Stability in Gagliardo-Nirenberg inequalities

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Geometric and functional inequalities and applications


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The stability result of G. Bianchi and H. Egnell

A question: [Brezis, Lieb (1985)] *Is there a natural way to bound*

$$\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

from below in terms of a "distance" to the set of optimal [Aubin-Talenti] functions when $d \geq 3$?

▷ [Bianchi, Egnell (1991)] There is a positive constant α such that

$$\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \alpha \inf_{\varphi \in \mathscr{M}} \|\nabla u - \nabla \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ Various improvements, *e.g.*, [Cianchi, Fusco, Maggi, Pratelli (2009)] there are constants α and κ and $u \mapsto \lambda(u)$ such that

$$\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \left(1 + \kappa \, \lambda(u)^\alpha\right) \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

The question of ***constructive*** *estimates is still widely open*

Entropy – entropy production inequalities

Free energy functional

$$\mathscr{F} = \int_{\mathbb{R}^d} \varphi\left(\frac{u}{u_\infty}\right) u_\infty \, \mathrm{d}x$$

- $s \mapsto \varphi(s)$ is a nonnegative convex function such that $\varphi(1) = 0$
- u_∞ is an attractor of an associated flow, typically

$$\frac{\partial u}{\partial t} = \mathscr{L} u \quad \text{or} \quad \frac{\partial u}{\partial t} = \mathscr{L} u^m$$

Fisher information functional

$$\mathscr{I} := \int_{\mathbb{R}^d} \varphi''(v) \, |\nabla v|^2 \, d\mu = -\frac{d\mathscr{F}}{dt}$$

where $v = u/u_\infty$

$$\mathscr{F}(t) \le \mathscr{F}(0)\, e^{-\Lambda t} \quad \Longleftrightarrow \quad \boxed{\mathscr{I} \ge \Lambda \mathscr{F}}$$

An example of *entropy method*

- Logarithmic entropy $s \mapsto \varphi(s) = s \log s + s - 1$
- Gaussian equilibrium $u_\infty(x) = (2\pi)^{-d/2}\, e^{-|x|^2/2}$ on $\mathbb{R}^d$
- Fokker-Planck equation $\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (x\, u)$

Entropy-entropy production (Gross' logarithmic Sobolev) inequality: $\Lambda = 2$, $w = \sqrt{u}$, $u \ge 0$ and $\int_{\mathbb{R}^d} u\,dx = \int_{\mathbb{R}^d} |w|^2\,dx = 1$

$$\int_{\mathbb{R}^d} |\nabla w|^2\,dx \ge \int_{\mathbb{R}^d} |w|^2 \log |w|^2\,dx + \frac{d}{4} \log\left(2\pi\, e^2\right)$$

[Jordan, Kinderlehrer, Otto (1998)] The Fokker-Planck equation is the gradient flow of the free energy $\mathscr{F}$ with respect to Wasserstein's distance

From the carré du champ method to stability results

Carré du champ method (D. Bakry and M. Emery) From

$$\frac{d\mathscr{I}}{dt} \leq -\Lambda\,\mathscr{I}$$

deduce that $\mathscr{I} - \Lambda\mathscr{F}$ is monotone non-increasing with limit 0

▷ ***Improved constant*** means ***stability***
Under some restrictions on the functions, there is some $\Lambda_\star \geq \Lambda$ such that

$$\mathscr{I} - \Lambda\mathscr{F} \geq (\Lambda_\star - \Lambda)\,\mathscr{F}$$

▷ ***Improved entropy – entropy production inequality***

$$\mathscr{I} \geq \Lambda\,\psi(\mathscr{F})$$

for some ψ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi'' > 0$

$$\mathscr{I} - \Lambda\mathscr{F} \geq \Lambda\big(\psi(\mathscr{F}) - \mathscr{F}\big) \geq 0$$

Outline

Part I: *Two examples of stability results by entropy methods*

▷ ***Sobolev and Hardy-Littlewood-Sobolev inequalities***
joint work with G. Jankowiak

▷ ***Subcritical interpolation inequalities on the sphere***
joint work with M.J. Esteban and M. Loss

Part II: *A constructive result based on entropy and parabolic regularity*
joint work with M. Bonforte, B. Nazaret and N. Simonov

- The ***fast diffusion flow*** and ***entropy methods***

▷ *Rényi entropy powers:* a word on the *carré du champ* method
▷ the *entropy-entropy production inequality*
▷ spectral gap: the *asymptotic time layer*
▷ the *initial time layer*, a backward nonlinear estimate

- The ***uniform convergence in relative error***

▷ the *threshold time*
▷ a quantitative global Harnack principle and Hölder regularity
▷ the stability result in the entropy framework

Part I
Two examples of stability results by entropy methods

Example 1
Sobolev and Hardy-Littlewood-Sobolev inequalities

▷ Stability in a weaker norm but with explicit constants

▷ From duality to improved estimates based on Yamabe's flow

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \ge 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \le \mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathscr{D}^{1,2}(\mathbb{R}^d)$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \ge \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx \quad \forall\, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$

Improved Sobolev inequality by duality

Theorem

[JD, G. Jankowiak] *Assume that* $d \ge 3$ *and let* $q = \frac{d+2}{d-2}$. *There exists a positive constant* $\mathscr{C} \le 1$ *such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \le \mathscr{C} \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \mathscr{D}^{1,2}(\mathbb{R}^d)$

Proof: the completion of a square

Integrations by parts show that

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)^{-1} v|^2 \, dx = \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v \, dx$$

and, if $v = u^q$ with $q = \frac{d+2}{d-2}$,

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla(-\Delta)^{-1} v \, dx = \int_{\mathbb{R}^d} u\, v \, dx = \int_{\mathbb{R}^d} u^{2^*} \, dx$$

Hence the expansion of the square

$$0 \le \int_{\mathbb{R}^d} \left| \mathsf{S}_d \, \|u\|^{\frac{4}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \nabla u - \nabla(-\Delta)^{-1} v \right|^2 dx$$

shows that

$$0 \le \mathsf{S}_d \, \|u\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right] - \left[\mathsf{S}_d \, \|u^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} u^q \,(-\Delta)^{-1} u^q \, dx \right]$$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t>0\,, \quad x \in \mathbb{R}^d$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\,dx + \mathsf{S}_d \left(\int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\,dx$$

Our choice: the ***Yamabe*** flow with $m = \frac{d-2}{d+2}$, $m+1 = \frac{2d}{d+2}$

The first step in the entropy method

Proposition

Assume that $d \geq 3$ and $m = \frac{d-2}{d+2}$. If v is a solution the Yamabe flow with nonnegative initial datum in $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, then

$$\frac{1}{2}\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}\right]$$
$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$

▷ $u = v^m$ is an optimal function for Sobolev if v is optimal for HLS

An improvement

$$\mathsf{J}_d[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx \quad \text{and} \quad \mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

Theorem (J.D., G. Jankowiak)

Assume that $d \ge 3$. Then we have

$$0 \le \mathsf{H}_d[v] + \mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\psi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]\right)$$

$$\forall\, u \in \mathscr{D}\,,\ v = u^{\frac{d+2}{d-2}}$$

where $\psi(x) := \sqrt{\mathscr{C}^2 + 2\,\mathscr{C}\,x} - \mathscr{C}$ *for any* $x \ge 0$

