Stability in Gagliardo-Nirenberg inequalities

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The stability result of G. Bianchi and H. Egnell

A question: [Brezis, Lieb (1985)] *Is there a natural way to bound*

$$\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

from below in terms of a "distance" to the set of optimal [Aubin-Talenti] functions when $d\geq 3$?

▷ [Bianchi, Egnell (1991)] There is a positive constant α such that

$$\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\geq\alpha\inf_{\varphi\in\mathscr{M}}\|\nabla u-\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ Various improvements, *e.g.*, [Cianchi, Fusco, Maggi, Pratelli (2009)] there are constants α and κ and $u\mapsto\lambda(u)$ such that

$$\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\geq\left(1+\kappa\,\lambda(u)^\alpha\right)\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

*The question of **constructive** estimates is still widely open*

From the carré du champ method to stability results

$$\boxed{\mathcal{I}[u] \geq \Lambda\,\mathcal{F}[u]}$$

Carré du champ method (D. Bakry and M. Emery) From

$$\frac{\partial u}{\partial t} = \mathcal{L}\,u^m \text{ (typically)}\,, \quad \frac{d\mathcal{I}}{dt} \leq -\Lambda\,\mathcal{I}$$

deduce that $\mathcal{I} - \Lambda\,\mathcal{F}$ is monotone non-increasing with limit 0

▷ ***Improved constant*** means ***stability***

Under some restrictions on the functions, there is some $\Lambda_\star \geq \Lambda$ such that

$$\mathcal{I} - \Lambda\,\mathcal{F} \geq (\Lambda_\star - \Lambda)\,\mathcal{F}$$

▷ ***Improved entropy – entropy production inequality***

$$\mathcal{I} \geq \Lambda\,\psi(\mathcal{F})$$

for some ψ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi'' > 0$

$$\mathcal{I} - \Lambda\,\mathcal{F} \geq \Lambda\big(\psi(\mathcal{F}) - \mathcal{F}\big) \geq 0$$

Outline

Part I: *Two examples of stability results by entropy methods*

▷ ***Sobolev and Hardy-Littlewood-Sobolev inequalities***
joint work with G. Jankowiak

▷ ***Subcritical interpolation inequalities on the sphere***
joint work with M.J. Esteban and M. Loss

Part II: *A constructive result based on entropy and parabolic regularity*
joint work with M. Bonforte, B. Nazaret and N. Simonov

- The ***fast diffusion flow*** and ***entropy methods***

▷ *Rényi entropy powers:* a word on the *carré du champ* method
▷ the *entropy-entropy production inequality*
▷ spectral gap: the *asymptotic time layer*
▷ the *initial time layer*, a backward nonlinear estimate

- The ***uniform convergence in relative error***

▷ the *threshold time*
▷ a quantitative global Harnack principle and Hölder regularity
▷ the stability result in the entropy framework

Part I

Two examples of stability results by entropy methods

Example 1
Sobolev and Hardy-Littlewood-Sobolev inequalities

▷ Stability in a weaker norm but with explicit constants

▷ From duality to improved estimates based on Yamabe's flow

Sobolev and HLS

As it has been noticed by E. Lieb (1983) Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathscr{D}^{1,2}(\mathbb{R}^d)$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\, dx \quad \forall\, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$

The first step in the entropy method

Proposition

Assume that $d \geq 3$ *and* $m = \frac{d-2}{d+2}$. *If* v *is a solution the Yamabe flow*

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d\,, \quad m = \frac{d-2}{d+2}$$

with nonnegative initial datum in $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,\mathrm{d}x - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}\right]$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,\mathrm{d}x\right)^{\frac{2}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

An improvement

$$\mathsf{J}_d[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx \quad \text{and} \quad \mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

Theorem (J.D., G. Jankowiak)

Assume that $d \ge 3$. Then we have

$$0 \le \mathsf{H}_d[v] + \mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\psi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]\right)$$

$$\forall\, u \in \mathscr{D}\,,\; v = u^{\frac{d+2}{d-2}}$$

where $\psi(x) := \sqrt{\mathscr{C}^2 + 2\,\mathscr{C}\,x} - \mathscr{C}$ for any $x \ge 0$, with $\mathscr{C} = 1$

