Entropy methods and nonlinear diffusions: functional inequalities on manifolds and on weighted Euclidean spaces

Jean Dolbeault

http://www.ceremade.dauphine.fr/∼dolbeaul

Ceremade, Université Paris-Dauphine

June 14, 2016

Workshop on Entropy methods, dissipative systems, and applications
Thematic Program: Nonlinear Flows
ESI Vienna, June 13 – 17, 2016

Outline

- The *fast diffusion equation on the sphere* (or on a compact manifold): a tool based on the Bakry-Emery method for the investigation of some sharp functional inequalities

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

- Bifurcations, entropies and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities

$$\left(\int_{\mathbb{R}^d} |w|^{2p}\,\frac{dx}{|x|^\gamma}\right)^{\frac{1}{2p}} \leq \mathsf{C}_{\beta,\gamma,p}\left(\int_{\mathbb{R}^d} |\nabla w|^2\,\frac{dx}{|x|^\beta}\right)^{\frac{1}{2\vartheta}}\left(\int_{\mathbb{R}^d} |w|^{p+1}\,\frac{dx}{|x|^\gamma}\right)^{\frac{1-\vartheta}{p+1}}$$

- The *fast diffusion equation with weights*

$$u_t + |x|^\gamma\,\nabla\cdot\left(|x|^{-\beta}\,u\,\nabla u^{m-1}\right) = 0 \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$$

  - Simple facts
  - Large time asymptotics
  - A symmetry result: Rényi entropy powers

Inequalities and flows on compact manifolds

▷ Flows on the sphere

▷ Can one prove Sobolev's inequalities with a heat flow ?

▷ The *bifurcation* point of view

▷ Some open problems: constraints and improved inequalities

[Bakry, Emery, 1984]
[Bidault-Véron, Véron, 1991], [Bakry, Ledoux, 1996]
[Demange, 2008][JD, Esteban, Loss, 2014 & 2015]

The interpolation inequalities

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \ge 1$, $p \neq 2$, is such that

$$p \le 2^* := \frac{2\,d}{d-2}$$

if $d \ge 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \setminus \{0\}$$

The Bakry-Emery method

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and compute $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$ and $\frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

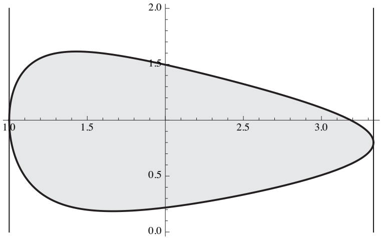
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m \,. \tag{1}$$

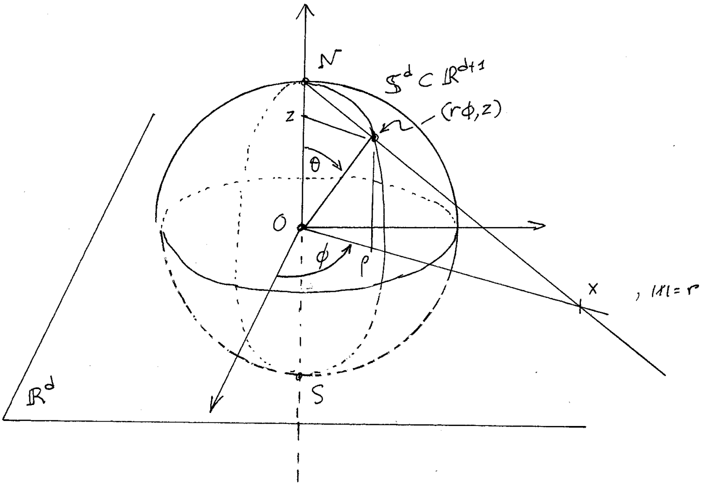
[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) *admissible region*, $d = 5$

Cylindrical coordinates, Schwarz symmetrization, stereographic projection...



... and the ultra-spherical operator

Change of variables $z=\cos\theta$, $v(\theta)=f(z)$, $d\nu_d:=\nu^{\frac{d}{2}-1}\,dz/Z_d$, $\nu(z):=1-z^2$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\,f:=(1-z^2)\,f''-d\,z\,f'=\nu\,f''+\frac{d}{2}\,\nu'\,f'$$

which satisfies $\langle f_1,\mathcal{L}\,f_2\rangle=-\int_{-1}^1 f_1'\,f_2'\,\nu\,d\nu_d$

Proposition

Let $p\in[1,2)\cup(2,2^*]$, $d\ge1$. For any $f\in\mathrm{H}^1([-1,1],d\nu_d)$,

$$-\langle f,\mathcal{L}\,f\rangle=\int_{-1}^1|f'|^2\,\nu\,d\nu_d\ge d\,\frac{\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2}$$

The heat equation $\frac{\partial g}{\partial t} = \mathcal{L}\, g$ for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f, \mathcal{L}\, f\rangle = \langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\left\langle \frac{|f'|^2}{f}\,\nu, \mathcal{L}\, f\right\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\,d\,\mathcal{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$

$$= -\,2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\,\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} = 2^\# < \frac{2\,d}{d-2} = 2^*$$

The rigidity point of view (nonlinear flow)

$u_t = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$... Which computation do we have to do ?

$$-\mathcal{L}\,u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\;d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