Proof: $\mathsf{H}(t) = -\mathsf{Y}(\mathsf{J}(t))\ \forall\, t \in [0,T)$, $\kappa_0 := \frac{\mathsf{H}_0'}{\mathsf{J}_0}$ and consider the differential inequality

$$\mathsf{Y}'\left(\mathscr{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \le \frac{d+2}{2\,d}\,\mathscr{C}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,, \quad \mathsf{Y}(0) = 0\,, \quad \mathsf{Y}(\mathsf{J}_0) = -\mathsf{H}_0$$

... and a consequence: $\mathscr{C} = 1$ is not optimal

Theorem

[JD, G. Jankowiak] *In the inequality*

$$S_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx$$
$$\le C_d \, S_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

we have

$$\frac{d}{d+4} \le C_d < 1$$

based on a (painful) linearization

Extensions:
- fractional Laplacian operator [Jankowiak, Nguyen]
- Moser-Trudinger-Onofri inequality

Example 2
Improved interpolation inequalities on the sphere

The interpolation inequalities on $\mathbb{S}^d$

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \ge 1$, $p \neq 2$, is such that

$$p \le 2^* := \frac{2\,d}{d-2}$$

if $d \ge 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2}\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}$$

The Bakry-Emery method

Entropy functional

$$\mathscr{F}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathscr{F}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathscr{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and compute $\frac{d}{dt}\mathscr{F}_p[\rho] = -\mathscr{I}_p[\rho]$ and $\frac{d}{dt}\mathscr{I}_p[\rho] \le -d\,\mathscr{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathscr{I}_p[\rho] - d\,\mathscr{F}_p[\rho]\right) \le 0 \quad \Longrightarrow \quad \mathscr{I}_p[\rho] \ge d\,\mathscr{F}_p[\rho]$$

with $\rho = |u|^p$, if $p \le 2^{\#} := \frac{2\,d^2+1}{(d-1)^2}$

A refined interpolation inequality on the sphere

Theorem

Assume that

$$p\neq 2\,,\quad \text{and}\quad 1\le p\le 2^\#\quad \text{if}\quad d\ge 2\,,\qquad\qquad p\ge 1\quad \text{if}\quad d=1$$

$$\gamma=\left(\frac{d-1}{d+2}\right)^2(p-1)\,(2^\#-p)\quad \text{if}\quad d\ge 2\,,\qquad \gamma=\frac{p-1}{3}\quad \text{if}\quad d=1$$

Then for any $u\in\mathrm H^1(\mathbb S^d)$,

$$\|\nabla u\|^2_{\mathrm L^2(\mathbb S^d)}\ge\frac{d}{2-p-\gamma}\left(\|u\|^2_{\mathrm L^2(\mathbb S^d)}-\|u\|^{2-\frac{2\gamma}{2-p}}_{\mathrm L^p(\mathbb S^d)}\,\|u\|^{\frac{2\gamma}{2-p}}_{\mathrm L^2(\mathbb S^d)}\right)\ \text{if}\ \gamma\neq 2-p$$

$$\|\nabla u\|^2_{\mathrm L^2(\mathbb S^d)}\ge\frac{2\,d}{p-2}\,\|u\|^2_{\mathrm L^2(\mathbb S^d)}\,\log\left(\frac{\|u\|^2_{\mathrm L^2(\mathbb S^d)}}{\|u\|^2_{\mathrm L^p(\mathbb S^d)}}\right)\qquad\forall\,u\in\mathrm H^1(\mathbb S^d)$$

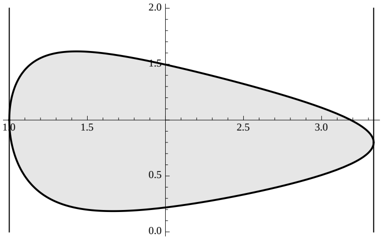
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m . \tag{1}$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{F}_p[\rho]\Big) \le 0$$



(p, m) *admissible region*, $d = 5$

Improved interpolation inequalities on the sphere

$$\lambda^\star \quad := \inf_{\substack{v \in \mathrm{H}^1_+(\mathbb{S}^d, d\mu) \\ \int_{\mathbb{S}^d} v \, d\mu = 1 \\ \int_{\mathbb{S}^d} x \, |v|^p \, d\mu = 0}} \frac{\int_{\mathbb{S}^d} (\Delta v)^2 \, d\mu}{\int_{\mathbb{S}^d} |\nabla v|^2 \, v \, d\mu} > d$$

For any $f \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ s.t. $\int_{\mathbb{S}^d} x \, |f|^p \, d\mu = 0$, consider the inequality

$$\int_{\mathbb{S}^d} |\nabla f|^2 \, v \, d\mu + \frac{\lambda}{p-2} \, \|f\|_2^2 \ge \frac{\lambda}{p-2} \, \|f\|_p^2$$

Proposition

If $p \in (2, 2^\#)$, the inequality holds with

$$\lambda \ge d + \frac{(d-1)^2}{d\,(d+2)} \, (2^\# - p) \, (\lambda^\star - d)$$

$p=2$: the logarithmic Sobolev case

$$\lambda^\star = d + \frac{2\,(d+2)}{2\,(d+3)+\sqrt{2\,(d+3)\,(2\,d+3)}}$$

Proposition

Let $d \ge 2$. For any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}$ such that $\int_{\mathbb{S}^d} x\,|u|^2\,d\mu = 0$, we have

$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\mu \ge \frac{\delta}{2}\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|_2^2}\right) d\mu$$

with $\delta := d + \frac{2}{d}\,\frac{4\,d-1}{2\,(d+3)+\sqrt{2\,(d+3)\,(2\,d+3)}}$

Stability under antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x)=u(x)\quad\forall\,x\in\mathbb{S}^d$$

Theorem

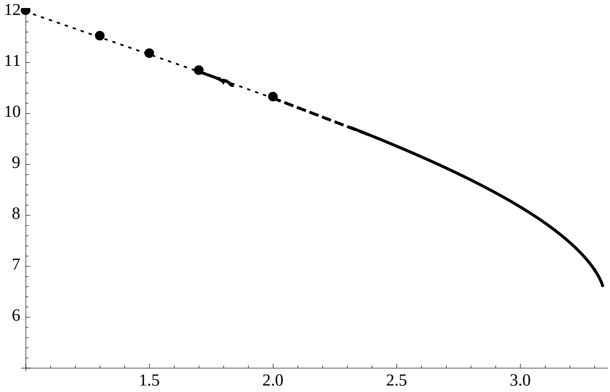
If $p\in(1,2)\cup(2,2^*)$, we have

$$\int_{\mathbb{S}^d}|\nabla u|^2\,d\mu\ge\frac{d}{p-2}\left[1+\frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1}\right]\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $u\in\mathrm{H}^1(\mathbb{S}^d,d\mu)$ with antipodal symmetry. The limit case $p=2$ corresponds to the improved logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d}|\nabla u|^2\,d\mu\ge\frac{d}{2}\,\frac{(d+3)^2}{(d+1)^2}\int_{\mathbb{S}^d}|u|^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)d\mu$$

