Stability in Gagliardo-Nirenberg inequalities

Jean Dolbeault

http://www.ceremade.dauphine.fr/~dolbeaul

Ceremade, Université Paris-Dauphine

October 28, 2020

Online analysis and PDE seminar (Spain)
Two examples of entropy methods applied to stability
The fast diffusion equation
Regularity and stability

J. Dolbeault
Stability in Gagliardo-Nirenberg inequalities
The stability result of G. Bianchi and H. Egnell

A question: [Brezis, Lieb (1985)] *Is there a natural way to bound*

$$S_d \| \nabla u \|^2_{L^2(\mathbb{R}^d)} - \| u \|^2_{L^2^* (\mathbb{R}^d)}$$

*from below in terms of a “distance” to the set of optimal [Aubin-Talenti] functions* when $d \geq 3$?

▷ [Bianchi, Egnell (1991)] There is a positive constant $\alpha$ such that

$$S_d \| \nabla u \|^2_{L^2(\mathbb{R}^d)} - \| u \|^2_{L^2^* (\mathbb{R}^d)} \geq \alpha \inf_{\varphi \in \mathcal{M}} \| \nabla u - \nabla \varphi \|^2_{L^2(\mathbb{R}^d)}$$

▷ Various improvements, e.g., [Cianchi, Fusco, Maggi, Pratelli (2009)] there are constants $\alpha$ and $\kappa$ and $u \mapsto \lambda(u)$ such that

$$S_d \| \nabla u \|^2_{L^2(\mathbb{R}^d)} \geq (1 + \kappa \lambda(u)^\alpha) \| u \|^2_{L^2^* (\mathbb{R}^d)}$$

*The question of constructive estimates is still widely open*
From the carré du champ method to stability results

\[ \mathcal{I}[u] \geq \Lambda F[u] \]

**Carré du champ method** (D. Bakry and M. Emery) From

\[ \frac{\partial u}{\partial t} = \mathcal{L} u^m \text{ (typically),} \quad \frac{d \mathcal{I}}{dt} \leq -\Lambda \mathcal{I} \]

deduce that \( \mathcal{I} - \Lambda F \) is monotone non-increasing with limit 0

▷ **Improved constant** means **stability**

Under some restrictions on the functions, there is some \( \Lambda^* \geq \Lambda \) such that

\[ \mathcal{I} - \Lambda F \geq (\Lambda^* - \Lambda) F \]

▷ **Improved entropy – entropy production inequality**

\[ \mathcal{I} \geq \Lambda \psi(F) \]

for some \( \psi \) such that \( \psi(0) = 0, \psi'(0) = 1 \) and \( \psi'' > 0 \)

\[ \mathcal{I} - \Lambda F \geq \Lambda \left( \psi(F) - F \right) \geq 0 \]
Part I: *Two examples of stability results by entropy methods*

- Sobolev and Hardy-Littlewood-Sobolev inequalities  
  joint work with G. Jankowiak
- Subcritical interpolation inequalities on the sphere  
  joint work with M.J. Esteban and M. Loss

Part II: *A constructive result based on entropy and parabolic regularity*

  joint work with M. Bonforte, B. Nazaret and N. Simonov

- The *fast diffusion flow* and *entropy methods*
  - Rényi entropy powers: a word on the *carré du champ* method
  - the *entropy-entropy production inequality*
  - spectral gap: the *asymptotic time layer*
  - the *initial time layer*, a backward nonlinear estimate

- The *uniform convergence in relative error*
  - the *threshold time*
  - a quantitative global Harnack principle and Hölder regularity
  - the stability result in the entropy framework
Part I

Two examples of stability results by entropy methods
Example 1
Sobolev and Hardy-Littlewood-Sobolev inequalities

- Stability in a weaker norm but with explicit constants
- From duality to improved estimates based on Yamabe’s flow
As it has been noticed by E. Lieb (1983) Sobolev’s inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\| u \|_{L^{2^*}(\mathbb{R}^d)}^2 \leq S_d \| \nabla u \|_{L^2(\mathbb{R}^d)}^2 \quad \forall \ u \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

and the Hardy-Littlewood-Sobolev inequality

$$S_d \| v \|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}^{\frac{2d}{d+2}} \geq \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx \quad \forall \ v \in L^{\frac{2d}{d+2}}(\mathbb{R}^d)$$

are dual of each other. Here $S_d$ is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$
Two examples of entropy methods applied to stability
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Critical Sobolev and HLS inequalities
Improved interpolation inequalities on the sphere

The first step in the entropy method

Proposition

Assume that $d \geq 3$ and $m = \frac{d-2}{d+2}$. If $v$ is a solution the Yamabe flow

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0, \quad x \in \mathbb{R}^d, \quad m = \frac{d-2}{d+2}$$

with nonnegative initial datum in $L^{2d/(d+2)}(\mathbb{R}^d)$, then

$$\frac{1}{2} \frac{d}{dt} \left[ \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx - S_d \| v \|^2_{L^{2d/(d+2)}(\mathbb{R}^d)} \right]$$

$$= \left( \int_{\mathbb{R}^d} v^{m+1} \, dx \right)^{\frac{2}{d}} \left[ S_d \| \nabla u \|^2_{L^2(\mathbb{R}^d)} - \| u \|^2_{L^{2^*}(\mathbb{R}^d)} \right] \geq 0$$
An improvement

\[ J_d[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}} \, dx \quad \text{and} \quad H_d[v] := \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx - S_d \| v \|_{L^{d+2}(\mathbb{R}^d)}^{2d} \]

