equilibrium
  - [J.D., Markowich, Ölz, Schmeiser]: **Diffusion limit** of a time-relaxation equation which has **polytropic states** as stationary solutions
  - [J.D., Mouhot, Schmeiser]: **a L^2 hypocoercivity theory**

... a diffusion limit related to fast diffusion equations

BGK models

- BGK model of gas dynamics

$$\partial_t f + v \cdot \nabla_x f = \frac{\rho(x,t)}{(2\,\pi\,T)^{n/2}} \exp\left(-\frac{|v-u(x,t)|^2}{2\,T(x,t)}\right) - f$$

  where $\rho(x,t)$ (position density), $u(x,t)$ (local mean velocity) and $T(x,t)$ (temperature) are chosen such that they equal the corresponding quantities associated to f

  [Perthame, Pulvirenti]: Weighted L^∞ bounds and uniqueness for the Boltzmann BGK model, 1993

- Linear BGK model in semiconductor physics

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \frac{\rho(x,t)}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}|v|^2\right) - f$$

  where $\rho(x,t) = \int_{\mathbb{R}^d} f(t,x,v)\,dv$ is the position density of f

  [Poupaud]: Mathematical theory of kinetic equations for transport modelling in semiconductors, 1994

- For applications in astrophysics, we are interested in collision kernels (BGK type / time-relaxation) such that stationary states are *polytropic equilibria*

$$f(x,v) = \left(\frac{1}{2}\,|v|^2 + V(x) - \mu\right)_+^k$$

Diffusion limit for a time-relaxation model for polytropes

$$\begin{aligned} \varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon - \varepsilon \nabla_x V(x) \cdot \nabla_v f^\varepsilon &= G_{f^\varepsilon} - f^\varepsilon \\ f^\varepsilon(x,v,t=0) &= f_I(x,v)\,, \quad x,v \in \mathbb{R}^3 \end{aligned}$$

$$\text{with Gibbs equilibrium} \quad G_f := \gamma\left(\frac{|v|^2}{2} + V(x) - \mu_{\rho_f}(x,t)\right)$$

The Fermi energy $\mu_{\rho_f}(x,t)$ is implicitly defined by

$$\int_{\mathbb{R}^3} \gamma\left(\frac{|v|^2}{2} + V(x) - \mu_{\rho_f}(x,t)\right) dv = \int_{\mathbb{R}^3} f(x,v,t)\, dv =: \rho_f(x,t)$$

$$\begin{aligned} f^\varepsilon(x,v,t) &\quad \ldots \quad \text{phase space particle density} \\ V(x) &\quad \ldots \quad \text{potential} \\ \varepsilon &\quad \ldots \quad \text{mean free path} \end{aligned}$$

- Formal expansions (generalized Smoluchowski equation): [Ben Abdallah, J.D.], [Chavanis, Laurençot, Lemou], [Chavanis et al.], [Degond, Ringhofer]
- Astrophysics: [Binney, Tremaine], [Guo, Rein], [Chavanis et al.]
- Fermi-Dirac statistics in semiconductors models: [Goudon, Poupaud]

Fast diffusion and porous media as a diffusion limit

Theorem 14. *Under assumptions on V and the initial data f_I, for any $\varepsilon > 0$, there is a unique weak solution $f^\varepsilon \in C(0,\infty; L^1 \cap L^p(\mathbb{R}^6))$ for all $p < \infty$ which converges to*

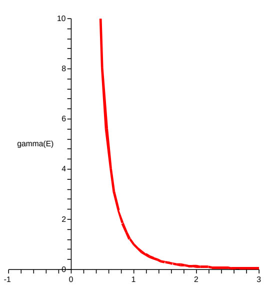
$$f^0(x,v,t) = \gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}(\rho(x,t))\right) \quad \text{with} \int_{\mathbb{R}^d} \gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}(\rho)\right) dv = \rho$$

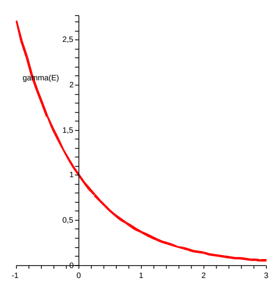
as $\varepsilon \to 0$, where ρ is a solution of the nonlinear diffusion equation