... and a consequence: $\mathscr{C} = 1$ is not optimal

Theorem

[JD, G. Jankowiak] *In the inequality*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx$$
$$\le \mathsf{C}_d \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

we have

$$\frac{d}{d+4} \le \mathsf{C}_d < 1$$

Example 2
Improved interpolation inequalities on the sphere

The interpolation inequalities on $\mathbb{S}^d$

On the d-dimensional sphere, let us consider the interpolation inequality

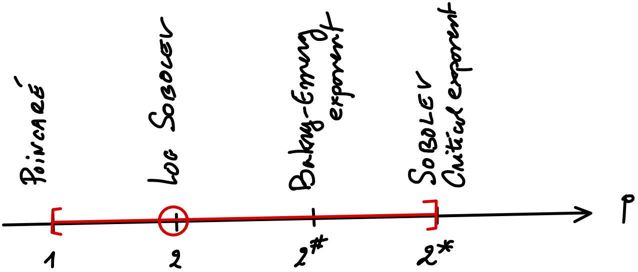
$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \ge 1$, $p \ne 2$, is such that

$$p \le 2^* := \frac{2\,d}{d-2}$$

if $d \ge 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2}\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \setminus \{0\}$$



$$\|\nabla u\|^2_{L^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{L^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{L^p(\mathbb{S}^d)} \quad \forall\, u \in H^1(\mathbb{S}^d, d\mu)$$

The Bakry-Emery method

Entropy functional

$$\mathscr{F}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathscr{F}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathscr{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and compute $\frac{d}{dt}\mathscr{F}_p[\rho] = -\mathscr{I}_p[\rho]$ and $\frac{d}{dt}\mathscr{I}_p[\rho] \le -d\,\mathscr{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathscr{I}_p[\rho] - d\,\mathscr{F}_p[\rho]\right) \le 0 \quad \Longrightarrow \quad \mathscr{I}_p[\rho] \ge d\,\mathscr{F}_p[\rho]$$

with $\rho = |u|^p$, if $p \le 2^\# := \frac{2d^2+1}{(d-1)^2}$

A refined interpolation inequality on the sphere

Theorem

Assume that

$$p \neq 2, \quad \text{and} \quad 1 \le p \le 2^\# \quad \text{if} \quad d \ge 2, \qquad p \ge 1 \quad \text{if} \quad d = 1$$

$$\gamma = \left(\frac{d-1}{d+2}\right)^2 (p-1)(2^\# - p) \quad \text{if} \quad d \ge 2, \qquad \gamma = \frac{p-1}{3} \quad \text{if} \quad d = 1$$

Then for any $u \in \mathrm{H}^1(\mathbb{S}^d)$,

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2-p-\gamma}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^{2-\frac{2\gamma}{2-p}}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u\|^{\frac{2\gamma}{2-p}}_{\mathrm{L}^2(\mathbb{S}^d)}\right) \text{ if } \gamma \neq 2-p$$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{2d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \log\left(\frac{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

Improved interpolation inequalities on the sphere

$$\lambda^\star \quad := \inf_{\substack{v \in \mathrm{H}^1_+(\mathbb{S}^d, d\mu) \\ \int_{\mathbb{S}^d} v \, d\mu = 1 \\ \int_{\mathbb{S}^d} x \, |v|^p \, d\mu = 0}} \frac{\int_{\mathbb{S}^d} (\Delta v)^2 \, d\mu}{\int_{\mathbb{S}^d} |\nabla v|^2 \, v \, d\mu} > d$$

For any $f \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ s.t. $\int_{\mathbb{S}^d} x \, |f|^p \, d\mu = 0$, consider the inequality

$$\int_{\mathbb{S}^d} |\nabla f|^2 \, v \, d\mu + \frac{\lambda}{p-2} \, \|f\|_2^2 \geq \frac{\lambda}{p-2} \, \|f\|_p^2$$

Proposition

If $p \in (2, 2^\#)$, the inequality holds with

$$\lambda \geq d + \frac{(d-1)^2}{d\,(d+2)} \, (2^\# - p) \, (\lambda^\star - d)$$

$p=2$: the logarithmic Sobolev case

$$\lambda^\star = d + \frac{2\,(d+2)}{2\,(d+3)+\sqrt{2\,(d+3)\,(2\,d+3)}}$$

Proposition

Let $d \ge 2$. For any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}$ such that $\int_{\mathbb{S}^d} x\,|u|^2\,d\mu = 0$, we have

$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\mu \ge \frac{\delta}{2}\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|_2^2}\right) d\mu$$

with $\delta := d + \frac{2}{d}\,\frac{4\,d-1}{2\,(d+3)+\sqrt{2\,(d+3)\,(2\,d+3)}}$

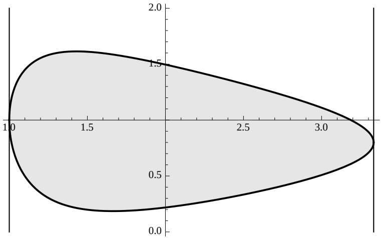
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m . \tag{1}$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{F}_p[\rho]\Big) \le 0$$



(p, m) admissible region, $d = 5$

Stability under antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x)=u(x)\quad\forall\,x\in\mathbb{S}^d$$