**Theorem (J.D., G. Jankowiak)**

Assume that \( d \geq 3 \). Then we have

\[
0 \leq H_d[v] + S_d J_d[v]^{1 + \frac{2}{d}} \psi \left( J_d[v]^{\frac{2}{d}} - 1 \right) \left[ S_d \| \nabla u \|_{L^{2}(\mathbb{R}^d)}^2 - \| u \|_{L^{2^*}(\mathbb{R}^d)}^2 \right]
\]

\[
\forall u \in \mathcal{D}, \ v = u^{\frac{d+2}{d-2}}
\]

where \( \psi(x) := \sqrt{\mathcal{C}^2 + 2\mathcal{C}x - \mathcal{C}} \) for any \( x \geq 0 \), with \( \mathcal{C} = 1 \)
... and a consequence: $C = 1$ is not optimal

Theorem

[JD, G. Jankowiak] In the inequality

$$S_d \|w^q\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}^2 - \int_{\mathbb{R}^d} w^q (-\Delta)^{-1} w^q \, dx$$

$$\leq C_d S_d \|w\|_{L^{\frac{8}{d-2}}(\mathbb{R}^d)}^{\frac{8}{d-2}} \left[ \|\nabla w\|_{L^2(\mathbb{R}^d)}^2 - S_d \|w\|_{L^{2*}(\mathbb{R}^d)}^2 \right]$$

we have

$$\frac{d}{d+4} \leq C_d < 1$$
Example 2
Improved interpolation inequalities on the sphere
The interpolation inequalities on $\mathbb{S}^d$

On the $d$-dimensional sphere, let us consider the interpolation inequality

$$\| \nabla u \|_{L^2(\mathbb{S}^d)}^2 + \frac{d}{p-2} \| u \|_{L^2(\mathbb{S}^d)}^2 \geq \frac{d}{p-2} \| u \|_{L^p(\mathbb{S}^d)}^2 \quad \forall \, u \in H^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \geq 1$, $p \neq 2$, is such that

$$p \leq 2^* := \frac{2d}{d-2}$$

if $d \geq 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\| \nabla u \|_{L^2(\mathbb{S}^d)}^2 \geq \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \log \left( \frac{|u|^2}{\| u \|_{L^2(\mathbb{S}^d)}^2} \right) \, d\mu \quad \forall \, u \in H^1(\mathbb{S}^d, d\mu) \setminus \{0\}$$
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Critical Sobolev and HLS inequalities
Improved interpolation inequalities on the sphere

\[ \|\nabla u\|_{L^2(\mathbb{R}^d)} + \frac{d}{p-2} \|u\|_{L^2(\mathbb{R}^d)}^2 \geq \frac{d}{p-2} \|u\|^2_{L^p(\mathbb{R}^d)} \quad \forall u \in H^1(\mathbb{R}^d, \mu) \]
The Bakry-Emery method

Entropy functional

\[ \mathcal{F}_p[\rho] := \frac{1}{p-2} \left[ \int_{\mathbb{S}^d} \rho^{\frac{2}{p}} \, d\mu - \left( \int_{\mathbb{S}^d} \rho \, d\mu \right)^{\frac{2}{p}} \right] \quad \text{if} \quad p \neq 2 \]

\[ \mathcal{F}_2[\rho] := \int_{\mathbb{S}^d} \rho \log \left( \frac{\rho}{\|\rho\|_{L^1(\mathbb{S}^d)}} \right) \, d\mu \]

Fisher information functional

\[ \mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\mu \]

Bakry-Emery (carré du champ) method: use the heat flow

\[ \frac{\partial \rho}{\partial t} = \Delta \rho \]

and compute \( \frac{d}{dt} \mathcal{F}_p[\rho] = -\mathcal{I}_p[\rho] \) and \( \frac{d}{dt} \mathcal{I}_p[\rho] \leq -d \mathcal{I}_p[\rho] \) to get

\[ \frac{d}{dt} \left( \mathcal{I}_p[\rho] - d \mathcal{F}_p[\rho] \right) \leq 0 \quad \Rightarrow \quad \mathcal{I}_p[\rho] \geq d \mathcal{F}_p[\rho] \]

with \( \rho = |u|^p \), if \( p \leq 2^\# := \frac{2d^2+1}{(d-1)^2} \).
Theorem

Assume that

\[ p \neq 2, \quad \text{and} \quad 1 \leq p \leq 2^# \quad \text{if} \quad d \geq 2, \quad p \geq 1 \quad \text{if} \quad d = 1 \]

\[ \gamma = \left( \frac{d-1}{d+2} \right)^2 (p-1)(2^# - p) \quad \text{if} \quad d \geq 2, \quad \gamma = \frac{p-1}{3} \quad \text{if} \quad d = 1 \]

Then for any \( u \in H^1(\mathbb{S}^d) \),

\[
\| \nabla u \|_{L^2(\mathbb{S}^d)}^2 \geq \frac{d}{2 - p - \gamma} \left( \| u \|_{L^2(\mathbb{S}^d)}^2 - \| u \|_{L^p(\mathbb{S}^d)}^{2 - \frac{2\gamma}{2-p}} \| u \|_{L^2(\mathbb{S}^d)}^{\frac{2\gamma}{2-p}} \right) \quad \text{if} \quad \gamma \neq 2 - p
\]

\[
\| \nabla u \|_{L^2(\mathbb{S}^d)}^2 \geq \frac{2d}{p-2} \| u \|_{L^2(\mathbb{S}^d)}^2 \log \left( \frac{\| u \|_{L^2(\mathbb{S}^d)}^2}{\| u \|_{L^p(\mathbb{S}^d)}^2} \right) \quad \forall \ u \in H^1(\mathbb{S}^d)
\]
Improved interpolation inequalities on the sphere

\[ \lambda^* := \inf_{\nu \in H^1_+(S^d, d\mu)} \frac{\int_{S^d} (\Delta \nu)^2 \, d\mu}{\int_{S^d} |\nabla \nu|^2 \nu \, d\mu} > \frac{d}{d^2} \int_{S^d} |\nabla \nu|^2 \nu \, d\mu = 1 \]
\[ \int_{S^d} \nu \, d\mu = 1 \]
\[ \int_{S^d} |\nabla \nu|^2 \nu \, d\mu = 0 \]