The optimal constant in the antipodal framework



Numerical computation of the optimal constant when $d = 5$ *and* $1 \leq p \leq 10/3 \approx 3.33$. *The limiting value of the constant is numerically found to be equal to* $\lambda_\star = 2^{1-2/p}\, d \approx 6.59754$ *with* $d = 5$ *and* $p = 10/3$

Part II

A constructive result of stability based on entropy and parabolic regularity

Main results (part II) have been obtained in collaboration with

Matteo Bonforte
▷ *Universidad Autónoma de Madrid and ICMAT*

Bruno Nazaret
▷ *Université Paris 1 Panthéon-Sorbonne and Mokaplan team*

Nikita Simonov
▷ *Ceremade, Université Paris-Dauphine (PSL)*

Introduction

- A special family of Gagliardo-Nirenberg inequalities
- Optimal functions
- A stability result

Gagliardo-Nirenberg inequalities

For any smooth f on $\mathbb{R}^d$ with compact support

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\rm GN} \, \|f\|_{2p} \tag{2}$$

[Gagliardo, 1958] [Nirenberg, 1959] $\quad \theta = \frac{d\,(p-1)}{p\,(d+2-p\,(d-2))}$

- if $d \ge 3$, the exponent p is in the range $1 < p \le \frac{d}{d-2}$ and $2p = \frac{2\,d}{d-2} = 2^* =: 2\,p^*$ is the critical Sobolev exponent, corresponding to *Sobolev's inequality* with $(\theta = 1)$ [Rodemich, 1968] [Aubin & Talenti, 1976]

$$\|\nabla f\|_2^2 \ge \mathsf{S}_d \, \|f\|_{2^*}^2$$

$\triangleright$ if $d = 1$ or 2, the exponent p is in the range $1 < p < +\infty =: p^*$

- the limit case as $p \to 1_+$ is *Euclidean logarithmic Sobolev inequality in scale invariant form* [Blachman, 1965] [Stam, 1959] [Weissler, 1978]

$$\frac{d}{2} \log\left(\frac{2}{\pi\,d\,e} \int_{\mathbb{R}^d} |\nabla f|^2 \, dx\right) \ge \int_{\mathbb{R}^d} |f|^2 \log |f|^2 \, dx$$

for any function $f \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|f\|_2 = 1$

Optimal functions and scalings

$$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\rm GN}\,\|f\|_{2p}\qquad(1)$$

[del Pino, JD, 2002] Equality is achieved by the *Aubin-Talenti* type function

$$\mathrm{g}(x)=\left(1+|x|^2\right)^{-\frac{1}{p-1}}\quad\forall\,x\in\mathbb{R}^d$$

By homogeneity, translation, scalings, equality is also achieved by

$$\mathrm{g}_{\lambda,\mu,y}(x):=\mu\,\lambda^{-\frac{d}{2p}}\,\mathrm{g}\left(\frac{x-y}{\lambda}\right)\quad(\lambda,\mu,y)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d$$

▷ A non-scale invariant form of the inequality

$$a\,\|\nabla f\|_2^2+b\,\|f\|_{p+1}^{p+1}\ge\mathscr{K}_{\rm GN}\,\|f\|_{2p}^{2p\gamma}$$

$a=\frac12\,(p-1)^2$, $b=2\,\frac{d-p(d-2)}{p+1}$, $\mathscr{K}_{\rm GN}=\|\mathrm{g}\|_{2p}^{2p(1-\gamma)}$ and $\gamma=\frac{d+2-p(d-2)}{d-p(d-4)}$

If $p=1$: standard *Euclidean logarithmic Sobolev inequality* [Gross, 1975]

$$\int_{\mathbb{R}^d}|\nabla f|^2\,dx\ge\frac12\int_{\mathbb{R}^d}|v|^2\log\left(\frac{|f|^2}{\|f\|_2^2}\right)dx+\frac{d}{4}\log\left(2\pi\,e^2\right)\|f\|_2^2$$

The stability issue

What kind of distance to the manifold $\mathfrak{M}$ of the Aubin-Talenti type functions is measured by the *deficit functional* δ ?

$$\delta[f] := a\,\|\nabla f\|_2^2 + b\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GN}\,\|f\|_{2p}^{2p\gamma}$$

Some (not completely satisfactory) answers:

▷ In the critical case $p = d/(d-2)$, $d \geq 3$, [Bianchi, Egnell, 1991]: there is a positive constant $\mathscr{C}$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \geq \mathscr{C}\,\inf_{\mathfrak{M}}\|\nabla f - \nabla g\|_2^2$$

▷ [JD, Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a constant $\mathscr{C}$ with $1 < \mathscr{C} \leq 1 + \frac{4}{d}$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \geq \frac{\mathscr{C}}{\mathsf{S}_d\,\|f\|_{2^*}^{2(2^*-2)}}\left(\mathsf{S}_d\,\|f^q\|_{\frac{2d}{d+2}}^2 - \int_{\mathbb{R}^d}|f|^q\,(-\Delta)^{-1}|f|^q\,\mathrm{d}x\right)$$

▷ [Blanchet, Bonforte, JD, Grillo, Vázquez] [JD, Toscani]... various improvements based on entropy methods and fast diffusion flows

A stability result

The *relative entropy*

$$\mathscr{F}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(|f|^{p+1} - \mathrm{g}^{p+1} - \tfrac{1+p}{2p}\,\mathrm{g}^{1-p}\left(|f|^{2p} - \mathrm{g}^{2p}\right)\right)dx$$

Theorem

Let $d \ge 1$, $p \in (1, p^*)$, $A > 0$ and $G > 0$. There is a $\mathscr{C} > 0$ such that

$$\delta[f] \ge \mathscr{C}\,\mathscr{F}[f]$$

for any $f \in \mathscr{W} := \left\{f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2\,dx) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, dx)\right\}$ such that

$$\int_{\mathbb{R}^d} |f|^{2p}\,dx = \int_{\mathbb{R}^d} |\mathrm{g}|^{2p}\,dx\,, \quad \int_{\mathbb{R}^d} x\,|f|^{2p}\,dx = 0$$

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f|^{2p}\,dx \le A \quad \text{and} \quad \mathscr{F}[f] \le G$$

Reminder: $\delta[f] := a\,\|\nabla f\|_2^2 + b\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GN}}\,\|f\|_{2p}^{2p\gamma}$

Some comments

▷ *The constant $\mathscr{C}$ is explicit*

▷ *A Csiszár-Kullback inequality.* There exists a constant $C_p > 0$ such that

$$\Big\| |f|^{2p} - g^{2p} \Big\|_{\mathrm{L}^1(\mathbb{R}^d)} \le C_p \sqrt{\mathscr{F}[f]} \quad \text{if} \quad \|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} = \|g\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}$$