The two terms cancel and we are left only with

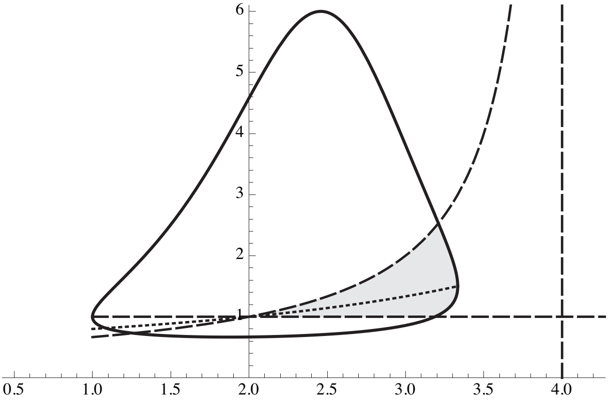
$$\int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\;d\nu_d = 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

Improved functional inequalities

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

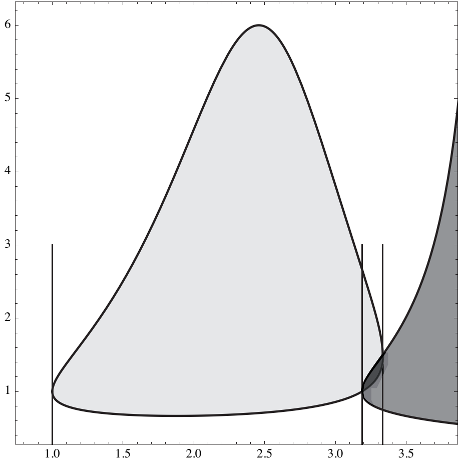
$$\rho = |u|^{\beta p}$$

$$m = 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right)$$



(p, β) *representation of the admissible range of parameters when* $d = 5$
[JD, Esteban, Kowalczyk, Loss]

Can one prove Sobolev's inequalities with a heat flow ?

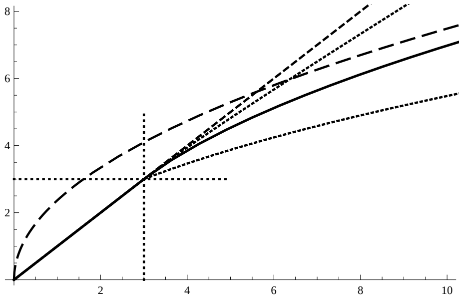


(p, β) *representation when* $d = 5$*. In the dark grey area, the functional is not monotone under the action of the heat flow [JD, Esteban, Loss]*

The bifurcation point of view

$\mu(\lambda)$ is the optimal constant in the functional inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$



Here $d = 3$ *and* $p = 4$

- A critical point of $u \mapsto \mathcal{Q}_\lambda[u] := \dfrac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}$ solves

$$-\Delta u + \lambda\, u = |u|^{p-2}\, u \qquad \text{(EL)}$$

up to a multiplication by a constant (and a conformal transformation if $p = 2^*$)

- The best constant $\mu(\lambda) = \inf_{u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}} \mathcal{Q}_\lambda[u]$ is such that $\mu(\lambda) < \lambda$ if $\lambda > \frac{d}{p-2}$, and $\mu(\lambda) = \lambda$ if $\lambda \le \frac{d}{p-2}$ so that

$$\frac{d}{p-2} = \min\{\lambda > 0 \,:\, \mu(\lambda) < \lambda\}$$

- *Rigidity* : the unique positive solution of (EL) is $u = \lambda^{1/(p-2)}$ if $\lambda \le \frac{d}{p-2}$ **[...]**

Constraints and improvements

- Taylor expansion:

$$d = \inf_{u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \setminus \{0\}} \frac{(p-2)\, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

is achieved in the limit as $\varepsilon \to 0$ with $u = 1 + \varepsilon\, \varphi_1$ such that

$$-\Delta \varphi_1 = d\, \varphi_1$$

▷ This suggest that improved inequalities can be obtained under appropriate orthogonality constraints...

Integral constraints

With the heat flow...

Proposition

For any $p \in (2, 2^\#)$, *the inequality*

$$\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{\lambda}{p-2} \, \|f\|_2^2 \geq \frac{\lambda}{p-2} \, \|f\|_p^2$$

$$\forall \, f \in \mathrm{H}^1((-1,1), d\nu_d) \ s.t. \int_{-1}^{1} z \, |f|^p \, d\nu_d = 0$$

holds with

$$\lambda \geq d + \frac{(d-1)^2}{d\,(d+2)} \, (2^\# - p) \, (\lambda^\star - d)$$

... and with a nonlinear diffusion flow ?

Antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x) = u(x) \quad \forall\, x \in \mathbb{S}^d$$

Theorem

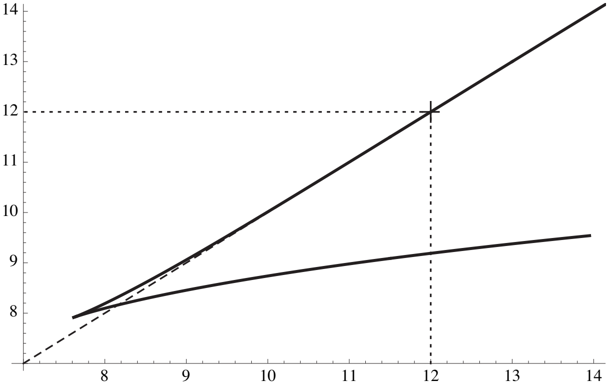
If $p \in (1,2) \cup (2,2^*)$, we have

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \ge \frac{d}{p-2} \left[1 + \frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1}\right] \left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ with antipodal symmetry. The limit case $p=2$ corresponds to the improved logarithmic Sobolev inequality

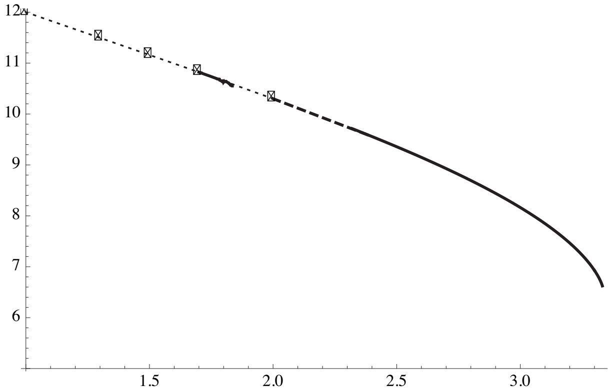
$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \ge \frac{d}{2} \frac{(d+3)^2}{(d+1)^2} \int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$

The larger picture: branches of antipodal solutions



Case $d = 5$, $p = 3$: values of the shooting parameter a as a function of λ

The optimal constant in the antipodal framework



Numerical computation of the optimal constant when $d = 5$ *and* $1 \leq p \leq 10/3 \approx 3.33$. *The limiting value of the constant is numerically found to be equal to* $\lambda_\star = 2^{1-2/p}\, d \approx 6.59754$ *with* $d = 5$ *and* $p = 10/3$

Fast diffusion equations with weights: simple facts

- The equation and the self-similar solutions
- Without weights
- A perturbation result
- Symmetry breaking

New results: joint work with M. Bonforte, M. Muratori and B. Nazaret

Fast diffusion equations with weights: self-similar solutions

Let us consider the *fast diffusion equation with weights*

$$u_t + |x|^\gamma \, \nabla \cdot \left(|x|^{-\beta} \, u \, \nabla u^{m-1} \right) = 0 \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$$

Here β and γ are two real parameters, and $m \in [m_1, 1)$ with

$$m_1 := \frac{2\,d-2-\beta-\gamma}{2\,(d-\gamma)}$$

Generalized *Barenblatt self-similar solutions*

$$u_\star(\rho\, t, x) = t^{-\,\rho\,(d-\gamma)} \, \mathfrak{B}_{\beta,\gamma}\left(t^{-\rho}\, x\right)\,, \quad \mathfrak{B}_{\beta,\gamma}(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{\frac{1}{m-1}}$$

where $1/\rho = (d-\gamma)\,(m-m_c)$ with $m_c := \frac{d-2-\beta}{d-\gamma} < m_1 < 1$
Self-similar solutions are known to govern the asymptotic behavior of the solutions when $(\beta,\gamma) = (0,0)$

- Mass conservation

$$\frac{d}{dt}\int_{\mathbb{R}^d} u\,\frac{dx}{|x|^\gamma}=0$$

and self-similar solutions suggest to introduce the

- Time-dependent rescaling

$$u(t,x)=R^{\gamma-d}\,v\left((2+\beta-\gamma)^{-1}\log R,\frac{x}{R}\right)$$

with $R=R(t)$ defined by

$$\frac{dR}{dt}=(2+\beta-\gamma)\,R^{(m-1)(\gamma-d)-(2+\beta-\gamma)+1}\,,\quad R(0)=1$$

$$R(t)=\left(1+\tfrac{2+\beta-\gamma}{\rho}\,t\right)^\rho$$

with $1/\rho=(1-m)\,(\gamma-d)+2+\beta-\gamma=(d-\gamma)\,(m-m_c)$

- A *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[\,|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0$$

with initial condition $v(t=0,\cdot)=u_0$

Without weights: time-dependent rescaling, free energy

- *Time-dependent rescaling*: Take $u(\tau, y) = R^{-d}(\tau)\, v\,(t, y/R(\tau))$ where

$$\frac{dR}{d\tau} = R^{d(1-m)-1}\;, \quad R(0) = 1\;, \quad t = \log R$$

- The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x\, v)\;, \quad v_{|t=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{F}[v] := \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac12 |x|^2 v \right) dx - \mathcal{F}_0$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{F}[v] = -\mathcal{I}[v]\;, \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$

Without weights: relative entropy, entropy production

- *Stationary solution:* choose C such that $\|v_\infty\|_{\mathrm{L}^1} = \|u\|_{\mathrm{L}^1} = M > 0$

$$v_\infty(x) := \left(C + \tfrac{1-m}{2\,m}\,|x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{F}_0$ so that $\mathcal{F}[v_\infty] = 0$

- *Entropy – entropy production inequality*

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[v] \geq 2\,\mathcal{F}[v]$$

Corollary

A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[u_0]\,e^{-\,2\,t}$

More simple facts...