For any \( f \in H^1(S^d, d\mu) \) s.t. \( \int_{S^d} |f|^p \, d\mu = 0 \), consider the inequality

\[ \int_{S^d} |\nabla f|^2 \nu \, d\mu + \frac{\lambda}{p-2} \| f \|_2^2 \geq \frac{\lambda}{p-2} \| f \|_p^2 \]

**Proposition**

*If \( p \in (2, 2\#) \), the inequality holds with*

\[ \lambda \geq d + \frac{(d-1)^2}{d(d+2)} (2\# - p)(\lambda^* - d) \]
\( p = 2: \) the logarithmic Sobolev case

\[ \lambda^* = d + \frac{2(d + 2)}{2(d + 3) + \sqrt{2(d + 3)(2d + 3)}} \]

**Proposition**

Let \( d \geq 2 \). For any \( u \in H^1(\mathbb{S}^d, d\mu) \setminus \{0\} \) such that \( \int_{\mathbb{S}^d} |u|^2 \, d\mu = 0 \), we have

\[ \int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{\delta}{2} \int_{\mathbb{S}^d} |u|^2 \log \left( \frac{|u|^2}{\|u\|_2^2} \right) \, d\mu \]

with \( \delta := d + \frac{2}{d} \frac{4d - 1}{2(d + 3) + \sqrt{2(d + 3)(2d + 3)}} \).
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^*$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m. \tag{1}$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{H}_p[\rho] := \frac{d}{dt} \left( \mathcal{I}_p[\rho] - d \mathcal{F}_p[\rho] \right) \leq 0$$

$(p, m)$ admissible region, $d = 5$
Stability under antipodal symmetry

With the additional restriction of antipodal symmetry, that is

\[ u(-x) = u(x) \quad \forall x \in \mathbb{S}^d \]

**Theorem**

If \( p \in (1, 2) \cup (2, 2^*) \), we have

\[
\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{d}{p-2} \left[ 1 + \frac{(d^2 - 4)(2^* - p)}{d(d + 2) + p - 1} \right] \left( \|u\|_{L^p(\mathbb{S}^d)}^2 - \|u\|_{L^2(\mathbb{S}^d)}^2 \right)
\]

for any \( u \in H^1(\mathbb{S}^d, d\mu) \) with antipodal symmetry. The limit case \( p = 2 \) corresponds to the improved logarithmic Sobolev inequality

\[
\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{d}{2} \frac{(d + 3)^2}{(d + 1)^2} \int_{\mathbb{S}^d} |u|^2 \log \left( \frac{|u|^2}{\|u\|_{L^2(\mathbb{S}^d)}^2} \right) \, d\mu
\]
Numerical computation of the optimal constant when $d = 5$ and $1 \leq p \leq 10/3 \approx 3.33$. The limiting value of the constant is numerically found to be equal to $\lambda_* = 2^{1-2/p} d \approx 6.59754$ with $d = 5$ and $p = 10/3$.
Part II

A constructive result of stability based on entropy and parabolic regularity

▷ An introduction

▷ The fast diffusion equation

▷ Regularity and stability
Main results (part II) have been obtained in collaboration with

Matteo Bonforte
▶ Universidad Autónoma de Madrid and ICMAT

Bruno Nazaret
▶ Université Paris 1 Panthéon-Sorbonne and Mokaplan team

Nikita Simonov
▶ Ceremade, Université Paris-Dauphine (PSL)
Introduction

- A special family of Gagliardo-Nirenberg inequalities
- Optimal functions
- A stability result
Gagliardo-Nirenberg inequalities

For any smooth $f$ on $\mathbb{R}^d$ with compact support

$$\|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta} \geq C_{GN} \|f\|_{2p}$$

(2)

[Gagliardo, 1958] [Nirenberg, 1959] \quad \theta = \frac{d(p-1)}{p(d+2-p(d-2))}

if $d \geq 3$, the exponent $p$ is in the range $1 < p \leq \frac{d}{d-2}$ and

$2p = \frac{2d}{d-2} = 2^* =: 2p^*$ is the critical Sobolev exponent, corresponding to Sobolev's inequality with $(\theta = 1)$ [Rodemich, 1968] [Aubin & Talenti, 1976]

$$\|\nabla f\|_2^2 \geq S_d \|f\|_{2^*}^2$$

if $d = 1$ or 2, the exponent $p$ is in the range $1 < p < +\infty =: p^*$

the limit case as $p \to 1^+$ is Euclidean logarithmic Sobolev inequality in scale invariant form [Blachman, 1965] [Stam, 1959] [Weissler, 1978]

$$\frac{d}{2} \log \left( \frac{2}{\pi d e} \int_{\mathbb{R}^d} |\nabla f|^2 \, dx \right) \geq \int_{\mathbb{R}^d} |f|^2 \log |f|^2 \, dx$$

for any function $f \in H^1(\mathbb{R}^d, \, dx)$ such that $\|f\|_2 = 1$
Optimal functions and scalings

\[ \| \nabla f \|_2^\theta \| f \|_p^{1-\theta} \geq C_{GN} \| f \|_{2p} \]  