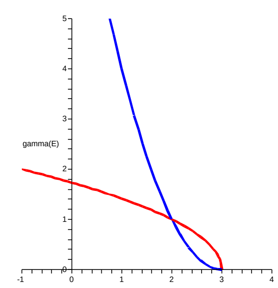
$$\partial_t \rho = \nabla_x \cdot (\nabla_x \nu(\rho) + \rho \nabla_x V(x)) \,, \quad \nu(\rho) = \int_0^\rho s\,\bar{\mu}'(s)\,ds$$

with initial data $\rho(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v)\,dv$

- Fast diffusion case: $\gamma(E) := D\,E_+^{-k}$, $D > 0$ and $k > 5/2$, $\nu(\rho) = \rho^{\frac{k-5/2}{k-3/2}}$
- Linear case: $\gamma(E) := D\,\exp(-E)$, $D > 0$, $\nu(\rho) = \rho$
- Porous medium case: $\gamma(E) = D\,(-E)_+^k$, $D > 0$ and $k > 0$, $\nu(\rho) = \rho^{\frac{k+5/2}{k+3/2}}$







The relative entropy...

... or **free energy functional:** with $\beta(s) = \int_s^0 \gamma^{-1}(\sigma)\, d\sigma$ convex (γ monotone decreasing), we get

$$\mathcal{F}[f] := \iint_{\mathbb{R}^2\times\mathbb{R}^2} \left[f \left(\frac{1}{2}\,|v|^2 + V \right) + \beta(f) \right] dx\, dv$$

is such that

$$\frac{d}{dt}\mathcal{F}[f(t,\cdot,\cdot)] := \iint_{\mathbb{R}^2\times\mathbb{R}^2} \Big(G_f - f \Big)\Big(\gamma^{-1}(G_f) - \gamma^{-1}(f) \Big)\, dx\, dv \le 0$$

If $f = G_\rho$ is a local Gibbs state, we can define a **reduced free energy** by $\mathcal{F}[G_\rho] = \mathcal{G}[\rho]$ with $G_\rho(x,v) := \gamma\left(\frac{1}{2}\,|v|^2 + \bar{\mu}(\rho)\right)$:

$$\mathcal{G}[\rho] = \int_{\mathbb{R}^2} [h(\rho) + V\,\rho]\; dx \quad \text{with} \quad \rho\, h''(\rho) = \nu'(\rho)$$

Polytropes: $h(\rho) = \frac{1}{m-1}\,\rho^m$

Hypocoercivity

- The goal is to understand the *rate of relaxation* of the solutions of a kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \mathsf{L} f$$

towards a global equilibrium when the collision term acts only on the velocity space. Here $f = f(t, x, v)$ is the distribution function. It can be seen as a probability distribution on the phase space, where x is the position and v the velocity. However, since we are in a linear framework, the fact that f has a constant sign plays no role.

- A key feature of our approach [J.D., Mouhot, Schmeiser] is that it distinguishes the mechanisms of relaxation at *microscopic level* (convergence towards a local equilibrium, in velocity space) and *macroscopic level* (convergence of the spatial density to a steady state), where the rate is given by a spectral gap which has to do with the underlying diffusion equation for the spatial density

A very brief review of the literature

- Non constructive decay results: [Ukai (1974)] [Desvillettes (1990)]
- Explicit $t^{-\infty}$-decay, no spectral gap: [Desvillettes, Villani (2001-05)], [Fellner, Miljanovic, Neumann, Schmeiser (2004)], [Cáceres, Carrillo, Goudon (2003)]
- *hypoelliptic theory*:
  [Hérau, Nier (2004)]: spectral analysis of the Vlasov-Fokker-Planck equation
  [Hérau (2006)]: linear Boltzmann relaxation operator
  [Pravda-Starov], [Hérau, Pravda-Starov]
- Hypoelliptic theory vs. *hypocoercivity* (Gallay) approach and generalized entropies:
  [Mouhot, Neumann (2006)], [Villani (2007, 2008)]
- Other related approaches: non-linear Boltzmann and Landau equations:
  micro-macro decomposition: [Guo]
  hydrodynamic limits (fluid-kinetic decomposition): [Yu]