Theorem

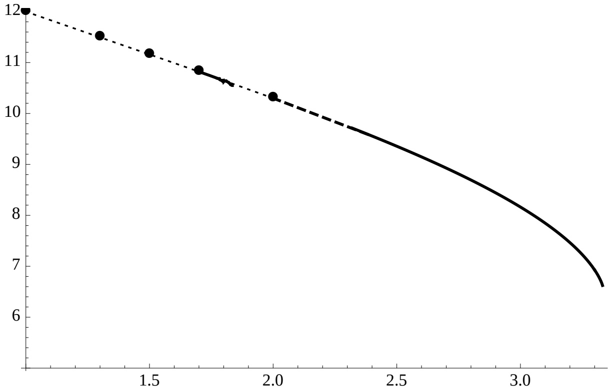
If $p\in(1,2)\cup(2,2^*)$, we have

$$\int_{\mathbb{S}^d}|\nabla u|^2\,d\mu\ge\frac{d}{p-2}\left[1+\frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1}\right]\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $u\in\mathrm{H}^1(\mathbb{S}^d,d\mu)$ with antipodal symmetry. The limit case $p=2$ corresponds to the improved logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d}|\nabla u|^2\,d\mu\ge\frac{d}{2}\,\frac{(d+3)^2}{(d+1)^2}\int_{\mathbb{S}^d}|u|^2\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)d\mu$$

The optimal constant in the antipodal framework



Numerical computation of the optimal constant when $d = 5$ *and* $1 \le p \le 10/3 \approx 3.33$. *The limiting value of the constant is numerically found to be equal to* $\lambda_\star = 2^{1-2/p}\, d \approx 6.59754$ *with* $d = 5$ *and* $p = 10/3$

Part II

A constructive result of stability based on entropy and parabolic regularity

▷ An introduction

▷ The fast diffusion equation

▷ Regularity and stability

Main results (part II) have been obtained in collaboration with

Matteo Bonforte
▷ *Universidad Autónoma de Madrid and ICMAT*

Bruno Nazaret
▷ *Université Paris 1 Panthéon-Sorbonne and Mokaplan team*

Nikita Simonov
▷ *Ceremade, Université Paris-Dauphine (PSL)*

Introduction

- A special family of Gagliardo-Nirenberg inequalities
- Optimal functions
- A stability result

Gagliardo-Nirenberg inequalities

For any smooth f on $\mathbb{R}^d$ with compact support

$$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p} \tag{2}$$

[Gagliardo, 1958] [Nirenberg, 1959] $\quad \theta=\frac{d\,(p-1)}{p\,(d+2-p\,(d-2))}$

- if $d\ge3$, the exponent p is in the range $1<p\le\frac{d}{d-2}$ and $2p=\frac{2\,d}{d-2}=2^*=:2\,p^*$ is the critical Sobolev exponent, corresponding to *Sobolev's inequality* with ($\theta=1$) [Rodemich, 1968] [Aubin & Talenti, 1976]

$$\|\nabla f\|_2^2\ge\mathsf{S}_d\,\|f\|_{2^*}^2$$

▷ if $d=1$ or 2, the exponent p is in the range $1<p<+\infty=:p^*$

- the limit case as $p\to1_+$ is *Euclidean logarithmic Sobolev inequality in scale invariant form* [Blachman, 1965] [Stam, 1959] [Weissler, 1978]

$$\frac{d}{2}\log\left(\frac{2}{\pi\,d\,e}\int_{\mathbb{R}^d}|\nabla f|^2\,\mathrm{d}x\right)\ge\int_{\mathbb{R}^d}|f|^2\log|f|^2\,\mathrm{d}x$$

for any function $f\in\mathrm{H}^1(\mathbb{R}^d,\mathrm{d}x)$ such that $\|f\|_2=1$

Optimal functions and scalings

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p}\tag{1}$$

[del Pino, JD, 2002] Equality is achieved by the *Aubin-Talenti* type function

$$\mathrm{g}(x)=\left(1+|x|^2\right)^{-\frac{1}{p-1}}\quad\forall\,x\in\mathbb{R}^d$$

By homogeneity, translation, scalings, equality is also achieved by

$$\mathrm{g}_{\lambda,\mu,y}(x):=\mu\,\lambda^{-\frac{d}{2p}}\,\mathrm{g}\left(\frac{x-y}{\lambda}\right)\quad(\lambda,\mu,y)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d$$

▷ A non-scale invariant form of the inequality

$$a\,\|\nabla f\|_2^2+b\,\|f\|_{p+1}^{p+1}\ge\mathscr{K}_{\mathrm{GN}}\,\|f\|_{2p}^{2p\gamma}$$

$a=\frac12\,(p-1)^2$, $b=2\,\frac{d-p(d-2)}{p+1}$, $\mathscr{K}_{\mathrm{GN}}=\|\mathrm{g}\|_{2p}^{2p(1-\gamma)}$ and $\gamma=\frac{d+2-p(d-2)}{d-p(d-4)}$