▷ Literature on stability of Sobolev type inequalities is huge:
– Weak $\mathrm{L}^{2^*/2}$-remainder term in bounded domains [Brezis, Lieb, 1985]
– Fractional versions and $(-\Delta)^s$ [Lu, Wei, 2000] [Gazzola, Grunau, 2001] [Bartsch, Weth, Willem, 2003] [Chen, Frank, Weth, 2013]
– Inverse stereographic projection (eigenvalues): [Ding, 1986] [Beckner, 1993] [Morpurgo, 2002] [Bartsch, Schneider, Weth, 2004]
– Symmetrization [Cianchi, Fusco, Maggi, Pratelli, 2009] and [Figalli, Maggi, Pratelli, 2010]

... to be continued

▷ On stability and flows (continued)

– Many other papers by Figalli and his collaborators, among which (most recent ones): [Figalli, Neumayer, 2018] [Neumayer, 2020] [Figalli, Zhang, 2020] [Figalli, Glaudo, 2020]

– Stability for Gagliardo-Nirenberg inequalities [Carlen, Figalli, 2013] [Seuffert, 2017] [Nguyen, 2019]

– Gradient flow issues [Otto, 2001] and many subsequent papers

– Carré du champ applied to the fast diffusion equation [Carrillo, Toscani, 2000] [Carrillo and Vázquez, 2003] [CJMTU, 2001] [Jüngel, 2016]

– Spectral gap properties [Scheffer, 2001] [Denzler, McCann, 2003 & 2005]

▷ On entropy methods

– Carré du champ: the semi-group and Markov precesses point of view [Bakry, Gentil, Ledoux, 2014]

– The PDE point of view (+ some applications to numerical analysis) [Jüngel, 2016]

▷ Global Harnack principle: [Vázquez, 2003] [Bonforte, Vázquez, 2006] [Vázquez, 2006] [Bonforte, Simonov, 2020]

$\Longrightarrow$ Our tool: the fast diffusion equation

The fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{3}$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy-entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)

The fast diffusion equation in original variables

Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \geq 1$, $m \in (0,1)$

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

with initial datum $u(t=0,x) = u_0(x) \geq 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, dx = \mathcal{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \, dx < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$\mathcal{U}(t,x) := \frac{1}{\left(\kappa\, t^{1/\mu}\right)^d} \, \mathcal{B}\left(\frac{x}{\kappa\, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$, $\kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$ and $\mathcal{B}$ is the Barenblatt profile

$$\mathcal{B}(x) := \left(C + |x|^2\right)^{-\frac{1}{1-m}}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} u^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} u \, |\nabla \mathsf{P}|^2 \, dx \quad \text{with} \quad \mathsf{P} = \frac{m}{m-1} \, u^{m-1} \text{ is the \textit{pressure variable}}$$

If u solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m)\,\mathsf{I}$$

The *Rényi entropy power*

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{2}{d} \, \frac{1}{1-m} - 1$$

applied to self-similar Barenblatt solutions has a linear growth in t

The variation of the Fisher information

Lemma

If u solves $\frac{\partial u}{\partial t} = \Delta u^m$ with $\frac{d-1}{d} =: m_1 \le m < 1$, then

$$\mathrm{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} u\,|\nabla \mathrm{P}|^2\,dx = -2 \int_{\mathbb{R}^d} u^m \left(\left\| \mathrm{D}^2 \mathrm{P} - \frac{1}{d}\,\Delta \mathrm{P}\,\mathrm{Id} \right\|^2 + (m - m_1)\,(\Delta \mathrm{P})^2 \right) dx$$

▷ This is where the limitation $m \ge m_1 := \frac{d-1}{d}$ appears

.... there are no boundary terms in the integrations by parts ?

The concavity property

Theorem

[Toscani, Savaré, 2014] *Assume that $m_1 \le m < 1$ if $d > 1$ and $m > 1/2$ if $d = 1$. Then $F(t)$ is increasing, $(1-m)\,\mathsf{F}''(t) \le 0$ and*

$$\lim_{t\to+\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t\to+\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I}$$

[Dolbeault, Toscani, 2016] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \ge \mathsf{E}[\mathscr{B}]^{\sigma-1}\,\mathsf{I}[\mathscr{B}]$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p} \tag{1}$$

$u^{m-1/2} = \frac{f}{\|f\|_{2p}}$ and $p = \frac{1}{2m-1} \in (1, p^*) \Longleftrightarrow \max\left\{\frac{1}{2}, m_1\right\} < m < 1$

Self-similar variables: entropy-entropy production method

The fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^m$$

has a self-similar solution

$$\mathcal{U}(t,x) := \frac{1}{\kappa^d (\mu\, t)^{d/\mu}}\, \mathcal{B}\left(\frac{x}{\kappa\,(\mu\, t)^{1/\mu}}\right) \quad \text{where} \quad \mathcal{B}(x) := \left(1+|x|^2\right)^{-\frac{1}{1-m}}$$

A time-dependent rescaling based on *self-similar variables*

$$u(t,x) = \frac{1}{\kappa^d\, R^d}\, v\left(\tau, \frac{x}{\kappa\, R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \tfrac{1}{2} \log\left(\frac{R(t)}{R_0}\right)$$

Then the function v solves *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \left[v \left(\nabla u^{m-1} - 2\, x \right) \right] = 0$$

with *same initial datum* $v_0 = u_0$ if $R_0 = R(0) = 1$

Free energy and Fisher information

The function v and $\mathscr{B}$ (same mass) solve the Fokker-Planck type equation

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\,x \right) \right] = 0 \tag{4}$$

A Lyapunov functional [Ralston, Newman, 1984]

Generalized entropy or *Free energy*

$$\mathscr{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathscr{B}^m - m\,\mathscr{B}^{m-1} (v - \mathscr{B}) \right) \mathrm{d}x$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt} \mathscr{F}[v] = -\mathscr{I}[v]\,, \quad \mathscr{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2 \mathrm{d}x$$

The entropy - entropy production inequality

$$\mathscr{B}(x) := \left(1+|x|^2\right)^{-\frac{1}{1-m}}$$

Theorem

[del Pino, JD, 2002] $d \ge 3$, $m \in [m_1, 1)$, $m > \frac{1}{2}$, $\int_{\mathbb{R}^d} v_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B} \, \mathrm{d}x$

$$\int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2 \mathrm{d}x = \mathscr{I}[v] \ge 4\,\mathscr{F}[v] = 4 \int_{\mathbb{R}^d} \left(\frac{\mathscr{B}^m}{m} - \frac{v^m}{m} + |x|^2 \left(v - \mathscr{B}\right) \right) \mathrm{d}x$$

$p = \frac{1}{2m-1}$, $v = f^{2p}$

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\mathrm{GN}} \, \|f\|_{2p} \Longleftrightarrow \delta[f] = \mathscr{I}[v] - 4\,\mathscr{F}[v] \ge 0$$