(1)

[del Pino, JD, 2002] Equality is achieved by the Aubin-Talenti type function

\[ g(x) = \left( 1 + |x|^2 \right)^{-\frac{1}{p-1}} \quad \forall \ x \in \mathbb{R}^d \]

By homogeneity, translation, scalings, equality is also achieved by

\[ g_{\lambda, \mu, y}(x) := \mu \lambda^{-\frac{d}{2p}} g \left( \frac{x-y}{\lambda} \right) \quad (\lambda, \mu, y) \in (0, +\infty) \times \mathbb{R} \times \mathbb{R}^d \]

▷ A non-scale invariant form of the inequality

\[ a \| \nabla f \|_2^2 + b \| f \|_{p+1}^{p+1} \geq \mathcal{K}_{GN} \| f \|_{2p}^{2p \gamma} \]

\[ a = \frac{1}{2} (p - 1)^2, \quad b = 2 \frac{d-p(d-2)}{p+1} \]

\[ \mathcal{K}_{GN} = \| g \|_{2p}^{2p(1-\gamma)} \quad \text{and} \quad \gamma = \frac{d+2-p(d-2)}{d-p(d-4)} \]

If \( p = 1 \): standard Euclidean logarithmic Sobolev inequality [Gross, 1975]

\[ \int_{\mathbb{R}^d} |\nabla f|^2 \, dx \geq \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 \log \left( \frac{|f|^2}{\| f \|_2^2} \right) \, dx + \frac{d}{4} \log(2 \pi e^2) \| f \|_2^2 \]
The stability issue

What kind of distance to the manifold $\mathcal{M}$ of the Aubin-Talenti type functions is measured by the \textit{deficit functional} $\delta$?

$$\delta[f] := a \| \nabla f \|_2^2 + b \| f \|_{p+1}^{p+1} - \mathcal{H}_{\text{GN}} \| f \|_{2p}^{2p}$$

Some (not completely satisfactory) answers:

\[\uparrow\] In the critical case $p = d/(d-2)$, $d \geq 3$, [Bianchi, Egnell, 1991]: there is a positive constant $C$ such that

$$\| \nabla f \|_2^2 - S_d \| f \|_{2^*}^{2^*} \geq C \inf_{\mathcal{M}} \| \nabla f - \nabla g \|_2^2$$

\[\uparrow\] [JD, Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a constant $C$ with $1 < C \leq 1 + \frac{4}{d}$ such that

$$\| \nabla f \|_2^2 - S_d \| f \|_{2^*}^{2^*} \geq \frac{C}{S_d \| f \|_{2^*}^{2(2^*-2)}} \left( S_d \| f_q \|_{2^*}^{2d} - \int_{\mathbb{R}^d} |f|^q \left( (-\Delta)^{-1} |f|^q \right) dx \right)$$

\[\uparrow\] [ Blanchet, Bonforte, JD, Grillo, Vázquez ] [JD, Toscani] ... various improvements based on entropy methods and fast diffusion flows
A stability result

The relative entropy
\[ \mathcal{F}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left( |f|^{p+1} - g^{p+1} - \frac{1+p}{2p} g^{1-p} (|f|^{2p} - g^{2p}) \right) dx \]

The deficit functional
\[ \delta[f] := a \| \nabla f \|^2_2 + b \| f \|^{p+1}_{p+1} - \mathcal{H}_{GN} \| f \|^{2p\gamma}_{2p} \geq 0 \]

Theorem

Let \( d \geq 1, \ p \in (1, p^*) \), \( A > 0 \) and \( G > 0 \). There is a \( C > 0 \) such that
\[ \delta[f] \geq C \mathcal{F}[f] \]

for any \( f \in \mathcal{W} := \{ f \in L^1(\mathbb{R}^d, (1+|x|)^2 \, dx) : \nabla f \in L^2(\mathbb{R}^d, dx) \} \) such that
\[ \int_{\mathbb{R}^d} |f|^{2p} \, dx = \int_{\mathbb{R}^d} |g|^{2p} \, dx, \quad \int_{\mathbb{R}^d} x |f|^{2p} \, dx = 0 \]
\[ \sup_{r>0} r \frac{d-p(d-4)}{p-1} \int_{|x|>r} |f|^{2p} \, dx \leq A \quad \text{and} \quad \mathcal{F}[f] \leq G \]
Some comments

▷ **The constant $C$ is explicit**

▷ **A Csiszár-Kullback inequality.** There exists a constant $C_p > 0$ such that

$$
\left\| |f|^{2p} - g^{2p} \right\|_{L^1(\mathbb{R}^d)} \leq C_p \sqrt{\mathcal{F}[f]} \quad \text{if} \quad \|f\|_{L^2_p(\mathbb{R}^d)} = \|g\|_{L^2_p(\mathbb{R}^d)}
$$

▷ Literature on stability of Sobolev type inequalities is huge:
   – Weak $L^{2^*/2}$-remainder term in bounded domains [Brezis, Lieb, 1985]
   – Fractional versions and $(-\Delta)^s$ [Lu, Wei, 2000] [Gazzola, Grunau, 2001]
     [Bartsch, Weth, Willem, 2003] [Chen, Frank, Weth, 2013]
   – Inverse stereographic projection (eigenvalues): [Ding, 1986] [Beckner, 1993] [Morpurgo, 2002] [Bartsch, Schneider, Weth, 2004]
   – Symmetrization [Cianchi, Fusco, Maggi, Pratelli, 2009] and [Figalli, Maggi, Pratelli, 2010]