▷ The entropy – entropy production inequality is equivalent to the Gagliardo-Nirenberg inequality
[del Pino, J.D.] With $1 < p \leq \frac{d}{d-2}$ (fast diffusion case) and $d \geq 3$

$$\|w\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \leq \mathcal{C}_{p,d}^{\mathrm{GN}} \|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta} \|w\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

Proofs: variational methods [del Pino, J.D.], or *carré du champ method* (Bakry-Emery): [Carrillo, Toscani], [Carrillo, Vázquez], [CJMTU]

▷ Sharp asymptotic rates are determined by the spectral gap in the linearized entropy – entropy production (Hardy–Poincaré) inequality [Blanchet, Bonforte, J.D., Grillo, Vázquez]

▷ Higher order matching asymptotics can be achieved by *best matching* methods: [Bonforte, J.D., Grillo, Vázquez], [J.D., Toscani]
[...]

...

$\triangleright$ *Improved entropy – entropy production inequalities* $\varphi(\mathcal{F}[v]) \leq \mathcal{I}[v]$ can be proved [J.D., Toscani], [Carrillo, Toscani]

$\triangleright$ *Rényi entropy powers:* concavity, asymptotic regime (self-similar solutions) and Gagliardo-Nirenberg inequalities in scale invariant form [Savaré, Toscani], [J.D., Toscani]

$\triangleright$ Concavity of second moment estimates and *delays* [J.D., Toscani]

$\triangleright$ *Stability* of entropy – entropy production inequalities (scaling methods), and improved rates of convergence [Carrillo, Toscani], [J.D., Toscani]

With one weight: a perturbation result

On the space of smooth functions on $\mathbb{R}^d$ with compact support

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_\gamma \, \|\nabla w\|^{\vartheta}_{\mathrm{L}^2(\mathbb{R}^d)} \, \|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}$$

where $\vartheta := \frac{2^*_\gamma\,(p-1)}{2\,p\left(2^*_\gamma-p-1\right)} = \frac{(d-\gamma)\,(p-1)}{p\,\left(d+2-2\,\gamma-p\,(d-2)\right)}$ and

$$\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q} \quad \text{and} \quad \|w\|_{\mathrm{L}^q(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$$

and $d \ge 3$, $\gamma \in (0,2)$, $p \in \left(1, 2^*_\gamma/2\right)$ with $2^*_\gamma := 2\,\frac{d-\gamma}{d-2}$

Theorem

[J.D., Muratori, Nazaret] *Let $d \ge 3$. For any $p \in (1, d/(d-2))$, there exists a positive γ^* such that equality holds for all $\gamma \in (0,\gamma^*)$ with*

$$w_\star(x) := \left(1+|x|^{2-\gamma}\right)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

Caffarelli-Kohn-Nirenberg inequalities (with two weights)

Norms: $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q}$, $\|w\|_{\mathrm{L}^{q}(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$
(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p}\,\|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}\,\|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \qquad \text{(CKN)}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \ge 2\,, \quad \gamma-2<\beta<\frac{d-2}{d}\,\gamma\,, \quad \gamma\in(-\infty,d)\,, \quad p\in(1,p_\star] \quad \text{with } p_\star := \frac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

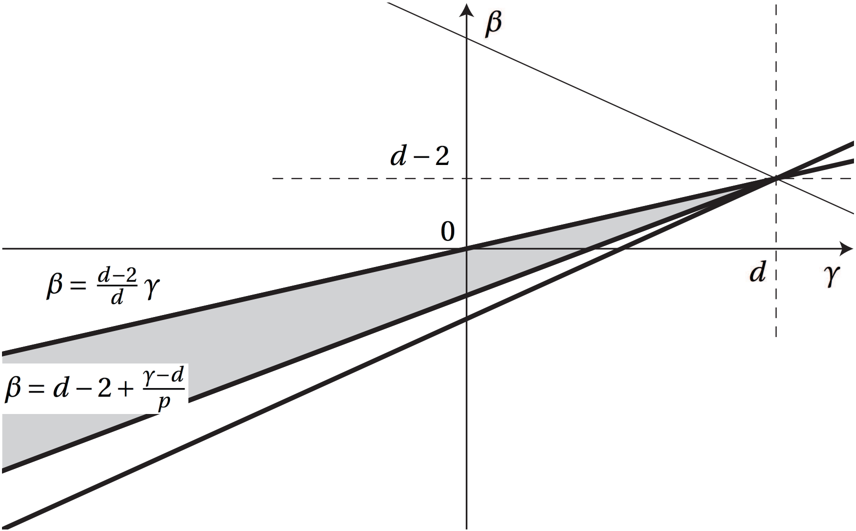
$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$$

- Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\,x\in\mathbb{R}^d \quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions?