If $p=1$: standard *Euclidean logarithmic Sobolev inequality* [Gross, 1975]

$$\int_{\mathbb{R}^d}|\nabla f|^2\,dx\ge\frac12\int_{\mathbb{R}^d}|v|^2\log\left(\frac{|f|^2}{\|f\|_2^2}\right)dx+\frac d4\,\log\!\left(2\pi\,e^2\right)\|f\|_2^2$$

The stability issue

What kind of distance to the manifold $\mathfrak{M}$ of the Aubin-Talenti type functions is measured by the *deficit functional* δ ?

$$\delta[f] := a\,\|\nabla f\|_2^2 + b\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GN}\,\|f\|_{2p}^{2p\gamma}$$

Some (not completely satisfactory) answers:

▷ In the critical case $p = d/(d-2)$, $d \geq 3$, [Bianchi, Egnell, 1991]: there is a positive constant $\mathscr{C}$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \geq \mathscr{C}\,\inf_{\mathfrak{M}}\|\nabla f - \nabla g\|_2^2$$

▷ [JD, Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a constant $\mathscr{C}$ with $1 < \mathscr{C} \leq 1 + \frac{4}{d}$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \geq \frac{\mathscr{C}}{\mathsf{S}_d\,\|f\|_{2^*}^{2(2^*-2)}}\left(\mathsf{S}_d\,\|f^q\|_{\frac{2d}{d+2}}^2 - \int_{\mathbb{R}^d}|f|^q\,(-\Delta)^{-1}|f|^q\,\mathrm{d}x\right)$$

▷ [Blanchet, Bonforte, JD, Grillo, Vázquez] [JD, Toscani]... various improvements based on entropy methods and fast diffusion flows

A stability result

The *relative entropy*

$$\mathscr{F}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(|f|^{p+1} - g^{p+1} - \frac{1+p}{2p}\, g^{1-p} \left(|f|^{2p} - g^{2p} \right) \right) dx$$

The *deficit functional*

$$\delta[f] := a \, \|\nabla f\|_2^2 + b \, \|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GN}} \, \|f\|_{2p}^{2p\gamma} \geq 0$$

Theorem

Let $d \geq 1$, $p \in (1, p^*)$, $A > 0$ and $G > 0$. There is a $\mathscr{C} > 0$ such that

$$\delta[f] \geq \mathscr{C}\, \mathscr{F}[f]$$

for any $f \in \mathscr{W} := \left\{ f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2 \, dx) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, dx) \right\}$ such that

$$\int_{\mathbb{R}^d} |f|^{2p} \, dx = \int_{\mathbb{R}^d} |g|^{2p} \, dx \,, \quad \int_{\mathbb{R}^d} x \, |f|^{2p} \, dx = 0$$

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f|^{2p} \, dx \leq A \quad \text{and} \quad \mathscr{F}[f] \leq G$$

Some comments

▷ *The constant $\mathscr{C}$ is explicit*

▷ *A Csiszár-Kullback inequality.* There exists a constant $C_p > 0$ such that

$$\left\| |f|^{2p} - g^{2p} \right\|_{\mathrm{L}^1(\mathbb{R}^d)} \le C_p \sqrt{\mathscr{F}[f]} \quad \text{if} \quad \|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} = \|g\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}$$

▷ Literature on stability of Sobolev type inequalities is huge:
– Weak $\mathrm{L}^{2^*/2}$-remainder term in bounded domains [Brezis, Lieb, 1985]
– Fractional versions and $(-\Delta)^s$ [Lu, Wei, 2000] [Gazzola, Grunau, 2001] [Bartsch, Weth, Willem, 2003] [Chen, Frank, Weth, 2013]
– Inverse stereographic projection (eigenvalues): [Ding, 1986] [Beckner, 1993] [Morpurgo, 2002] [Bartsch, Schneider, Weth, 2004]
– Symmetrization [Cianchi, Fusco, Maggi, Pratelli, 2009] and [Figalli, Maggi, Pratelli, 2010]

... to be continued

▷ On stability and flows (continued)

– Many other papers by Figalli and his collaborators, among which (most recent ones): [Figalli, Neumayer, 2018] [Neumayer, 2020] [Figalli, Zhang, 2020] [Figalli, Glaudo, 2020]

– Stability for Gagliardo-Nirenberg inequalities [Carlen, Figalli, 2013] [Seuffert, 2017] [Nguyen, 2019]

– Gradient flow issues [Otto, 2001] and many subsequent papers

– Carré du champ applied to the fast diffusion equation [Carrillo, Toscani, 2000] [Carrillo and Vázquez, 2003] [CJMTU, 2001] [Jüngel, 2016]