Corollary

[del Pino, JD, 2002] *A solution v of* (4) *with initial data $v_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2 v_0 \in L^1(\mathbb{R}^d)$, $v_0^m \in L^1(\mathbb{R}^d)$ satisfies*

$$\mathscr{F}[v(t,\cdot)] \le \mathscr{F}[v_0] \, e^{-4\,t}$$

A computation on a large ball, with boundary terms

Carré du champ method [Carrillo, Toscani] [Carrillo, Vázquez] [Carrillo, Jüngel, Toscani, Markowich, Unterreiter]

$$\frac{\partial u}{\partial t} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\,x \right) \right] = 0 \quad t > 0\,, \quad x \in B_R$$

$$\left(\nabla v^{m-1} - 2\,x \right) \cdot \frac{x}{|x|} = 0 \quad t > 0\,, \quad x \in \partial B_R$$

$$\frac{d}{dt} \int_{B_R} v\,|\nabla v^{m-1} - 2\,x|^2\,dx + 4 \int_{B_R} v\,|\nabla v^{m-1} - 2\,x|^2\,dx$$

$$+\,2\,\frac{1-m}{m} \int_{B_R} v^m \left(\left\| \mathrm{D}^2 \big(v^{m-1} - \mathscr{B}^{m-1} \big) \right\|^2 - (1-m) \left| \Delta \big(v^{m-1} - \mathscr{B}^{m-1} \big) \right|^2 \right) dx$$

$$= \int_{\partial B_R} v^m \left(\omega \cdot \nabla |\big(v^{m-1} - \mathscr{B}^{m-1} \big)|^2 \right) d\sigma \le 0 \text{ (by Grisvard's lemma)}$$

Improvement: $\exists\,\phi$ such that $\phi'' > 0$, $\phi(0) = 0$ and $\phi'(0) = 4$ [Toscani, JD]

$$\mathscr{I}[v|\mathscr{B}_\sigma] \ge \phi(\mathscr{F}[v|\mathscr{B}_\sigma]) \quad \Longleftarrow \quad \text{idea: } \frac{d\mathscr{I}}{dt} + 4\,\mathscr{I} \le -\frac{\mathscr{I}}{\mathscr{F}^2}$$

Spectral gap: sharp asymptotic rates of convergence

Assumptions on the initial datum v_0

(H_1) $\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}}$

(H_2) *if* $d \ge 3$ *and* $m \le m_* := \frac{d-4}{d-2}$, *then* $(v_0-\mathscr{B})$ *is integrable*

Theorem

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009] *If* $m<1$ *and* $m \neq m_*$, *then*

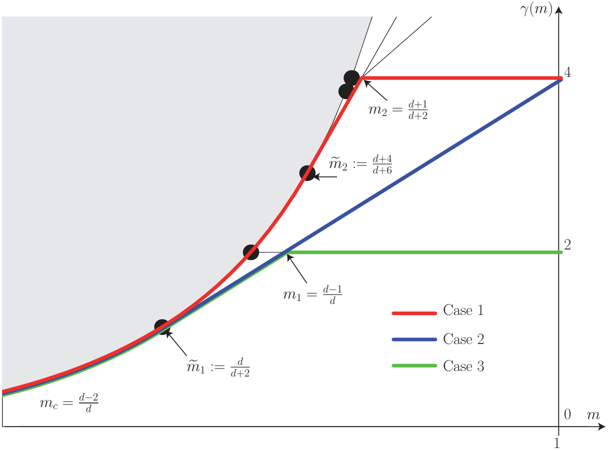
$$\mathscr{F}[v(t,\cdot)] \le C\, e^{-2\gamma(m)\,t} \quad \forall\, t \ge 0\,, \quad \gamma(m) := (1-m)\,\Lambda_{\alpha,d}$$

where $\Lambda_{\alpha,d}>0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d}|f|^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d}|\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f \in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,, \quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\alpha := \frac{1}{m-1}<0$, $\mathrm{d}\mu_\alpha := h_\alpha\,dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

Spectral gap and the asymptotic time layer



$$\mathscr{F}[v(t,\cdot)] \le C\, e^{-2\gamma(m)\,t} \quad \forall\, t \ge 0$$

[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]

Spectral gap and improvements... the details

▷ *Asymptotic time layer* [BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]

Corollary

Assume that v solves (4): $\partial_t v + \nabla\cdot\left[v\left(\nabla v^{m-1} - 2x\right)\right] = 0$ with initial datum $v_0 \ge 0$ such that $\int_{\mathbb{R}^d} v_0\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B}\,\mathrm{d}x$

(i) *there is a constant $\mathscr{C}_1 > 0$ such that $\mathscr{F}[v(t,\cdot)] \le \mathscr{C}_1\, e^{-2\gamma(m)\,t}$ with $\gamma(m) = 2$ if $m_1 \le m < 1$*

(ii) *if $m_1 \le m < 1$ and $\int_{\mathbb{R}^d} x\, v_0\,\mathrm{d}x = 0$, there is a constant $\mathscr{C}_2 > 0$ such that $\mathscr{F}[v(t,\cdot)] \le \mathscr{C}_2\, e^{-2\gamma(m)\,t}$ with $\gamma(m) = 4 - 2\,d\,(1-m)$*

(iii) *Assume that $\frac{d+1}{d+2} \le m < 1$ and $\int_{\mathbb{R}^d} x\, v_0\,\mathrm{d}x = 0$ and let*

$$\mathscr{B}_\sigma := \sigma^{-\frac{d}{2}}\,\mathscr{B}\big(x/\sqrt{\sigma}\big)$$

be such that $\int_{\mathbb{R}^d} |x|^2\, u(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d} |x|^2\, \mathscr{B}_\sigma(x)\,\mathrm{d}x$. Then there is a constant $\mathscr{C}_3 > 0$ such that $\mathscr{F}[v(t,\cdot)|\mathscr{B}_\sigma] \le \mathscr{C}_3\, e^{-4\,t}$

▷ *Initial time layer* $\mathscr{I}[v|\mathscr{B}_\sigma] \ge \phi(\mathscr{F}[v|\mathscr{B}_\sigma]) \Rightarrow$ faster decay for $t \sim 0$

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2} \int_{\mathbb{R}^d} |g|^2 \mathscr{B}^{2-m} \, dx \quad \text{and} \quad \mathsf{I}[g] := m(1-m) \int_{\mathbb{R}^d} |\nabla g|^2 \mathscr{B} \, dx$$

Hardy-Poincaré inequality. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in L^2(\mathbb{R}^d, \mathscr{B}^{2-m} dx)$ such that $\nabla g \in L^2(\mathbb{R}^d, \mathscr{B} \, dx)$, $\int_{\mathbb{R}^d} g \, \mathscr{B}^{2-m} dx = 0$ and $\int_{\mathbb{R}^d} x \, g \, \mathscr{B}^{2-m} dx = 0$

$$\mathsf{I}[g] \ge 4 \alpha \, \mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d(1-m)$$