Range of the parameters



CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\frac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{F}[v]\le\mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}$. Here the *free energy* and the *relative Fisher information* are defined by

$$\mathcal{F}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\frac{dx}{|x|^\beta}\,.$$

If v solves the *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0\qquad\text{(WFDE-FP)}$$

then

$$\frac{d}{dt}\mathcal{F}[v(t,\cdot)]=-\frac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$$

Proposition

Let $m=\frac{p+1}{2\,p}$ *and consider a solution to (WFDE-FP) with nonnegative initial datum* $u_0\in\mathrm L^{1,\gamma}(\mathbb R^d)$ *such that* $\|u_0^m\|_{\mathrm L^{1,\gamma}(\mathbb R^d)}$ *and* $\int_{\mathbb R^d}u_0\,|x|^{2+\beta-2\gamma}\,dx$ *are finite. Then*

$$\mathcal F[v(t,\cdot)]\le\mathcal F[u_0]\,e^{-\,(2+\beta-\gamma)^2\,t}\quad\forall\,t\ge0$$

if one of the following two conditions is satisfied:
(i) either u_0 *is a.e. radially symmetric*
(ii) or symmetry holds in (CKN)

With two weights: a symmetry breaking result

Let us define

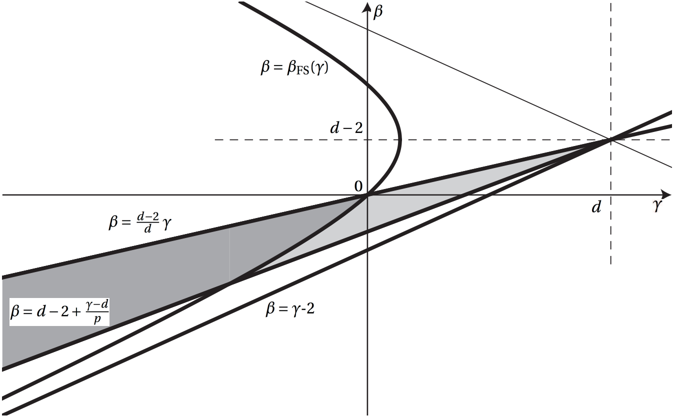
$$\beta_{\rm FS}(\gamma) := d-2-\sqrt{(d-\gamma)^2-4\,(d-1)}$$

Theorem

Symmetry breaking holds in (CKN) if

$$\gamma<0 \quad \text{and} \quad \beta_{\rm FS}(\gamma)<\beta<\frac{d-2}{d}\,\gamma$$

In the range $\beta_{\rm FS}(\gamma)<\beta<\frac{d-2}{d}\,\gamma$, $w_\star(x)=\left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$ *is not* optimal.



The grey area corresponds to the admissible cone. The light grey area is the region of symmetry, while the dark grey area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$$

A useful change of variables

With

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d - \gamma}{\beta + 2 - \gamma}\,,$$

(CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta}$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$. Parameters are in the range

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,, \quad p_\star := \frac{n}{n-2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha < \alpha_{\mathrm{FS}} \quad \text{with} \quad \alpha_{\mathrm{FS}} := \sqrt{\frac{d-1}{n-1}}$$

The second variation

$$\mathcal{J}[v] := \vartheta \log\left(\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\right) + (1-\vartheta) \log\left(\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}\right) + \log \mathsf{K}_{\alpha,n,p} - \log\left(\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\right)$$

Let us define $d\mu_\delta := \mu_\delta(x)\,dx$, where $\mu_\delta(x) := (1+|x|^2)^{-\delta}$. Since $v_\star$ is a critical point of $\mathcal{J}$, a Taylor expansion at order ε^2 shows that

$$\|\mathfrak{D}_\alpha v_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\mathcal{J}\left[v_\star + \varepsilon\,\mu_{\delta/2}\,f\right] = \tfrac{1}{2}\,\varepsilon^2\,\vartheta\,\mathcal{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2\,p}{p-1}$ and

$$\mathcal{Q}[f] = \int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\,|x|^{n-d}\,d\mu_\delta - \frac{4\,p\,\alpha^2}{p-1}\int_{\mathbb{R}^d} |f|^2\,|x|^{n-d}\,d\mu_{\delta+1}$$

We assume that $\int_{\mathbb{R}^d} f\,|x|^{n-d}\,d\mu_{\delta+1} = 0$ (mass conservation)

Symmetry breaking: the proof

Proposition (Hardy-Poincaré inequality)

Let $d \geq 2$, $\alpha \in (0, +\infty)$, $n > d$ and $\delta \geq n$. If f has 0 average, then

$$\int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2 \, |x|^{n-d} \, d\mu_\delta \geq \Lambda \int_{\mathbb{R}^d} |f|^2 \, |x|^{n-d} \, d\mu_{\delta+1}$$

with optimal constant $\Lambda = \min\{2\,\alpha^2\,(2\,\delta - n), 2\,\alpha^2\,\delta\,\eta\}$ where η is the unique positive solution to $\eta\,(\eta + n - 2) = (d-1)/\alpha^2$. The corresponding eigenfunction is not radially symmetric if $\alpha^2 > \frac{(d-1)\,\delta^2}{n\,(2\,\delta - n)\,(\delta - 1)}$.

$\mathcal{Q} \geq 0$ iff $\frac{4\,p\,\alpha^2}{p-1} \leq \Lambda$ and symmetry breaking occurs in (CKN) if

$$2\,\alpha^2\,\delta\,\eta < \frac{4\,p\,\alpha^2}{p-1} \iff \eta < 1$$

$$\iff \frac{d-1}{\alpha^2} = \eta\,(\eta + n - 2) < n - 1 \iff \alpha > \alpha_{\rm FS}$$

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \, \nabla \cdot \Big[|x|^{-\beta} \, v \, \nabla \big(v^{m-1} - |x|^{2+\beta-\gamma} \big) \Big] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