– Spectral gap properties [Scheffer, 2001] [Denzler, McCann, 2003 & 2005]

▷ On entropy methods

– Carré du champ: the semi-group and Markov precesses point of view [Bakry, Gentil, Ledoux, 2014]

– The PDE point of view (+ some applications to numerical analysis) [Jüngel, 2016]

▷ Global Harnack principle: [Vázquez, 2003] [Bonforte, Vázquez, 2006] [Vázquez, 2006] [Bonforte, Simonov, 2020]

$\Longrightarrow$ Our tool: the fast diffusion equation

The fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{3}$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy-entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)

The fast diffusion equation in original variables

Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \geq 1$, $m \in (0,1)$

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

with initial datum $u(t=0,x) = u_0(x) \geq 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, dx = \mathcal{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \, dx < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$\mathcal{U}(t,x) := \frac{1}{\left(\kappa\, t^{1/\mu}\right)^d} \, \mathcal{B}\left(\frac{x}{\kappa\, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$, $\kappa := \left|\frac{2\,\mu\, m}{m-1}\right|^{1/\mu}$ and $\mathcal{B}$ is the Barenblatt profile

$$\mathcal{B}(x) := \left(C + |x|^2\right)^{-\frac{1}{1-m}}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} u^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} u \, |\nabla \mathsf{P}|^2 \, dx \quad \text{with} \quad \mathsf{P} = \frac{m}{m-1} \, u^{m-1} \text{ is the \textit{pressure variable}}$$

If u solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

The *Rényi entropy power*

$$\mathsf{F} := \mathsf{E}^\sigma = \left(\int_{\mathbb{R}^d} u^m \, dx \right)^\sigma \quad \text{with} \quad \sigma = \frac{2}{d} \, \frac{1}{1-m} - 1$$

applied to self-similar Barenblatt solutions has a linear growth in t

The concavity property

Theorem

[Toscani, Savaré, 2014] *Assume that $m_1 \le m < 1$ if $d > 1$ and $m > 1/2$ if $d = 1$. Then $F(t)$ is concave, increasing, and*

$$\lim_{t\to+\infty} \mathsf{F}'(t) = (1-m)\,\sigma \lim_{t\to+\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I}$$

[Dolbeault, Toscani, 2016] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \ge \mathsf{E}[\mathscr{B}]^{\sigma-1}\,\mathsf{I}[\mathscr{B}]$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p} \tag{1}$$

$u^{m-1/2} = \frac{f}{\|f\|_{2p}}$ and $p = \frac{1}{2\,m-1} \in (1, p^*) \Longleftrightarrow \max\{\frac{1}{2}, m_1\} < m < 1$

Self-similar variables: entropy-entropy production method

The fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^m$$

has a self-similar solution

$$\mathscr{U}(t,x) := \frac{1}{\kappa^d (\mu\, t)^{d/\mu}} \mathscr{B}\left(\frac{x}{\kappa\,(\mu\, t)^{1/\mu}}\right) \quad \text{where} \quad \mathscr{B}(x) := \left(1+|x|^2\right)^{-\frac{1}{1-m}}$$

A time-dependent rescaling based on *self-similar variables*

$$u(t,x) = \frac{1}{\kappa^d\, R^d}\, v\left(\tau, \frac{x}{\kappa\, R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}\,, \quad \tau(t) := \tfrac{1}{2} \log\left(\frac{R(t)}{R_0}\right)$$

Then the function v solves *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \left[v \left(\nabla u^{m-1} - 2\, x \right) \right] = 0$$

with *same initial datum* $v_0 = u_0$ if $R_0 = R(0) = 1$

Free energy and Fisher information

The function v and $\mathscr{B}$ (same mass) solve the Fokker-Planck type equation

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\,x \right) \right] = 0 \tag{4}$$

A Lyapunov functional [Ralston, Newman, 1984]

Generalized entropy or *Free energy*

$$\mathscr{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathscr{B}^m - m\,\mathscr{B}^{m-1}\,(v - \mathscr{B}) \right) \mathrm{d}x$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathscr{F}[v] = -\mathscr{I}[v]\,, \quad \mathscr{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2 \mathrm{d}x$$

The entropy - entropy production inequality

$$\mathscr{B}(x) := \left(1+|x|^2\right)^{-\frac{1}{1-m}}$$

Theorem

[del Pino, JD, 2002] $d \geq 3$, $m \in [m_1, 1)$, $m > \frac{1}{2}$, $\int_{\mathbb{R}^d} v_0 \, dx = \int_{\mathbb{R}^d} \mathscr{B} \, dx$

$$\int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2x \right|^2 dx = \mathscr{I}[v] \geq 4\,\mathscr{F}[v] = 4 \int_{\mathbb{R}^d} \left(\frac{\mathscr{B}^m}{m} - \frac{v^m}{m} + |x|^2 (v - \mathscr{B}) \right) dx$$