A toy problem

$$\frac{du}{dt} = (L - T)\, u\,, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad k^2 \geq \Lambda > 0$$

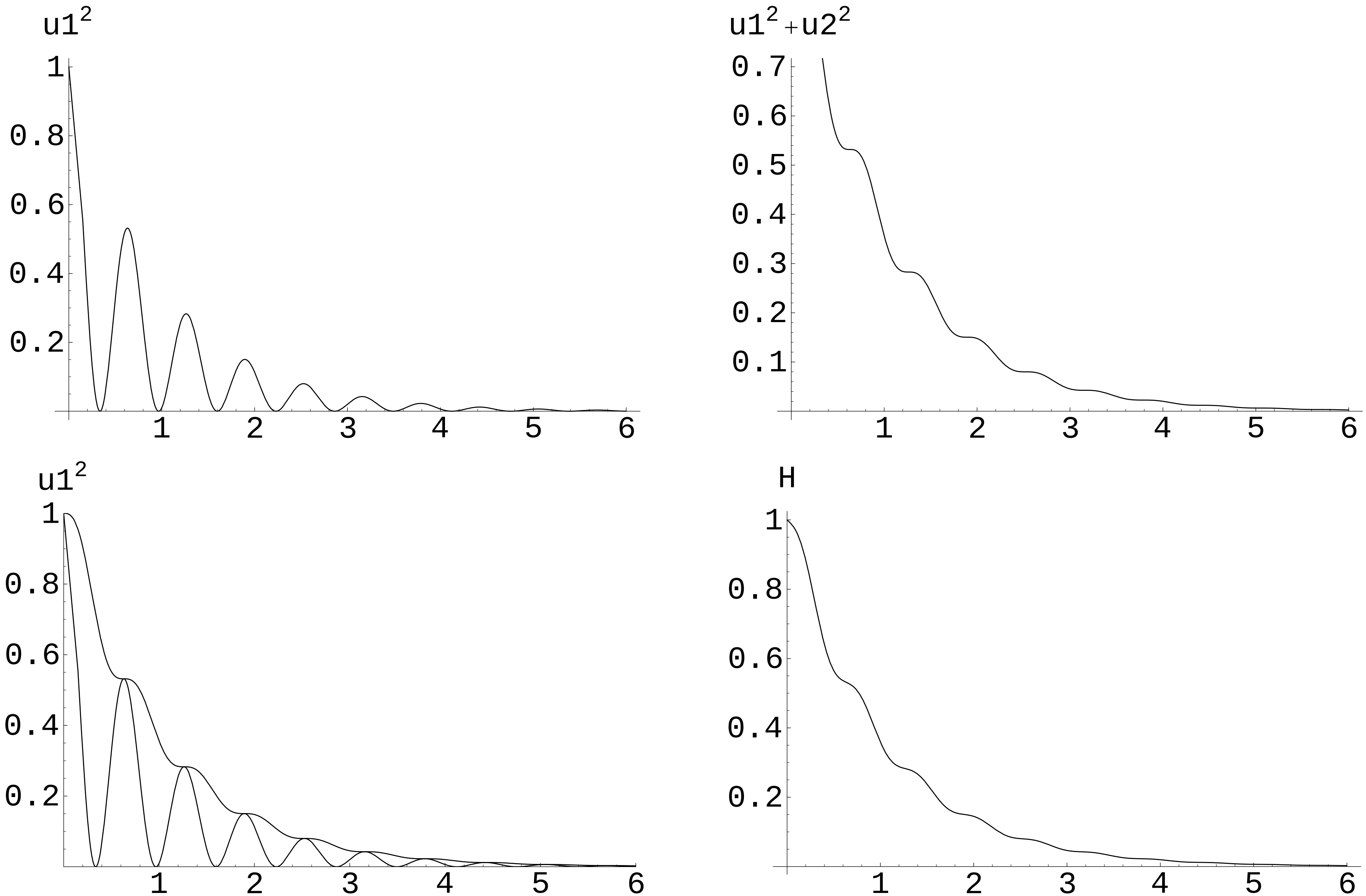
Nonmonotone decay, reminiscent of [Filbet, Mouhot, Pareschi (2006)]