Proposition

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2d(m - m_1)$ and $\chi = m/(266 + 56 m)$. If $\int_{\mathbb{R}^d} v \, dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x \, v \, dx = 0$ and

$$(1-\varepsilon) \mathscr{B} \le v \le (1+\varepsilon) \mathscr{B}$$

for some $\varepsilon \in (0, \chi \eta)$, then

$$\mathscr{Q}[v] \ge 4 + \eta$$

The initial time layer improvement: backward estimate

Rephrasing the *carré du champ* method, $\mathscr{Q}[v] := \frac{\mathscr{I}[v]}{\mathscr{F}[v]}$ is such that

$$\frac{d\mathscr{Q}}{dt} \le \mathscr{Q}\,(\mathscr{Q} - 4)$$

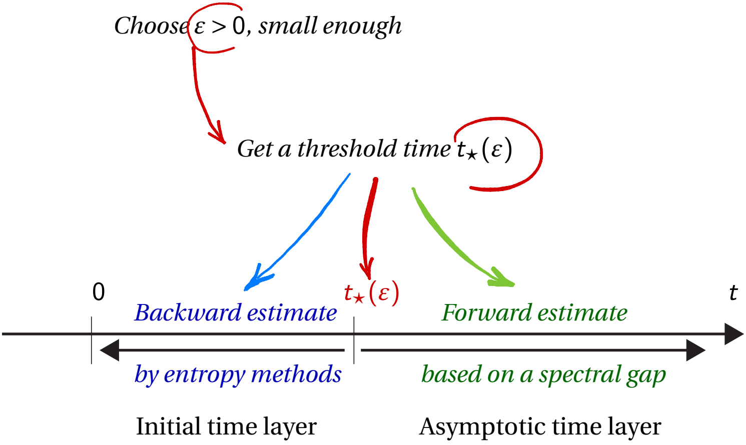
Lemma

Assume that $m > m_1$ and v is a solution to (4) with nonnegative initial datum v_0. If for some $\eta > 0$ and $T > 0$, we have $\mathscr{Q}[v(T,\cdot)] \ge 4 + \eta$, then

$$\mathscr{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\,e^{-4\,T}}{4 + \eta - \eta\,e^{-4\,T}} \quad \forall\, t \in [0,T]$$

Regularity and stability

Our strategy



Uniform convergence in relative error: statement

Theorem

Assume that $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$ *and let* $\varepsilon \in (0, 1/2)$, *small enough,* $A > 0$, *and* $G > 0$ *be given. There exists an explicit time* $t_\star \geq 0$ *such that, if* u *is a solution of*

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *satisfying*

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} u \, dx \leq A < \infty \tag{H_A}$$

$\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B} \, \mathrm{d}x = \mathscr{M}$ *and* $\mathscr{F}[u_0] \leq G$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall \, t \geq t_\star$$

The threshold time

Proposition

Let $m\in(m_1,1)$ *if* $d\ge2$, $m\in(1/3,1)$ *if* $d=1$, $\varepsilon\in(0,\varepsilon_{m,d})$, $A>0$ *and* $G>0$

$$t_\star=\mathsf c_\star\,\frac{1+A^{1-m}+G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf a}}$$

where $\mathsf a=\frac{\alpha}{\vartheta}\,\frac{2-m}{1-m}$ *and* $\vartheta=\nu/(d+\nu)$

$$\mathsf c_\star=\mathsf c_\star(m,d)=\sup_{\varepsilon\in(0,\varepsilon_{m,d})}\max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\,\varepsilon^{\mathsf a}\kappa_2(\varepsilon,m),\,\varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m):=\max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1},\frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m):=\frac{(4\,\alpha)^{\alpha-1}\,\mathsf K^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}}\quad\text{and}\quad\kappa_3(\varepsilon,m):=\frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

Improved entropy-entropy production inequality

Theorem

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. Then there is a positive number ζ such that

$$\mathscr{I}[v]\ge(4+\zeta)\,\mathscr{F}[v]$$

for any nonnegative function $v\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v]=G$, $\int_{\mathbb{R}^d}v\,\mathrm{d}x=\mathscr{M}$, $\int_{\mathbb{R}^d}x\,v\,\mathrm{d}x=0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon\in(0,2\varepsilon_\star)\,,\quad\varepsilon_\star:=\frac12\min\{\varepsilon_{m,d},\chi\,\eta\}\quad\text{with}\quad T=\frac12\log R(t_\star)$$

$$(1-\varepsilon)\,\mathscr{B}\le v(t,\cdot)\le(1+\varepsilon)\,\mathscr{B}\quad\forall\,t\ge T$$

and, as a consequence, the *initial time layer estimate*

$$\mathscr{I}[v(t,.)]\ge(4+\zeta)\,\mathscr{F}[v(t,.)]\quad\forall\,t\in[0,T],\quad\text{where}\quad\zeta=\frac{4\,\eta\,e^{-4\,T}}{4+\eta-\eta\,e^{-4\,T}}$$

Two consequences

$$\zeta = \mathsf{Z}(A, \mathscr{F}[u_0])\,, \quad \mathsf{Z}(A, G) := \frac{\zeta_\star}{1 + A^{(1-m)\,\frac{2}{\alpha}} + G}\,, \quad \zeta_\star := \frac{4\,\eta}{4+\eta}\left(\frac{\varepsilon_\star^{\mathsf{a}}}{2\,\alpha\,\mathsf{c}_\star}\right)^{\frac{2}{\alpha}} c_\alpha$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (4) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0 \, \mathrm{d}x = \mathscr{M}$, $\int_{\mathbb{R}^d} x\, v_0 \, \mathrm{d}x = 0$ and v_0 satisfies (H_A), then

$$\mathscr{F}[v(t,.)] \leq \mathscr{F}[v_0]\, e^{-(4+\zeta)\,t} \quad \forall\, t \geq 0$$

▷ The stability in the entropy - entropy production estimate $\mathscr{I}[v] - 4\,\mathscr{F}[v] \geq \zeta\,\mathscr{F}[v]$ also holds in a stronger sense

$$\mathscr{I}[v] - 4\,\mathscr{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathscr{I}[v]$$

A general stability result

$$\lambda[f]:=\left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2p}{d-p(d-4)}}$$

$$\mathsf{A}[f]:=\frac{\mathscr{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}}\,\|f\|_{2p}^{2p}}\,\sup_{r>0}r^{\frac{d-p(d-4)}{p-1}}\int_{|x|>r}|f(x+x_f)|^{2p}\,dx$$

$$\mathsf{E}[f]:=\frac{2p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}}\,|f|^{p+1}-\mathrm{g}^{p+1}-\frac{1+p}{2p}\,\mathrm{g}^{1-p}\left(\frac{\kappa[f]^{2p}}{\lambda[f]^2}\,|f|^{2p}-\mathrm{g}^{2p}\right)\right)\mathrm{d}x$$

$$\mathfrak{S}[f]:=\frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf{Z}\big(\mathsf{A}[f],\mathsf{E}[f]\big)$$

Theorem

Let $d\ge1$ and $p\in(1,p^*)$. For any $f\in\mathscr{W}$, we have

$$\left(\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta}\right)^{2p\gamma}-\left(\mathscr{C}_{\rm GN}\,\|f\|_{2p}\right)^{2p\gamma}\ge\mathfrak{S}[f]\,\|f\|_{2p}^{2p\gamma}\,\mathsf{E}[f]$$

References

- M. Bonforte, J. Dolbeault, B. Nazaret, and N. Simonov. *Stability in Gagliardo-Nirenberg inequalities.*
Preprint https://hal.archives-ouvertes.fr/hal-02887010

- M. Bonforte, J. Dolbeault, B. Nazaret, and N. Simonov. *Explicit constants in Harnack inequalities and regularity estimates, with an application to the fast diffusion equation* (supplementary material).
Preprint https://hal.archives-ouvertes.fr/hal-02887013

http://www.ceremade.dauphine.fr/~dolbeaul
(temporary links from the home page)

These slides can be found at

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For final versions, use `Dolbeault` as login and `Jean` as password

E-mail: dolbeault@ceremade.dauphine.fr

Thank you for your attention !