$p = \frac{1}{2m-1}$, $v = f^{2p}$

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \geq \mathscr{C}_{\mathrm{GN}} \, \|f\|_{2p} \iff \delta[f] = \mathscr{I}[v] - 4\,\mathscr{F}[v] \geq 0$$

Corollary

[del Pino, JD, 2002] *A solution v of (4) with initial data $v_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2 v_0 \in L^1(\mathbb{R}^d)$, $v_0^m \in L^1(\mathbb{R}^d)$ satisfies*

$$\mathscr{F}[v(t,\cdot)] \leq \mathscr{F}[v_0] \, e^{-4t}$$

A computation on a large ball, with boundary terms

Carré du champ method [Carrillo, Toscani] [Carrillo, Vázquez] [Carrillo, Jüngel, Toscani, Markowich, Unterreiter]

$$\frac{\partial u}{\partial t} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\,x \right) \right] = 0 \quad t > 0\,, \quad x \in B_R$$

$$\left(\nabla v^{m-1} - 2\,x \right) \cdot \frac{x}{|x|} = 0 \quad t > 0\,, \quad x \in \partial B_R$$

$$\frac{d}{dt} \int_{B_R} v\,|\nabla v^{m-1} - 2\,x|^2\,dx + 4 \int_{B_R} v\,|\nabla v^{m-1} - 2\,x|^2\,dx$$

$$+ 2\,\tfrac{1-m}{m} \int_{B_R} v^m \left(\left\| \mathrm{D}^2 (v^{m-1} - \mathscr{B}^{m-1}) \right\|^2 - (1-m) \left| \Delta (v^{m-1} - \mathscr{B}^{m-1}) \right|^2 \right) dx$$

$$= \int_{\partial B_R} v^m \left(\omega \cdot \nabla |(v^{m-1} - \mathscr{B}^{m-1})|^2 \right) d\sigma \le 0 \text{ (by Grisvard's lemma)}$$

Improvement: $\exists \phi$ such that $\phi'' > 0$, $\phi(0) = 0$ and $\phi'(0) = 4$ [Toscani, JD]

$$\mathscr{I}[v|\mathscr{B}_\sigma] \ge \phi(\mathscr{F}[v|\mathscr{B}_\sigma]) \quad \Longleftarrow \quad \text{idea: } \frac{d\mathscr{I}}{dt} + 4\,\mathscr{I} \lesssim -\frac{\mathscr{I}}{\mathscr{F}^2}$$

Spectral gap: sharp asymptotic rates of convergence

Assumptions on the initial datum v_0

(H_1) $\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}}$

(H_2) *if* $d \ge 3$ *and* $m \le m_* := \frac{d-4}{d-2}$, *then* $(v_0-\mathscr{B})$ *is integrable*

Theorem

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009] *If* $m<1$ *and* $m \neq m_*$, *then*

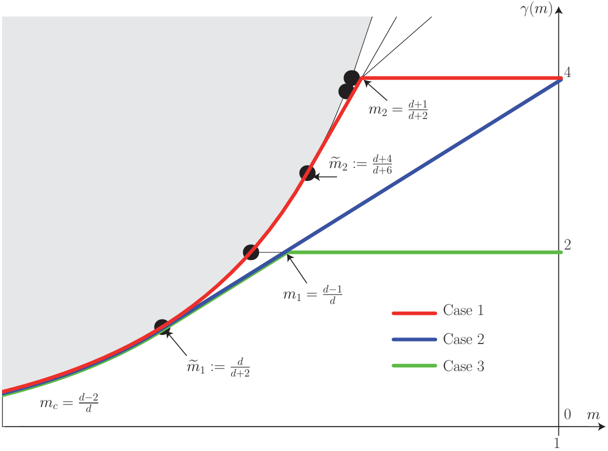
$$\mathscr{F}[v(t,\cdot)] \le C\,e^{-2\gamma(m)\,t} \quad \forall\, t \ge 0, \quad \gamma(m) := (1-m)\,\Lambda_{\alpha,d}$$

where $\Lambda_{\alpha,d}>0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d}|f|^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d}|\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f \in \mathrm{H}^1(\mathrm{d}\mu_\alpha), \quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\alpha := \frac{1}{m-1}<0$, $\mathrm{d}\mu_\alpha := h_\alpha\,dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

Spectral gap and the asymptotic time layer



$$\mathscr{F}[v(t,\cdot)] \le C\, e^{-2\gamma(m)\, t} \quad \forall\, t \ge 0$$

[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]

Spectral gap and improvements... the details

▷ *Asymptotic time layer* [BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]