... to be continued
On stability and flows (continued)
- Many other papers by Figalli and his collaborators, among which (most recent ones): [Figalli, Neumayer, 2018] [Neumayer, 2020] [Figalli, Zhang, 2020] [Figalli, Glaudo, 2020]
- Stability for Gagliardo-Nirenberg inequalities [Carlen, Figalli, 2013] [Seuffert, 2017] [Nguyen, 2019]
- Gradient flow issues [Otto, 2001] and many subsequent papers
- Carré du champ applied to the fast diffusion equation [Carrillo, Toscani, 2000] [Carrillo and Vázquez, 2003] [CJMTU, 2001] [Jüngel, 2016]
- Spectral gap properties [Scheffer, 2001] [Denzler, McCann, 2003 & 2005]

On entropy methods
- Carré du champ: the semi-group and Markov precesses point of view [Bakry, Gentil, Ledoux, 2014]
- The PDE point of view (+ some applications to numerical analysis) [Jüngel, 2016]

Global Harnack principle: [Vázquez, 2003] [Bonforte, Vázquez, 2006] [Vázquez, 2006] [Bonforte, Simonov, 2020]

⇒ Our tool: the fast diffusion equation
The fast diffusion equation

\[ \frac{\partial u}{\partial t} = \Delta u^m \quad (3) \]

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy-entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)
Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \geq 1$, $m \in (0, 1)$

$$\frac{\partial u}{\partial t} = \Delta u^m$$

(2)

with initial datum $u(t = 0, x) = u_0(x) \geq 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, dx = \mathcal{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 u_0 \, dx < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$\mathcal{U}(t, x) := \frac{1}{(\kappa t^{1/\mu})^d} \mathcal{B}\left(\frac{x}{\kappa t^{1/\mu}}\right)$$

where $\mu := 2 + d(m - 1)$, $\kappa := \left|\frac{2\mu m}{m-1}\right|^{1/\mu}$ and $\mathcal{B}$ is the Barenblatt profile

$$\mathcal{B}(x) := (C + |x|^2)^{-\frac{1}{1-m}}$$
The Rényi entropy power $F$

The entropy is defined by

$$ E := \int_{\mathbb{R}^d} u^m \, dx $$

and the Fisher information by

$$ I := \int_{\mathbb{R}^d} u |\nabla P|^2 \, dx \quad \text{with} \quad P = \frac{m}{m-1} u^{m-1} \text{is the pressure variable} $$

If $u$ solves the fast diffusion equation, then

$$ E' = (1 - m) I $$

The Rényi entropy power

$$ F := E^\sigma = \left( \int_{\mathbb{R}^d} u^m \, dx \right)^\sigma \quad \text{with} \quad \sigma = \frac{2}{d} \frac{1}{1-m} - 1 $$

applied to self-similar Barenblatt solutions has a linear growth in $t$
The concavity property

**Theorem**

[Toscani, Savaré, 2014] Assume that $m_1 \leq m < 1$ if $d > 1$ and $m > 1/2$ if $d = 1$. Then $F(t)$ is concave, increasing, and

$$
\lim_{t \to +\infty} F'(t) = (1 - m) \sigma \lim_{t \to +\infty} E^{\sigma - 1} I
$$

[Dolbeault, Toscani, 2016] The inequality

$$
E^{\sigma - 1} I \geq E[B]^{\sigma - 1} I[B]
$$

is equivalent to the Gagliardo-Nirenberg inequality

$$
\| \nabla f \|_2^\theta \| f \|^ {1-\theta} \|_{p+1} \geq C_{GN} \| f \|_2^p
$$

(1)

$$
u^{m-1/2} = \frac{f}{\| f \|_2^p} \quad \text{and} \quad p = \frac{1}{2m-1} \in (1, p^*) \iff \max \{ \frac{1}{2}, m_1 \} < m < 1
$$
Self-similar variables: entropy-entropy production method

The fast diffusion equation

\[ \frac{\partial u}{\partial t} = \Delta u^m \]

has a self-similar solution

\[ \mathcal{U}(t,x) := \frac{1}{\kappa^d (\mu t)^{d/\mu}} \mathcal{B} \left( \frac{x}{\kappa (\mu t)^{1/\mu}} \right) \quad \text{where} \quad \mathcal{B}(x) := (1 + |x|^2)^{-\frac{1}{1-m}} \]

A time-dependent rescaling based on self-similar variables

\[ u(t,x) = \frac{1}{\kappa^d R^d} v(\tau, \frac{x}{\kappa R}) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \frac{1}{2} \log \left( \frac{R(t)}{R_0} \right) \]

Then the function \( v \) solves a Fokker-Planck type equation

\[ \frac{\partial v}{\partial \tau} + \nabla \cdot \left[ v \left( \nabla u^{m-1} - 2x \right) \right] = 0 \]

with same initial datum \( v_0 = u_0 \) if \( R_0 = R(0) = 1 \)
Free energy and Fisher information

The function \( v \) and \( \mathcal{B} \) (same mass) solve the Fokker-Planck type equation

\[
\frac{\partial v}{\partial t} + \nabla \cdot \left[ v \left( \nabla v^{m-1} - 2x \right) \right] = 0 \tag{4}
\]

A Lyapunov functional [Ralston, Newman, 1984]

**Generalized entropy** or **Free energy**

\[
\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left( v^m - \mathcal{B}^m - m \mathcal{B}^{m-1} (v - \mathcal{B}) \right) dx
\]