Some details on the proof for the threshold time

Interpolation inequalities

Our regularity theory relies on the inequality

$$\|f\|^2_{\mathrm{L}^{p_m}(B)} \le \mathcal{K}\left(\|\nabla f\|^2_{\mathrm{L}^2(B)} + \|f\|^2_{\mathrm{L}^2(B)}\right)$$

where $B \subset \mathbb{R}^d$ is the unit ball

	p_m	$\mathcal{K}$	q	β
$d \ge 3$	$\frac{2d}{d-2}$	$\frac{2}{\pi}\,\Gamma\big(\frac{d}{2}+1\big)^{2/d}$	$\frac{d}{2}$	α
$d = 2$	4	$0.0564922... < 2/\frac{2}{\sqrt{\pi}} \approx 1.12838$	2	$2\,(\alpha-1)$
$d = 1$	$\frac{4}{m}$	$2^{1+\frac{m}{2}}\,\max\left(\frac{2\,(2-m)}{m\,\pi^2},\frac{1}{4}\right)$	$\frac{2}{2-m}$	$\frac{2\,m}{2-m}$

Local estimates (1/2): a Herrero-Pierre type lemma

Lemma

For all $(t,R)\in(0,+\infty)^2$, any solution u of (3) satisfies

$$\sup_{y\in B_{R/2}(x)} u(t,y)\le \overline{\kappa}\left(\frac{1}{t^{d/\alpha}}\left(\int_{B_R(x)} u_0(y)\,dy\right)^{2/\alpha}+\left(\frac{t}{R^2}\right)^{\frac{1}{1-m}}\right)$$

$$\overline{\kappa}=\mathrm{k}\,\mathscr{K}^{\frac{2q}{\beta}}$$

$$\mathrm{k}^\beta=\left(\tfrac{4\beta}{\beta+2}\right)^\beta\left(\tfrac{4}{\beta+2}\right)^2\pi^{8(q+1)}\,e^{8\sum_{j=0}^\infty\log(j+1)\left(\frac{q}{q+1}\right)^j}\,2^{\frac{2m}{1-m}}\,(1+\mathfrak{a}\,\omega_d)^2\,\mathfrak{b}$$

$$\omega_d:=|\mathbb{S}^{d-1}|=\frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$\mathfrak{a}=\frac{3\,(16\,(d+1)\,(3+m))^{\frac{1}{1-m}}}{(2-m)\,(1-m)^{\frac{m}{1-m}}}+\frac{2^{\frac{d-m(d+1)}{1-m}}}{3^d\,d}\quad\text{and}\quad\mathfrak{b}=\frac{38^{2(q+1)}}{\left(1-(2/3)^{\frac{\beta}{4(q+1)}}\right)^{4(q+1)}}$$

Local estimates (2/2)

Lemma

For all $R>0$, any solution u of (3) satisfies

$$\inf_{|x-x_0|\le R} u(t,x) \ge \kappa \left(R^{-2}\,t\right)^{\frac{1}{1-m}} \quad \text{if} \quad t \ge \kappa_\star \, \|u_0\|_{\mathrm{L}^1(B_R(x_0))}^{1-m} R^\alpha =: 2\,\underline{t}$$

$$\kappa_\star = 2^{3\,\alpha+2}\,d^{\,\alpha} \quad \text{and} \quad \kappa = \alpha\,\omega_d \left(\frac{(1-m)^4}{2^{38}\,d^{\,4}\,\pi^{16\,(1-m)\,\alpha}\,\overline{\kappa}\,\alpha^2\,(1-m)}\right)^{\frac{2}{(1-m)^2\,\alpha\,d}}$$

Global Harnack Principle

Proposition

Any solution u of (3) satisfies

$$u(t,x) \le B\left(t+\overline{t}-\frac{1}{\alpha},x;\overline{M}\right) \quad \forall\,(t,x)\in[\overline{t},+\infty)\times\mathbb{R}^d$$

$$u(t,x) \ge B\left(t-\underline{t}-\frac{1}{\alpha},x;\underline{M}\right) \quad \forall\,(t,x)\in[2\,\underline{t},+\infty)\times\mathbb{R}^d$$

$$c:=\max\left\{1,2^{5-m}\overline{\kappa}^{1-m}\,\mathsf{b}^{\alpha}\right\},\quad \overline{t}:=c\,t_0\,A^{1-m}$$

$$\overline{M}:=2^{\frac{\alpha}{2(1-m)}}\,\overline{\kappa}^{\frac{\alpha}{2}}\,(1+c)^{\frac{d}{2}}\,\mathsf{b}^{-\frac{d\,\alpha}{2}}\,\mathcal{M}^2$$

$$\underline{M}:=\min\left\{2^{-d/2}\left(\frac{\kappa}{\mathsf{b}^d}\right)^{\alpha/2},\frac{\kappa}{(d\,(1-m))^{d/2}\,\alpha^{\frac{\alpha}{2(1-m)}}}\right\}\kappa_\star^{\frac{1}{1-m}}\,\mathcal{M}^2$$

The outer estimates

Corollary

For any $\varepsilon \in (0, \underline{\varepsilon})$, *any solution* u *of* (3) *satisfies*

$$u(t,x) \geq (1-\varepsilon)\,B(t,x) \quad \text{if} \quad |x| \geq R(t)\,\underline{\rho}(\varepsilon) \quad \text{and} \quad t \geq \underline{T}(\varepsilon)$$

$$u(t,x) \leq (1+\varepsilon)\,B(t,x) \quad \text{if} \quad |x| \geq R(t)\,\overline{\rho}(\varepsilon) \quad \text{and} \quad t \geq \overline{T}(\varepsilon)$$

The inner estimate

Proposition

There exist a numerical constant $\mathsf{K}>0$ and an exponent $\vartheta\in(0,1)$ such that, for any $\varepsilon\in(0,\varepsilon_{m,d})$ and for any $t\ge 4\,T(\varepsilon)$, any solution u of (3) satisfies

$$\left|\frac{u(t,x)}{B(t,x)}-1\right|\le\frac{\mathsf{K}}{\varepsilon^{\frac{1}{1-m}}}\left(\frac{1}{t}+\frac{\sqrt{G}}{R(t)}\right)^{\vartheta}\quad \text{if}\quad |x|\le 2\,\rho(\varepsilon)\,R(t)$$

$$\mathsf{K}:=2^{\frac{3d}{\alpha}+\frac{3+6\alpha}{\alpha(1-m)}+\vartheta+10}\,\frac{(\alpha+\mathcal{M})^{\vartheta}}{m^{\vartheta}(1-m)^{2(1+\vartheta)+\frac{2}{1-m}}}$$
$$\times\left[1+\mathsf{b}^d\,\mathcal{C}_{d,\nu,1}\left(\left(\overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,\frac{2^{\nu}}{2^{\nu}-1}+c\right)^{\frac{d}{d+\nu}}+\frac{\mu^{2d}}{\alpha^{\frac{d}{\alpha}}}\,\mathcal{M}^{\frac{d}{d+\nu}}\right)\right]$$

b and c are numerical constants, $\vartheta=\nu/(d+\nu)$ and ν is a Hölder regularity exponent which arises from Harnack's inequality in J. Moser's proof