Corollary

Assume that v solves (4): $\partial_t v + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2x \right) \right] = 0$ with initial datum $v_0 \ge 0$ such that $\int_{\mathbb{R}^d} v_0 \,\mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B} \,\mathrm{d}x$

(i) *there is a constant $\mathscr{C}_1 > 0$ such that $\mathscr{F}[v(t,\cdot)] \le \mathscr{C}_1 \, e^{-2\gamma(m)\,t}$ with $\gamma(m) = 2$ if $m_1 \le m < 1$*

(ii) *if $m_1 \le m < 1$ and $\int_{\mathbb{R}^d} x \, v_0 \,\mathrm{d}x = 0$, there is a constant $\mathscr{C}_2 > 0$ such that $\mathscr{F}[v(t,\cdot)] \le \mathscr{C}_2 \, e^{-2\gamma(m)\,t}$ with $\gamma(m) = 4 - 2\,d\,(1-m)$*

(iii) *Assume that $\frac{d+1}{d+2} \le m < 1$ and $\int_{\mathbb{R}^d} x \, v_0 \,\mathrm{d}x = 0$ and let*

$$\mathscr{B}_\sigma := \sigma^{-\frac{d}{2}} \, \mathscr{B}\big(x/\sqrt{\sigma}\big)$$

be such that $\int_{\mathbb{R}^d} |x|^2 \, u(t,x) \,\mathrm{d}x = \int_{\mathbb{R}^d} |x|^2 \, \mathscr{B}_\sigma(x) \,\mathrm{d}x$. Then there is a constant $\mathscr{C}_3 > 0$ such that $\mathscr{F}[v(t,\cdot)|\mathscr{B}_\sigma] \le \mathscr{C}_3 \, e^{-4\,t}$

▷ *Initial time layer* $\mathscr{I}[v|\mathscr{B}_\sigma] \ge \phi(\mathscr{F}[v|\mathscr{B}_\sigma]) \Rightarrow$ faster decay for $t \sim 0$

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d}|g|^2\,\mathscr{B}^{2-m}\,\mathrm{d}x \quad\text{and}\quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d}|\nabla g|^2\,\mathscr{B}\,\mathrm{d}x$$

Hardy-Poincaré inequality. Let $d\ge 1$, $m\in(m_1,1)$ and $g\in\mathrm{L}^2(\mathbb{R}^d,\mathscr{B}^{2-m}\,dx)$ such that $\nabla g\in\mathrm{L}^2(\mathbb{R}^d,\mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,\mathrm{d}x=0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,\mathrm{d}x=0$

$$\mathsf{I}[g]\ge 4\,\alpha\,\mathsf{F}[g] \quad\text{where}\quad \alpha=2-d\,(1-m)$$

Proposition

Let $m\in(m_1,1)$ if $d\ge 2$, $m\in(1/3,1)$ if $d=1$, $\eta=2\,d\,(m-m_1)$ and $\chi=m/(266+56\,m)$. If $\int_{\mathbb{R}^d} v\,\mathrm{d}x=\mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,\mathrm{d}x=0$ and

$$(1-\varepsilon)\,\mathscr{B}\le v\le(1+\varepsilon)\,\mathscr{B}$$

for some $\varepsilon\in(0,\chi\,\eta)$, then

$$\mathscr{Q}[v] := \frac{\mathscr{I}[v]}{\mathscr{F}[v]}\ge 4+\eta$$

The initial time layer improvement: backward estimate

Rephrasing the *carré du champ* method, $\mathcal{Q}[v] := \frac{\mathscr{I}[v]}{\mathscr{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt} \le \mathcal{Q}\,(\mathcal{Q} - 4)$$

Lemma

Assume that $m > m_1$ and v is a solution to (4) with nonnegative initial datum v_0. If for some $\eta > 0$ and $T > 0$, we have $\mathcal{Q}[v(T,\cdot)] \ge 4 + \eta$, then

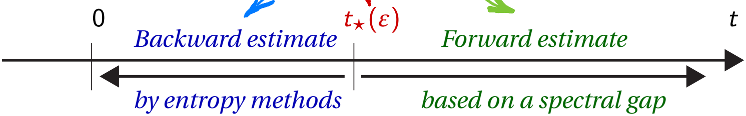
$$\mathcal{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\,e^{-4\,T}}{4 + \eta - \eta\,e^{-4\,T}} \quad \forall\, t \in [0, T]$$

Regularity and stability

Our strategy

Choose $\varepsilon > 0$*, small enough*

Get a threshold time $t_\star(\varepsilon)$



Initial time layer

Asymptotic time layer

Uniform convergence in relative error: statement

Theorem

Assume that $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$ *and let* $\varepsilon \in (0, 1/2)$, *small enough,* $A > 0$, *and* $G > 0$ *be given. There exists an explicit time* $t_\star \geq 0$ *such that, if* u *is a solution of*