Entropy production is measured by the **Generalized Fisher information**

\[
\frac{d}{dt} \mathcal{F}[v] = -\mathcal{J}[v], \quad \mathcal{J}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2x \right|^2 dx
\]
The entropy - entropy production inequality

\[ \mathcal{B}(x) := (1 + |x|^2)^{-\frac{1}{1-m}} \]

**Theorem**

[del Pino, JD, 2002] \( d \geq 3, \ m \in [m_1, 1), \ m > \frac{1}{2}, \ \int_{\mathbb{R}^d} v_0 \, dx = \int_{\mathbb{R}^d} \mathcal{B} \, dx \)

\[
\int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2x \right|^2 \, dx = \mathcal{I}[v] \geq 4 \mathcal{F}[v] = 4 \int_{\mathbb{R}^d} \left( \frac{\mathcal{B}^m}{m} - \frac{v^m}{m} + |x|^2 (v - \mathcal{B}) \right) \, dx
\]

\( p = \frac{1}{2m-1}, \ v = f^{2p} \)

\[
\| \nabla f \|_2^\theta \| f \|_{p+1}^{1-\theta} \geq C_{GN} \| f \|_{2p} \iff \delta[f] = \mathcal{I}[v] - 4 \mathcal{F}[v] \geq 0
\]

**Corollary**

[del Pino, JD, 2002] A solution \( v \) of (4) with initial data \( v_0 \in L^1_+(\mathbb{R}^d) \) such that \( |x|^2 v_0 \in L^1(\mathbb{R}^d), \ v_0^m \in L^1(\mathbb{R}^d) \) satisfies

\[ \mathcal{F}[v(t, \cdot)] \leq \mathcal{F}[v_0] e^{-4t} \]
A computation on a large ball, with boundary terms

**Carré du champ method** [Carrillo, Toscani] [Carrillo, Vázquez] [Carrillo, Jüngel, Toscani, Markowich, Unterreiter]

\[
\begin{align*}
\frac{\partial u}{\partial t} + \nabla \cdot \left[ v \left( \nabla v^{m-1} - 2x \right) \right] &= 0 \quad t > 0, \quad x \in B_R \\
\left( \nabla v^{m-1} - 2x \right) \cdot \frac{x}{|x|} &= 0 \quad t > 0, \quad x \in \partial B_R
\end{align*}
\]

\[
\frac{d}{dt} \int_{B_R} v |\nabla v^{m-1} - 2x|^2 \, dx + 4 \int_{B_R} v |\nabla v^{m-1} - 2x|^2 \, dx
\]

\[+ 2 \frac{1-m}{m} \int_{B_R} v^m \left( \|D^2(v^{m-1} - B^{m-1})\|^2 - (1-m) \left| \Delta(v^{m-1} - B^{m-1}) \right|^2 \right) \, dx = \int_{\partial B_R} v^m (\omega \cdot \nabla |(v^{m-1} - B^{m-1})|^2) \, d\sigma \leq 0 \text{ (by Grisvard’s lemma)}
\]

**Improvement:** \( \exists \phi \) such that \( \phi'' > 0, \phi(0) = 0 \) and \( \phi'(0) = 4 \) [Toscani, JD]

\[
\mathcal{I}[v|\mathcal{B}_\sigma] \geq \phi(\mathcal{F}[v|\mathcal{B}_\sigma]) \quad \iff \quad \text{idea:} \quad \frac{d\mathcal{I}}{dt} + 4 \mathcal{I} \lesssim -\frac{\mathcal{I}}{\mathcal{I}^2}
\]
Spectral gap: sharp asymptotic rates of convergence

Assumptions on the initial datum $v_0$

$$(H_1) \ (C_0 + |x|^2)^{-\frac{1}{1-m}} \leq v_0 \leq (C_1 + |x|^2)^{-\frac{1}{1-m}}$$

$$(H_2) \text{ if } d \geq 3 \text{ and } m \leq m_* := \frac{d-4}{d-2}, \text{ then } (v_0 - B) \text{ is integrable}$$

**Theorem**

*Blanchet, Bonforte, JD, Grillo, Vázquez, 2009*  
If $m < 1$ and $m \neq m_*$, then

$$ \mathcal{F} [v(t, \cdot)] \leq C e^{-2\gamma(m) t} \quad \forall \ t \geq 0, \quad \gamma(m) := (1 - m) \Lambda_{\alpha,d} $$

where $\Lambda_{\alpha,d} > 0$ is the best constant in the Hardy–Poincaré inequality

$$ \Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 \ d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 \ d\mu_{\alpha} \quad \forall \ f \in H^1(d\mu_{\alpha}), \ \int_{\mathbb{R}^d} f \ d\mu_{\alpha-1} = 0 $$

with $\alpha := \frac{1}{m-1} < 0$, $d\mu_{\alpha} := h_{\alpha} \, dx$, $h_{\alpha}(x) := (1 + |x|^2)^{\alpha}$
Spectral gap and the asymptotic time layer

Two examples of entropy methods applied to stability
The fast diffusion equation
Regularity and stability
Gagliardo-Nirenberg inequalities
The fast diffusion equation
Spectral gap and asymptotics

\[ F[v(t, \cdot)] \leq C e^{-2\gamma(m)t} \quad \forall \ t \geq 0 \]

[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]
Spectral gap and improvements... the details

**Asymptotic time layer** [BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]

**Corollary**

Assume that \( v \) solves (4): \( \partial_t v + \nabla \cdot \left[ v (\nabla v^{m-1} - 2x) \right] = 0 \) with initial datum \( v_0 \geq 0 \) such that \( \int_{\mathbb{R}^d} v_0 \, dx = \int_{\mathbb{R}^d} B \, dx \)