- H-theorem: $\frac{d}{dt}|u|^2 = -2\, u_2^2$
- macroscopic limit: $\frac{du_1}{dt} = -k^2\, u_1$
- generalized entropy: $H(u) = |u|^2 - \frac{\varepsilon\, k}{1+k^2}\, u_1\, u_2$

$$(1-\varepsilon)\,|u|^2 \leq H(u) \leq (1-\varepsilon)\,|u|^2$$

$$\begin{aligned} \frac{dH}{dt} &= -\left(2 - \frac{\varepsilon\, k^2}{1+k^2}\right) u_2^2 - \frac{\varepsilon\, k^2}{1+k^2}\, u_1^2 + \frac{\varepsilon\, k}{1+k^2}\, u_1\, u_2 \\ &\leq -(2-\varepsilon)\, u_2^2 - \frac{\varepsilon \Lambda}{1+\Lambda}\, u_1^2 + \frac{\varepsilon}{2}\, u_1 u_2 \end{aligned}$$

Plots for the toy problem



...compared to plots for the Boltzmann equation

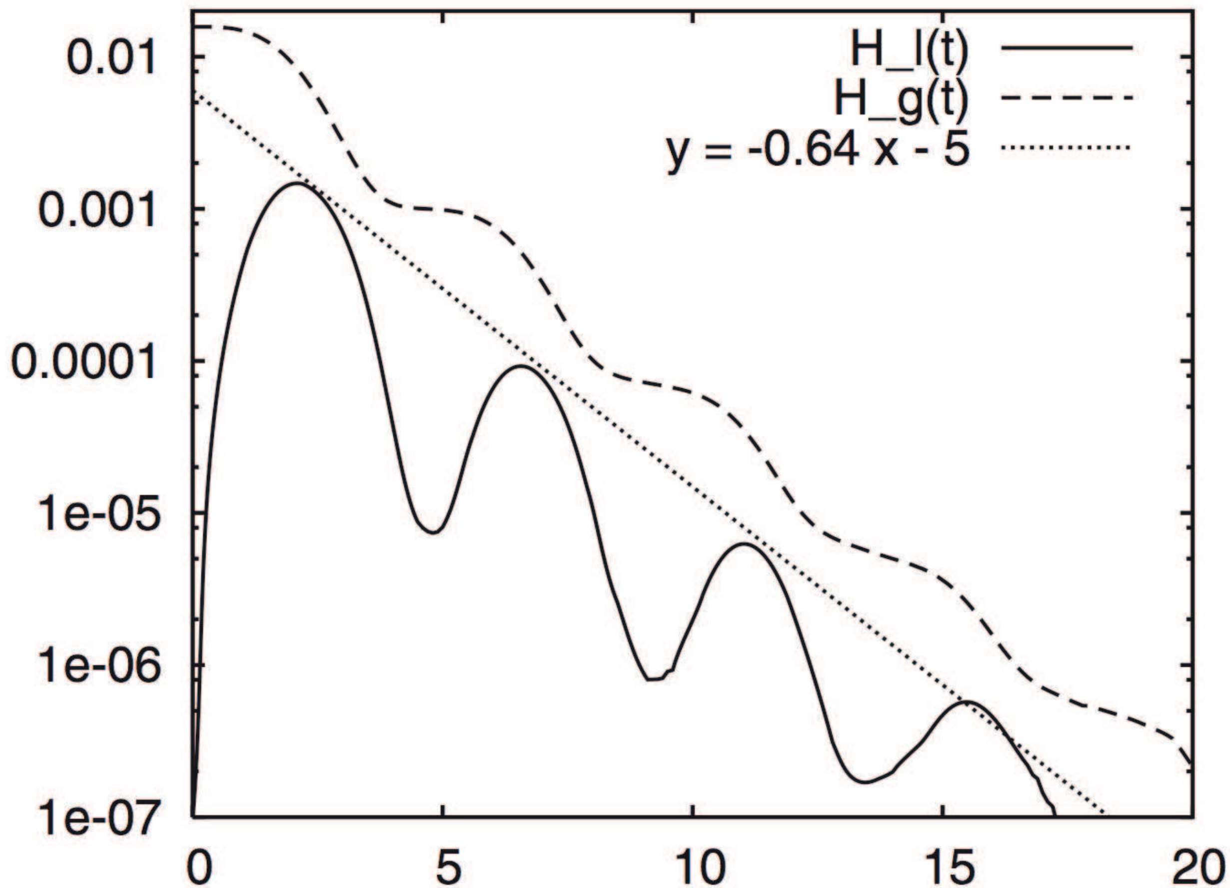


Figure 1: [Filbet, Mouhot, Pareschi (2006)]

The kinetic equation

$$\partial_t f + \mathsf{T}\, f = \mathsf{L}\, f\,, \quad f = f(t,x,v)\,,\ t>0,\ \ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d \tag{5}$$

- L is a linear collision operator
- V is a given *external potential* on $\mathbb{R}^d$, $d \geq 1$
- $\mathsf{T} := v\cdot\nabla_x - \nabla_x V\cdot\nabla_v$ is a transport operator

There exists a scalar product $\langle\cdot,\cdot\rangle$, such that L is symmetric and T is antisymmetric

$$\frac{d}{dt}\,\|f-F\|^2 = -2\,\|\mathsf{L}\,f\|^2$$

... seems to imply that the decay stops when $f\in\mathcal{N}(L)$
but we expect $f\to F$ as $t\to\infty$ since F generates $\mathcal{N}(L)\cap\mathcal{N}(T)$
Hypocoercivity: prove an H-theorem for a generalized entropy

$$H(f) := \frac{1}{2}\,\|f\|^2 + \varepsilon\,\langle \mathsf{A}\,f, f\rangle$$