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *satisfying*

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} u \, dx \leq A < \infty \tag{H_A}$$

$\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B} \, \mathrm{d}x = \mathscr{M}$ *and* $\mathscr{F}[u_0] \leq G$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall \, t \geq t_\star$$

The threshold time

Proposition

Let $m\in(m_1,1)$ *if* $d\ge2$, $m\in(1/3,1)$ *if* $d=1$, $\varepsilon\in(0,\varepsilon_{m,d})$, $A>0$ *and* $G>0$

$$t_\star=\mathsf{c}_\star\,\frac{1+A^{1-m}+G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a}=\frac{\alpha}{\vartheta}\,\frac{2-m}{1-m}$ *and* $\vartheta=\nu/(d+\nu)$

$$\mathsf{c}_\star=\mathsf{c}_\star(m,d)=\sup_{\varepsilon\in(0,\varepsilon_{m,d})}\max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\,\varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\,\varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m):=\max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1},\frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m):=\frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}}\quad\text{and}\quad\kappa_3(\varepsilon,m):=\frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

Improved entropy-entropy production inequality

Theorem

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. Then there is a positive number ζ such that

$$\mathscr{I}[v] \geq (4 + \zeta)\, \mathscr{F}[v]$$

for any nonnegative function $v \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v] = G$, $\int_{\mathbb{R}^d} v\, \mathrm{d}x = \mathscr{M}$, $\int_{\mathbb{R}^d} x\, v\, \mathrm{d}x = 0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon \in (0, 2\varepsilon_\star)\,, \quad \varepsilon_\star := \frac{1}{2} \min\{\varepsilon_{m,d}, \chi\,\eta\} \quad \text{with} \quad T = \frac{1}{2} \log R(t_\star)$$

$$(1 - \varepsilon)\, \mathscr{B} \leq v(t, \cdot) \leq (1 + \varepsilon)\, \mathscr{B} \quad \forall\, t \geq T$$

and, as a consequence, the *initial time layer estimate*

$$\mathscr{I}[v(t,.)] \geq (4 + \zeta)\, \mathscr{F}[v(t,.)] \quad \forall\, t \in [0, T]\,, \quad \text{where} \quad \zeta = \frac{4\,\eta\, e^{-4\,T}}{4 + \eta - \eta\, e^{-4\,T}}$$

Two consequences

$$\zeta = \mathsf{Z}(A,\mathscr{F}[u_0])\,,\quad \mathsf{Z}(A,G) := \frac{\zeta_\star}{1+A^{(1-m)\,\frac{2}{\alpha}}+G}\,,\quad \zeta_\star := \frac{4\,\eta}{4+\eta}\left(\frac{\varepsilon_\star^{\mathsf{a}}}{2\,\alpha\,\mathsf{c}_\star}\right)^{\frac{2}{\alpha}} c_\alpha$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. If v is a solution of (4) with nonnegative initial datum $v_0\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v_0]=G$, $\int_{\mathbb{R}^d} v_0\,\mathrm{d}x=\mathscr{M}$, $\int_{\mathbb{R}^d} x\,v_0\,\mathrm{d}x=0$ and v_0 satisfies (H_A), then

$$\mathscr{F}[v(t,.)]\le\mathscr{F}[v_0]\,e^{-(4+\zeta)\,t}\quad\forall\,t\ge0$$

▷ The stability in the entropy - entropy production estimate $\mathscr{I}[v]-4\,\mathscr{F}[v]\ge\zeta\,\mathscr{F}[v]$ also holds in a stronger sense

$$\mathscr{I}[v]-4\,\mathscr{F}[v]\ge\frac{\zeta}{4+\zeta}\,\mathscr{I}[v]$$

A general stability result

$$\lambda[f] := \left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2p}{d-p(d-4)}}$$

$$\mathsf{A}[f] := \frac{\mathscr{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}}\,\|f\|_{2p}^{2p}}\,\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f(x+x_f)|^{2p}\,dx$$

$$\mathsf{E}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}}\,|f|^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p}\,\mathrm{g}^{1-p}\left(\frac{\kappa[f]^{2p}}{\lambda[f]^2}\,|f|^{2p} - \mathrm{g}^{2p}\right)\right)\mathrm{d}x$$

$$\mathfrak{S}[f] := \frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf{Z}(\mathsf{A}[f],\mathsf{E}[f])$$

Theorem

Let $d \ge 1$ and $p \in (1,p^*)$. For any $f \in \mathscr{W}$, we have

$$\left(\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\right)^{2p\gamma} - \left(\mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p}\right)^{2p\gamma} \ge \mathfrak{S}[f]\,\|f\|_{2p}^{2p\gamma}\,\mathsf{E}[f]$$

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Preprint https://hal.archives-ouvertes.fr/hal-02887013

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Thank you for your attention !