(i) there is a constant \( C_1 > 0 \) such that \( \mathcal{F}[v(t, \cdot)] \leq C_1 e^{-2\gamma(m)t} \) with \( \gamma(m) = 2 \) if \( m_1 \leq m < 1 \)

(ii) if \( m_1 \leq m < 1 \) and \( \int_{\mathbb{R}^d} x \cdot v_0 \, dx = 0 \), there is a constant \( C_2 > 0 \) such that \( \mathcal{F}[v(t, \cdot)] \leq C_2 e^{-2\gamma(m)t} \) with \( \gamma(m) = 4 - 2d(1 - m) \)

(iii) Assume that \( \frac{d+1}{d+2} \leq m < 1 \) and \( \int_{\mathbb{R}^d} x \cdot v_0 \, dx = 0 \) and let

\[
B_\sigma := \sigma^{-\frac{d}{2}} B(x/\sqrt{\sigma})
\]

be such that \( \int_{\mathbb{R}^d} |x|^2 u(t, x) \, dx = \int_{\mathbb{R}^d} |x|^2 B_\sigma(x) \, dx \). Then there is a constant \( C_3 > 0 \) such that \( \mathcal{F}[v(t, \cdot)|B_\sigma] \leq C_3 e^{-4t} \)

**Initial time layer** \( \mathcal{I}[v|B_\sigma] \geq \phi(\mathcal{F}[v|B_\sigma]) \) ⇒ faster decay for \( t \sim 0 \)
The asymptotic time layer improvement

**Linearized free energy and linearized Fisher information**

\[
F[g] := \frac{m}{2} \int_{\mathbb{R}^d} |g|^2 B^{2-m} \, dx \quad \text{and} \quad I[g] := m(1-m) \int_{\mathbb{R}^d} |\nabla g|^2 B \, dx
\]

**Hardy-Poincaré inequality.** Let \( d \geq 1, m \in (m_1, 1) \) and \( g \in L^2(\mathbb{R}^d, B^{2-m} \, dx) \) such that \( \nabla g \in L^2(\mathbb{R}^d, B \, dx), \int_{\mathbb{R}^d} g \, B^{2-m} \, dx = 0 \) and \( \int_{\mathbb{R}^d} \nabla g \cdot B^{2-m} \, dx = 0 \)

\[
I[g] \geq 4 \alpha F[g] \quad \text{where} \quad \alpha = 2 - d(1-m)
\]

**Proposition**

Let \( m \in (m_1, 1) \) if \( d \geq 2, \ m \in (1/3, 1) \) if \( d = 1, \ \eta = 2d(m - m_1) \) and \( \chi = m/(266 + 56m) \). If \( \int_{\mathbb{R}^d} \nu \, dx = M, \ \int_{\mathbb{R}^d} \nabla \nu \, dx = 0 \) and

\[
(1 - \varepsilon) B \leq \nu \leq (1 + \varepsilon) B
\]

for some \( \varepsilon \in (0, \chi \eta) \), then

\[
\mathcal{D}[\nu] := \frac{\mathcal{I}[\nu]}{\mathcal{F}[\nu]} \geq 4 + \eta
\]
The initial time layer improvement: backward estimate

Rephrasing the carré du champ method, \( \mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]} \) is such that

\[
\frac{d \mathcal{Q}}{dt} \leq \mathcal{Q}(\mathcal{Q} - 4)
\]

Lemma

Assume that \( m > m_1 \) and \( v \) is a solution to (4) with nonnegative initial datum \( v_0 \). If for some \( \eta > 0 \) and \( T > 0 \), we have \( \mathcal{Q}[v(T, \cdot)] \geq 4 + \eta \), then

\[
\mathcal{Q}[v(t, \cdot)] \geq 4 + \frac{4\eta e^{-4T}}{4 + \eta - \eta e^{-4T}} \quad \forall \ t \in [0, T]
\]
Regularity and stability

Our strategy

Choose $\varepsilon > 0$, small enough

Get a threshold time $t_*(\varepsilon)$

Initial time layer

Backward estimate by entropy methods

Forward estimate based on a spectral gap

Asymptotic time layer
Uniform convergence in relative error: statement

**Theorem**

Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\epsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit time $t_\star \geq 0$ such that, if $u$ is a solution of

$$\frac{\partial u}{\partial t} = \Delta u^m \quad (2)$$

with nonnegative initial datum $u_0 \in L^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} \frac{d(m-mc)}{(1-m)} \int_{|x|>r} u \ dx \leq A < \infty \quad (H_A)$$

$$\int_{\mathbb{R}^d} u_0 \ dx = \int_{\mathbb{R}^d} B \ dx = M \text{ and } \mathcal{F}[u_0] \leq G, \text{ then}$$

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \epsilon \quad \forall \ t \geq t_\star$$
The threshold time