Examples, conventions

- L is a linear relaxation operator L

$$\mathsf{L}\, f = \Pi\, f - f\,, \quad \Pi\, f := \frac{\rho}{\rho_F}\, F(x,v)$$
$$\rho = \rho_f := \int_{\mathbb{R}^d} f\, dv$$

  - Maxwellian case: $F(x,v) := M(v)\, e^{-V(x)}$ with $M(v) := (2\pi)^{-d/2}\, e^{-|v|^2/2} \Longrightarrow \Pi f = \rho_f\, M(v)$
  - **Linearized fast diffusion case:** $F(x,v) := \omega\left(\frac{1}{2}\,|v|^2 + V(x)\right)^{-(k+1)}$
- L is a Fokker-Planck operator: $\mathsf{L}\, f = \Delta_v f + \nabla \cdot (v\, f)$
- L is a linear scattering operator (including the case of non-elastic collisions)

Conventions

- F is a positive probability distribution
- Measure: $d\mu(x,v) = F(x,v)^{-1}\, dx\, dv$ on $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v)$
- Scalar product and norm $\langle f, g\rangle = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f\, g\, d\mu$ and $\|f\|^2 = \langle f, f\rangle$

Maxwellian case: Assumptions

We assume that $F(x,v) := M(v)\, e^{-V(x)}$ with $M(v) := (2\pi)^{-d/2}\, e^{-|v|^2/2}$ where V satisfies the following assumptions

(H1) *Regularity:* $V \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$

(H2) *Normalization:* $\int_{\mathbb{R}^d} e^{-V}\, dx = 1$

(H3) *Spectral gap condition: there exists a positive constant* Λ *such that*

$$\int_{\mathbb{R}^d} |u|^2\, e^{-V}\, dx \le \Lambda \int_{\mathbb{R}^d} |\nabla_x u|^2\, e^{-V}\, dx$$

for any $u \in H^1(e^{-V}\, dx)$ *such that* $\int_{\mathbb{R}^d} u\, e^{-V}\, dx = 0$