Relative uniform convergence

$$\zeta := 1 - \left(1 - \tfrac{2-m}{(1-m)\,q}\right)\left(1 - \tfrac{2-m}{1-m}\,\theta\right)$$

$$\theta := \tfrac{(1-m)\,(2+\beta-\gamma)}{(1-m)\,(2+\beta)+2+\beta-\gamma} \text{ is in the range } 0 < \theta < \tfrac{1-m}{2-m} < 1$$

Theorem

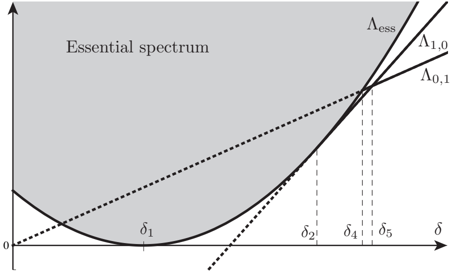
For "good" initial data, there exist positive constants $\mathcal{K}$ and t_0 such that, for all $q \in \left[\frac{2-m}{1-m}, \infty\right]$, the function $w = v/\mathfrak{B}$ satisfies

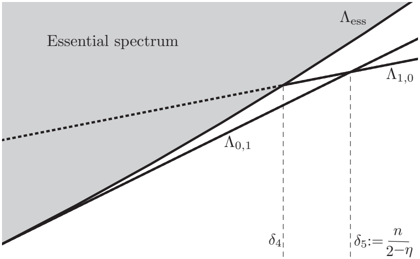
$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,\zeta\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \in (0, d)$, and

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \le 0$





The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\rm ess}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions **[...]**

Main steps of the proof:

- Existence of weak solutions, $\mathrm{L}^{1,\gamma}$ contraction, Comparison Principle, conservation of relative mass
- Self-similar variables and the Ornstein-Uhlenbeck equation in relative variables: the ratio $w(t,x) := v(t,x)/\mathfrak{B}(x)$ solves

$$\begin{cases} |x|^{-\gamma}\,w_t = -\,\frac{1}{\mathfrak{B}}\,\nabla\cdot\Big(|x|^{-\beta}\,\mathfrak{B}\,w\,\nabla\left(\left(w^{m-1}-1\right)\mathfrak{B}^{m-1}\right)\Big) & \text{in } \mathbb{R}^+\times\mathbb{R}^d\\ w(0,\cdot)=w_0:=v_0/\mathfrak{B} & \text{in } \mathbb{R}^d \end{cases}$$

- *Regularity*, relative uniform convergence (without rates) and asymptotic rates (linearization)
- The relative free energy and the relative Fisher information: linearized free energy and linearized Fisher information
- A Duhamel formula and a bootstrap

Regularity (1/2): Harnack inequality and Hölder regularity

We change variables: $x \mapsto |x|^{\alpha-1}\,x$ and adapt the ideas of F. Chiarenza and R. Serapioni to

$$u_t + \mathsf{D}_\alpha^*\Big[\mathsf{a}\,(\mathsf{D}_\alpha\,u + \mathsf{B}\,u)\Big] = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d$$

Proposition (A parabolic Harnack inequality)

Let $d \ge 2$, $\alpha > 0$ and $n > d$. If u is a bounded positive solution, then for all $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $r > 0$ such that $Q_r(t_0, x_0) \subset \mathbb{R}^+ \times B_1$, we have

$$\sup_{Q_r^-(t_0,x_0)} u \le H \inf_{Q_r^+(t_0,x_0)} u$$

The constant $H > 1$ depends only on the local bounds on the coefficients a, B *and on d, α, and n*

By adapting the classical method *à la De Giorgi* to our weighted

Regularity (1/2): from local to global estimates

Lemma

If w is a solution of the the Ornstein-Uhlenbeck equation with initial datum w_0 bounded from above and from below by a Barenblatt profile (+ relative mass condition) = "good solutions", then there exist $\nu \in (0,1)$ and a positive constant $\mathcal{K} > 0$, depending on d, m, β, γ, C, C_1, C_2 such that:

$$\|\nabla v(t)\|_{\mathrm{L}^\infty(B_{2\lambda}\setminus B_\lambda)} \le \frac{Q_1}{\lambda^{\frac{2+\beta-\gamma}{1-m}+1}} \quad \forall\, t \ge 1\,, \quad \forall\, \lambda > 1\,,$$

$$\sup_{t\ge1} \|w\|_{C^k((t,t+1)\times B_\varepsilon^c)} < \infty \quad \forall\, k \in \mathbb{N}\,,\ \forall\, \varepsilon > 0$$

$$\sup_{t\ge1} \|w(t)\|_{C^\nu(\mathbb{R}^d)} < \infty$$

$$\sup_{\tau\ge t} |w(\tau)-1|_{C^\nu(\mathbb{R}^d)} \le \mathcal{K} \sup_{\tau\ge t} \|w(\tau)-1\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \quad \forall\, t \ge 1$$

Asymptotic rates of convergence

Corollary

Assume that $m \in (0,1)$, with $m \neq m_*$ with $m_* :=$. Under the relative mass condition, for any "good solution" v there exists a positive constant $\mathcal{C}$ such that

$$\mathcal{F}[v(t)] \leq \mathcal{C}\, e^{-\,2\,(1-m)\,\Lambda\, t} \quad \forall\, t \geq 0\,.$$

- With Csiszár-Kullback-Pinsker inequalities, these estimates provide a rate of convergence in $\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$
- Improved estimates can be obtained using "best matching techniques"