**Proposition**

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\varepsilon \in (0, \varepsilon_{m,d})$, $A > 0$ and $G > 0$

$$t_\star = c_\star \frac{1 + A^{1-m} + G^{\frac{a}{2}}}{\varepsilon^a}$$

where $a = \frac{\alpha}{\vartheta} \frac{2-m}{1-m}$ and $\vartheta = \nu / (d + \nu)$

$$c_\star = c_\star(m, d) = \sup_{\varepsilon \in (0, \varepsilon_{m,d})} \max \{ \varepsilon \kappa_1(\varepsilon, m), \varepsilon^a \kappa_2(\varepsilon, m), \varepsilon \kappa_3(\varepsilon, m) \}$$

$$\kappa_1(\varepsilon, m) := \max \left\{ \frac{8c}{(1+\varepsilon)^{1-m} - 1}, \frac{2^{3-m} \kappa_\star}{1 - (1-\varepsilon)^{1-m}} \right\}$$

$$\kappa_2(\varepsilon, m) := \frac{(4 \alpha)^{\alpha-1}}{\varepsilon^{2-m} \frac{\alpha}{\vartheta}} \kappa_\frac{a}{\vartheta}$$

and

$$\kappa_3(\varepsilon, m) := \frac{8 \alpha^{-1}}{1 - (1-\varepsilon)^{1-m}}$$
Two examples of entropy methods applied to stability
The fast diffusion equation
Regularity and stability

Improved entropy-entropy production inequality

**Theorem**

Let \( m \in (m_1, 1) \) if \( d \geq 2 \), \( m \in (1/2, 1) \) if \( d = 1 \), \( A > 0 \) and \( G > 0 \). Then there is a positive number \( \zeta \) such that

\[
\mathcal{I}[v] \geq (4 + \zeta) \mathcal{F}[v]
\]

for any nonnegative function \( v \in L^1(\mathbb{R}^d) \) such that \( \mathcal{F}[v] = G \),
\[
\int_{\mathbb{R}^d} v \, dx = M, \quad \int_{\mathbb{R}^d} x \, v \, dx = 0
\]
and \( v \) satisfies \((H_A)\).

We have the *asymptotic time layer estimate*

\[
\varepsilon \in (0, 2 \varepsilon_\star), \quad \varepsilon_\star := \frac{1}{2} \min\{\varepsilon_{m,d}, \chi \eta\} \quad \text{with} \quad T = \frac{1}{2} \log R(t_\star)
\]

\[
(1 - \varepsilon) \mathcal{B} \leq v(t, \cdot) \leq (1 + \varepsilon) \mathcal{B} \quad \forall \ t \geq T
\]

and, as a consequence, the *initial time layer estimate*

\[
\mathcal{I}[v(t, \cdot)] \geq (4 + \zeta) \mathcal{F}[v(t, \cdot)] \quad \forall \ t \in [0, T], \quad \text{where} \quad \zeta = \frac{4 \eta e^{-4 T}}{4 + \eta - \eta e^{-4 T}}
\]
Two consequences

\[ \zeta = Z(A, \mathcal{F}(u_0)), \quad Z(A, G) := \frac{\zeta_\star}{1 + A(1-m)^{\frac{2}{\alpha}} + G}, \quad \zeta_\star := \frac{4\eta}{4 + \eta} \left( \frac{\varepsilon_{\star}^a}{2\alpha c_{\star}} \right)^{\frac{2}{\alpha}} c_{\alpha} \]

- Improved decay rate for the fast diffusion equation in rescaled variables

**Corollary**

Let \( m \in (m_1, 1) \) if \( d \geq 2 \), \( m \in (1/2, 1) \) if \( d = 1 \), \( A > 0 \) and \( G > 0 \). If \( v \) is a solution of (4) with nonnegative initial datum \( v_0 \in L^1(\mathbb{R}^d) \) such that \( \mathcal{F}(v_0) = G \), \( \int_{\mathbb{R}^d} v_0 \, dx = \mathcal{M} \), \( \int_{\mathbb{R}^d} \nabla v_0 \, dx = 0 \) and \( v_0 \) satisfies (\( H_A \)), then

\[ \mathcal{F}(v(t,.)) \leq \mathcal{F}(v_0) e^{-(4+\zeta)t} \quad \forall \ t \geq 0 \]

- The stability in the entropy - entropy production estimate \( \mathcal{I}[v] - 4\mathcal{F}[v] \geq \zeta \mathcal{F}[v] \) also holds in a stronger sense

\[ \mathcal{I}[v] - 4\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta} \mathcal{I}[v] \]
A general stability result

\[ \lambda[f] := \left( \frac{2d \kappa[f]^{p-1}}{p^2 - 1} \frac{\|f\|_p^{p+1}}{\|\nabla f\|_2^2} \right) \frac{2p}{d-p(d-4)} \]

\[ A[f] := \frac{M}{\lambda[f]^{\frac{p-1}{p-1}}} \sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f(x + x_f)|^{2p} \, dx \]

\[ E[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left( \frac{\kappa[f]^{p+1}}{\lambda[f]^d} \frac{p-1}{2p} \right) |f|^{p+1} - g^{p+1} - \frac{1+p}{2p} g^{1-p} \left( \frac{\kappa[f]^{2p}}{\lambda[f]^2} |f|^{2p} - g^{2p} \right) \, dx \]

\[ \mathcal{S}[f] := \frac{M}{\lambda^{2p}} \frac{1}{C(p,d)} Z(A[f], E[f]) \]

**Theorem**

Let \( d \geq 1 \) and \( p \in (1, p^*) \). For any \( f \in \mathcal{W} \), we have

\[ \left( \|\nabla f\|_2^{\theta} \|f\|_p^{1-\theta} \right)^{2p\gamma} - \left( \mathcal{C}_{GN} \|f\|_2^{p} \right)^{2p\gamma} \geq \mathcal{S}[f] \|f\|_2^{2p\gamma} E[f] \]
M. Bonforte, J. Dolbeault, B. Nazaret, and N. Simonov. *Stability in Gagliardo-Nirenberg inequalities.* Preprint https://hal.archives-ouvertes.fr/hal-02887010

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E-mail: dolbeault@ceremade.dauphine.fr

Thank you for your attention!