An L^p-C^ν interpolation

Let $\lfloor u \rfloor_{C^\nu(\Omega)} := \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\nu}$

Lemma

Let $p \ge 1$ and $\nu \in (0,1)$

$$\|u\|_{L^\infty(\mathbb{R}^d)} \le C_{d,\nu,p} \lfloor u \rfloor_{C^\nu(\mathbb{R}^d)}^{\frac{d}{d+p\nu}} \|u\|_{L^p(\mathbb{R}^d)}^{\frac{p\nu}{d+p\nu}} \quad \forall\, u \in L^p(\mathbb{R}^d) \cap C^\nu(\mathbb{R}^d)$$

$$C_{d,\nu,p} = 2^{\frac{(p-1)(d+p\nu)+dp}{p(d+p\nu)}} \left(1+\frac{d}{\omega_d}\right)^{\frac{1}{p}} \left(1+\left(\frac{d}{p\nu}\right)^{\frac{1}{p}}\right)^{\frac{d}{d+p\nu}} \left(\left(\frac{d}{p\nu}\right)^{\frac{p\nu}{d+p\nu}} + \left(\frac{p\nu}{d}\right)^{\frac{d}{d+p\nu}}\right)^{1/p}$$

Appendix: linear tools and corresponding constants

Linear parabolic equations

$$\frac{\partial v}{\partial t} = \nabla \cdot \big(A(t,x)\,\nabla v\big) \tag{5}$$

on $\Omega_T := (0,T) \times \Omega$, where Ω is an open domain, $A(t,x)$ is a real symmetric matrix such that

$$\lambda_0\,|\xi|^2 \le \sum_{i,j=1}^{d} A_{i,j}(t,x)\,\xi_i\,\xi_j \le \lambda_1\,|\xi|^2 \quad \forall\,(t,x,\xi) \in \mathbb{R}^+ \times \Omega_T \times \mathbb{R}^d$$

for some positive constants λ_0 and λ_1

$$D_R^+(t_0,x_0) := \big(t_0 + \tfrac{3}{4}\,R^2, t_0 + R^2\big) \times B_{R/2}(x_0)$$

$$D_R^-(t_0,x_0) := \left(t_0 - \tfrac{3}{4}\,R^2, t_0 - \tfrac{1}{4}\,R^2\right) \times B_{R/2}(x_0)$$

$$\mathsf{h} := \exp\left[2^{d+4}\,3^d\,d + c_0^3\,2^{2\,(d+2)+3}\left(1 + \frac{2^{d+2}}{(\sqrt{2}-1)^{2\,(d+2)}}\right)\sigma\right]$$

$$c_0 = 3^{\frac{2}{d}}\,2^{\frac{(d+2)\,(3\,d^2+18\,d+24)+13}{2\,d}} \left(\frac{(2+d)^{1+\frac{4}{d^2}}}{d^{1+\frac{2}{d^2}}}\right)^{(d+1)(d+2)} \mathcal{K}^{\frac{2\,d+4}{d}}$$

Harnack inequality (Moser)

Theorem

Let $T>0$, $R\in(0,\sqrt{T})$, and take $(t_0,x_0)\in(0,T)\times\Omega$ such that $(t_0-R^2,t_0+R^2)\times B_{2R}(x_0)\subset\Omega_T$. If u is a weak solution of (5), then

$$\sup_{D_R^-(t_0,x_0)} v \le \overline{\mathsf{h}} \inf_{D_R^+(t_0,x_0)} v$$

with $\overline{\mathsf{h}} := \mathsf{h}^{\lambda_1+1/\lambda_0}$

This Harnack inequality goes back to [Moser, 1964] [Moser1971]
The dependence of the constant on λ_0 and λ_1 is optimal [Moser, 1971]
The dependence of h on d is pointed out in [Gutierrez, Wheeden, 1990]
To our knowledge, this is the first explicit expression of h

Harnack inequality implies Hölder continuity

Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$ two bounded domains and let us consider
$Q_1 := (T_2, T_3) \times \Omega_1 \subset (T_1, T_4) \times \Omega_2 =: Q_2$, where $0 \le T_1 < T_2 < T_3 < T < 4$

We define the *parabolic distance* between Q_1 and Q_2 as

$$d(Q_1, Q_2) := \inf_{\substack{(t,x)\in Q_1 \\ (s,y)\in[T_1,T_4]\times\partial\Omega_2\cup\{T_1,T_4\}\times\Omega_2}} |x-y| + |t-s|^{\frac{1}{2}}$$

Theorem

If v is a nonnegative solution of (5) on Q_2, then

$$\sup_{(t,x),(s,y)\in Q_1} \frac{|v(t,x)-v(s,y)|}{\left(|x-y|+|t-s|^{1/2}\right)^{\nu}} \le 2\left(\frac{128}{d(Q_1,Q_2)}\right)^{\nu} \|v\|_{\mathrm{L}^\infty(Q_2)}$$

where $\nu := \log_4\left(\frac{\overline{\mathsf{h}}}{\overline{\mathsf{h}}-1}\right)$

Appendix: still another interpolation inequality

An L^p-C^ν interpolation

Let $\lfloor u \rfloor_{C^\nu(\Omega)} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\nu}$

Lemma

Let $p\ge 1$ and $\nu\in(0,1)$. Then there exists a positive constant $C_{d,\nu,p}$ such that, for any $R>0$ and $x\in\mathbb{R}^d$

$$\|u\|_{\mathrm{L}^\infty(B_R(x))} \le C_{d,\nu,p}\left(\lfloor u\rfloor_{C^\nu(B_{2R}(x))}^{\frac{d}{d+p\nu}}\,\|u\|_{\mathrm{L}^p(B_{2R}(x))}^{\frac{p\nu}{d+p\nu}} + R^{-\frac{d}{p}}\,\|u\|_{\mathrm{L}^p(B_{2R}(x))}\right)$$

for any $u\in\mathrm{L}^p(B_{2R}(x))\cap C^\nu(B_{2R}(x))$, and

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le C_{d,\nu,p}\,\lfloor u\rfloor_{C^\nu(\mathbb{R}^d)}^{\frac{d}{d+p\nu}}\,\|u\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{p\nu}{d+p\nu}}\quad \forall\, u\in\mathrm{L}^p(\mathbb{R}^d)\cap C^\nu(\mathbb{R}^d)$$

$$C_{d,\nu,p} = 2^{\frac{(p-1)(d+p\nu)+dp}{p(d+p\nu)}}\left(1+\frac{d}{\omega_d}\right)^{\frac{1}{p}}\left(1+\left(\frac{d}{p\nu}\right)^{\frac{1}{p}}\right)^{\frac{d}{d+p\nu}}\left(\left(\frac{d}{p\nu}\right)^{\frac{p\nu}{d+p\nu}}+\left(\frac{p\nu}{d}\right)^{\frac{d}{d+p\nu}}\right)^{1/p}$$