From asymptotic to global estimates

When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{F}[v]\le\frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{F}[v(t)]\le\mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t}\quad\forall\,t\ge0\quad\text{with}\quad\Lambda_\star:=\frac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{F}[v]\le\mathcal{I}[v]\quad\forall\,v\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d)\quad\text{such that}\quad\|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}=M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M):=\frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{F}[v(t)]\le\mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t}\quad\forall\,t\ge0$$

Symmetry breaking and global entropy – entropy production inequalities

Proposition

- *In the symmetry breaking range of (CKN), for any* $M > 0$*, we have*
$$0 < \mathcal{K}(M) \leq \frac{2}{m}\,(1-m)^2\,\Lambda_{0,1}$$
- *If symmetry holds in (CKN) then*
$$\mathcal{K}(M) \geq \frac{1-m}{m}\,(2+\beta-\gamma)^2$$

Corollary

Assume that $m \in [m_1, 1)$

(i) For any $M > 0$*, if* $\Lambda(M) = \Lambda_\star$ *then* $\beta = \beta_{\rm FS}(\gamma)$

(ii) If $\beta > \beta_{\rm FS}(\gamma)$ *then* $\Lambda_{0,1} < \Lambda_\star$ *and* $\Lambda(M) \in (0, \Lambda_{0,1}]$ *for any* $M > 0$

(iii) For any $M > 0$*, if* $\beta < \beta_{\rm FS}(\gamma)$ *and if symmetry holds in (CKN), then*
$$\Lambda(M) > \Lambda_\star$$

Fast diffusion equations with weights: a symmetry result

- Rényi entropy powers
- The symmetry result
- The strategy of the proof

Joint work with M.J. Esteban, M. Loss in the critical case $\beta = d - 2 + \frac{\gamma - d}{p}$

Joint work with M.J. Esteban, M. Loss and M. Muratori in the subcritical case $d - 2 + \frac{\gamma - d}{p} < \beta < \frac{d-2}{d}\,\gamma$

Rényi entropy powers

[Savaré, Toscani] We consider the flow $\frac{\partial u}{\partial t}=\Delta u^m$ and the Gagliardo-Nirenberg inequalities (GN)

$$\|w\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\le \mathcal{C}_{p,d}^{\mathrm{GN}}\,\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

where $u=w^{2p}$, that is, $w=u^{m-1/2}$ with $p=\frac{1}{2\,m-1}$. Straightforward computations show that (GN) can be brought into the form

$$\left(\int_{\mathbb{R}^d}u\,dx\right)^{(\sigma+1)\,m-1}\le C\,\mathcal{I}\,\mathcal{E}^{\sigma-1}\quad\text{where}\quad\sigma=\frac{2}{d\,(1-m)}-1$$

where $\mathcal{E}:=\int_{\mathbb{R}^d}u^m\,dx$ and $\mathcal{I}:=\int_{\mathbb{R}^d}u\,|\nabla\mathsf{P}|^2\,dx$, $\mathsf{P}=\frac{m}{1-m}\,u^{m-1}$ is the *pressure variable*. If $\mathcal{F}=\mathcal{E}^\sigma$ is the *Rényi entropy power* and $\sigma=\frac{2}{d}\,\frac{1}{1-m}-1$, then $\mathcal{F}''$ is proportional to

$$-2\,(1-m)\,\left\langle\mathrm{Tr}\left(\left(\mathrm{Hess}\,\mathsf{P}-\tfrac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right)^2\right)\right\rangle+(1-m)^2\,(1-\sigma)\,\left\langle\left(\Delta\mathsf{P}-\langle\Delta\mathsf{P}\rangle\right)^2\right\rangle$$

where we have used the notation $\langle A\rangle:=\int_{\mathbb{R}^d}u^m\,A\,dx/\int_{\mathbb{R}^d}u^m\,dx$

The symmetry result

▷ critical case: [J.D., Esteban, Loss; Inventiones]
▷ subcritical case: [J.D., Esteban, Loss, Muratori]

Theorem

Assume that $\beta \le \beta_{\rm FS}(\gamma)$. *Then all positive solutions in* $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ *of*

$$-\,div\left(|x|^{-\beta}\,\nabla w\right) = |x|^{-\gamma}\left(w^{2p-1} - \,w^p\right) \quad in \quad \mathbb{R}^d\setminus\{0\}$$

are radially symmetric and, up to a scaling and a multiplication by a constant, equal to $w_\star(x) = \left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$

The strategy of the proof (1/3)

The first step is based on a change of variables which amounts to rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise to consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$v(|x|^{\alpha-1}\,x) = w(x)\,,\quad \alpha = 1 + \frac{\beta-\gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta+2-\gamma}$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta} \quad \forall\,v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$ and

$$d \ge 2\,,\quad \alpha > 0\,,\quad n > d \quad \text{and} \quad p \in (1, p_\star]\,.$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall\,x \in \mathbb{R}^d$$