(H4) *Pointwise condition 1: there exists* $c_0 > 0$ *and* $\theta \in (0,1)$ *such that* $\Delta V \le \frac{\theta}{2}\, |\nabla_x V(x)|^2 + c_0\ \forall\, x \in \mathbb{R}^d$

(H5) *Pointwise condition 2: there exists* $c_1 > 0$ *such that* $|\nabla^2_x V(x)| \le c_1\, (1 + |\nabla_x V(x)|)\ \forall\, x \in \mathbb{R}^d$

(H6) *Growth condition:* $\int_{\mathbb{R}^d} |\nabla_x V|^2\, e^{-V}\, dx < \infty$

Maxwellian case

Theorem 15. *If $\partial_t f + \mathsf{T}\, f = \mathsf{L}\, f$, for $\varepsilon > 0$, small enough, there exists an explicit, positive constant $\lambda = \lambda(\varepsilon)$ such that*

$$\|f(t) - F\| \le (1+\varepsilon)\,\|f_0 - F\|\, e^{-\lambda\, t} \quad \forall\, t \ge 0$$

- The operator L has no regularization property: hypo-coercivity fundamentally differs from hypo-ellipticity
- Coercivity due to L is only on velocity variables

$$\frac{d}{dt}\|f(t) - F\|^2 = -\|(1-\Pi)f\|^2 = -\iint_{\mathbb{R}^d\times\mathbb{R}^d} |f - \rho_f\, M(v)|^2\, dv\, dx$$

- T and L do not commute: coercivity in v is transferred to the x variable. In the diffusion limit, ρ solves a Fokker-Planck equation

$$\partial_t \rho = \Delta\rho + \nabla\cdot(\rho\,\nabla V) \quad t > 0\,, \quad x\in\mathbb{R}^d$$

  The goal of the hypo-coercivity theory is to quantify the interaction of T and L and build a norm which controls $\|\cdot\|$ and decays exponentially

- Find a norm which is equivalent to $L^2(d\mu)$, for which we have coercivity, based on an operator A
- The operator A is determined by the diffusion limit

The linearized fast diffusion case

Consider a solution of $\partial_t f + \mathsf{T}\, f = \mathsf{L}\, f$ where $\mathsf{L}\, f = \Pi\, f - f$, $\Pi\, f := \frac{\rho}{\rho_F}\, F$

$$F(x,v) := \omega \left(\frac{1}{2}\, |v|^2 + V(x) \right)^{-(k+1)}, \quad V(x) = \left(1 + |x|^2\right)^{\beta}$$

where ω is a normalization constant chosen such that $\iint_{\mathbb{R}^d \times \mathbb{R}^d} F\, dx\, dv = 1$ and $\rho_F = \omega_0\, V^{d/2-k-1}$ for some $\omega_0 > 0$

Theorem 16. *Let $d \geq 1$, $k > d/2 + 1$. There exists a constant $\beta_0 > 1$ such that, for any $\beta \in (\min\{1, (d-4)/(2k-d-2)\}, \beta_0)$, there are two positive, explicit constants C and λ for which the solution satisfies:*

$$\forall\, t \geq 0\,, \quad \|f(t) - F\|^2 \leq C\, \|f_0 - F\|^2\, e^{-\lambda t}$$

- The Poincaré inequality is replaced by the **Hardy-Poincaré inequality** associated to the fast diffusion equation
- The nonlinear case can be reduced to the linear case [Schmeiser, work in progress]

Other applications, other models

An uncomplete list

- **Kinetic theory !**
- **Molecular motors** or models of Stokes' drift: existence, rate of convergence, homogenization, large-time asymptotics: [Kinderlehrer et al.], [J.D., Kowalczyk], [Blanchet, J.D., Kowalczyk], [Perthame, Souganidis]
- **Keller-Segel models** and models of aggregation in chemotaxis: existence *vs.* blow-up, critical mass, large time asymptotics, etc. [Blanchet, J.D., Fernández, Escobedo], [Campos, work in progress]
- Kinetic and thermodynamical models for **gravitating systems**
- **Systems of reaction-diffusion** equations in chemistry: [Carrillo, Desvillettes, Fellner]
- Models for polymers, fragmentation, etc.

… Thank you for your attention !