The strategy of the proof (2/3): concavity of the Rényi entropy power

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathfrak{D}_\alpha \mathsf{P}|^2 \, d\mu$$

where $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d} \, dx$ and the pressure is

$$\mathsf{P} := \frac{m}{1-m} \, u^{m-1}$$

Proving the symmetry in the inequality amounts to

proving the monotonicity of $\mathcal{G}[u]$

along a well chosen fast diffusion flow

With $\mathcal{L}_\alpha = -\mathcal{D}_\alpha^* \mathfrak{D}_\alpha = \alpha^2 \left(u'' + \frac{n-1}{s}\, u' \right) + \frac{1}{s^2}\, \Delta_\omega\, u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{L}_\alpha u^m$$

in the subcritical range $1 - 1/n < m < 1$. The key computation is the proof that

$$\begin{aligned} &- \tfrac{d}{dt}\, \mathcal{G}[u(t,\cdot)] \left(\textstyle\int_{\mathbb{R}^d} u^m \, d\mu \right)^{1-\sigma} \\ &\geq (1-m)\,(\sigma-1) \int_{\mathbb{R}^d} u^m \left| \mathcal{L}_\alpha \mathsf{P} - \frac{\int_{\mathbb{R}^d} u\, |\mathfrak{D}_\alpha \mathsf{P}|^2\, d\mu}{\int_{\mathbb{R}^d} u^m\, d\mu} \right|^2 d\mu \\ &+ 2 \int_{\mathbb{R}^d} \left(\alpha^4 \left(1 - \tfrac{1}{n}\right) \left| \mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega \mathsf{P}}{\alpha^2\,(n-1)\, s^2} \right|^2 + \frac{2\,\alpha^2}{s^2} \left| \nabla_\omega \mathsf{P}' - \frac{\nabla_\omega \mathsf{P}}{s} \right|^2 \right) u^m \, d\mu \\ &+ 2 \int_{\mathbb{R}^d} \left((n-2)\left(\alpha_{\rm FS}^2 - \alpha^2\right) |\nabla_\omega \mathsf{P}|^2 + c(n,m,d)\, \frac{|\nabla_\omega \mathsf{P}|^4}{\mathsf{P}^2} \right) u^m \, d\mu =: \mathcal{H}[u] \end{aligned}$$

for some numerical constant $c(n,m,d) > 0$. Hence if $\alpha \leq \alpha_{\rm FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result.

The strategy of the proof (3/3): integrations by parts

This method has a hidden difficulty: integrations by parts ! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings... to deduce decay estimates
- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere **[...]**

... without change of variables ?

if u solves the Euler-Lagrange equation, we can test it by $\mathcal{L}_\alpha u^m$

$$0=\int_{\mathbb{R}^d} \mathrm{d}\mathcal{G}[u]\cdot\mathcal{L}_\alpha u^m\,d\mu\ge\mathcal{H}[u]\ge 0$$

where the last inequality holds because $\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term). In original variables: test by $|x|^\gamma\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w^{1+p}\right)$ the equation

$$\frac{(p-1)^2}{p\,(p+1)}\,w^{1-3p}\,\mathrm{div}\left(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\right)+|\nabla w^{1-p}|^2+|x|^{-\gamma}\left(c_1\,w^{1-p}-c_2\right)=0$$

Remarks

- The fast diffusion equation (without weights) has a rich structure: a lot has been done (for instance, with parabolic methods or gradient flow techniques) and this is a fundamental equation to explore qualitative behaviors, sharp rates, *entropy methods in PDEs*, etc.

- With *weights*, self-similar Barenblatt solutions attract all solutions (in good spaces) on the long time range, the linearization of the entropy determines the sharp asymptotic rates... but when *symmetry breaking* occurs, there are other critical points and Barenblatt solutions are not optimal for entropy – entropy production ineq.

- Entropy methods can be used *as a tool* to produce symmetry / uniqueness / rigidity results which go well beyond results of elliptic PDEs (rearrangement, moving planes), energy / calculus of variations methods (concentration-compactness methods) and methods of spectral theory (so far).

- An example of doubly defective / degenerate operator, which is waiting for extension in (non-homogenous) kinetic equations !

Concluding remarks

▷ Adapted entropy methods (equivalence with a problem in a higher dimension, Rényi entropy powers) can be used to prove *symmetry results* in functional inequalities

▷ It is possible to adapt the *carré du champ* method to non-compact cases weighted Euclidean spaces in the case of the *Caffarelli-Kohn-Nirenberg inequalities*

$$\left(\int_{\mathbb{R}^d}|w|^{2p}\,\frac{dx}{|x|^\gamma}\right)^{\frac{1}{2p}}\le \mathsf{C}_{\beta,\gamma,p}\left(\int_{\mathbb{R}^d}|\nabla w|^2\,\frac{dx}{|x|^\beta}\right)^{\frac{1}{2\vartheta}}\left(\int_{\mathbb{R}^d}|w|^{p+1}\,\frac{dx}{|x|^\gamma}\right)^{\frac{1-\vartheta}{p+1}}$$

▷ The main difficulty is the justification of the integrations by parts

▷ The global rate of decay (measured in relative entropy) in the doubly weighted *fast diffusion equation with weights*

$$u_t+|x|^\gamma\,\nabla\cdot\left(|x|^{-\beta}\,u\,\nabla u^{m-1}\right)=0\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d$$

is in general not determined by the asymptotic regime (spectral gap associated with the linearized problem)

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
▷ Lectures

The papers can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/list/
▷ Preprints / papers

For final versions, use `Dolbeault` as login and `Jean` as password

Thank you for your attention !