On a result of symmetry based on nonlinear flows

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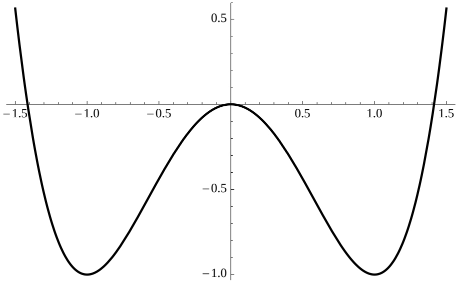
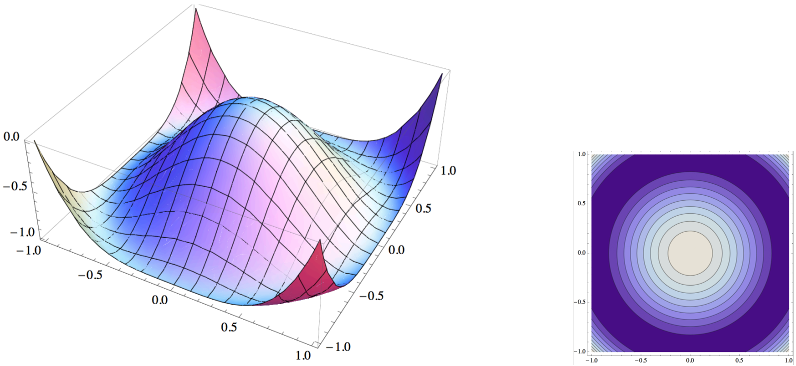
The *mexican hat* potential in Schrödinger equations

Let us consider a nonlinear Schrödinger equation in presence of a radial external potential with a minimum which is not at the origin

$$-\Delta u + V(x)\, u - f(u) = 0$$

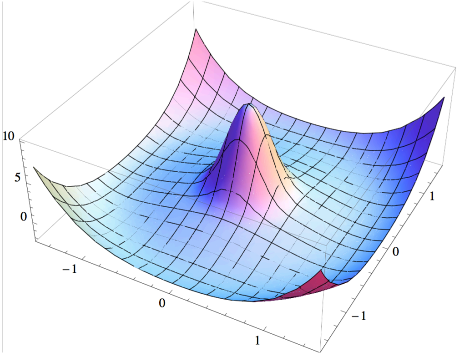


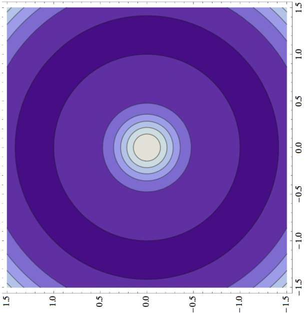
A one-dimensional potential $V(x)$



A two-dimensional potential $V(x)$ *with mexican hat shape*

Radial solutions to $-\Delta u + V(x)\,u - F'(u) = 0$

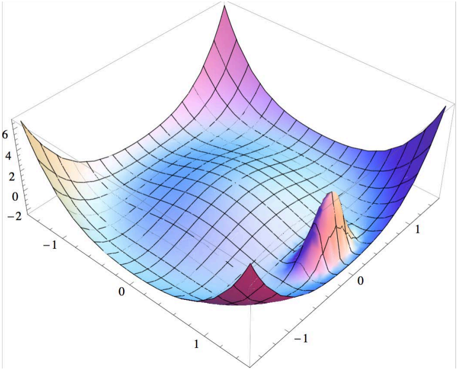


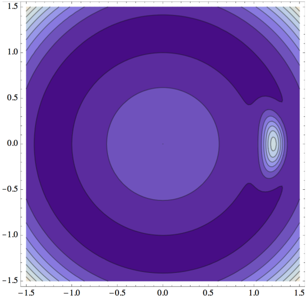


... *give rise to a radial density of energy* $x \mapsto V\,|u|^2 + F(u)$

symmetry breaking

... but in some cases minimal energy solutions





... *give rise to a non-radial density of energy* $x \mapsto V\,|u|^2 + F(u)$

Symmetry breaking may occur...

$\triangleright$ in presence of *non-cooperative* potentials

$\triangleright$ in weighted equations or weighted variational problems

$\triangleright$ in phase transition problems

$\triangleright$ in evolution equations (instability of symmetric solutions)

$\triangleright$ in various models of mathematical physics and quantum field theory

etc.

BGK-type kinetic equation as a motivation for nonlinear diffusions – polytropes and fast diffusion / porous medium

$$\begin{aligned} \varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon - \varepsilon \nabla_x V(x) \cdot \nabla_v f^\varepsilon &= G_{f^\varepsilon} - f^\varepsilon \\ f^\varepsilon(x, v, t=0) &= f_I(x, v)\,, \quad x, v \in \mathbb{R}^3 \end{aligned}$$

with the Gibbs equilibrium $G_f := \gamma\left(\frac{|v|^2}{2} + V(x) - \mu_{\rho_f}(x,t)\right)$

The Fermi energy $\mu_{\rho_f}(x,t)$ is implicitly defined by

$$\int_{\mathbb{R}^3} \gamma\left(\frac{|v|^2}{2} + V(x) - \mu_{\rho_f}(x,t)\right) dv = \int_{\mathbb{R}^3} f(x,v,t)\,dv =: \rho_f(x,t)$$

$$\begin{aligned} f^\varepsilon(x,v,t) \quad & \ldots \quad \text{phase space particle density} \\ V(x) \quad & \ldots \quad \text{potential} \\ \varepsilon \quad & \ldots \quad \text{mean free path} \end{aligned}$$

$$\Longrightarrow \qquad \mu_{\rho_f} = \bar{\mu}(\rho_f)$$

Diffusion limits

[J.D., P. Markowich, D. Ölz, C. Schmeiser]

Theorem

For any $\varepsilon > 0$*, the equation has a unique weak solution* $f^\varepsilon \in C(0,\infty; L^1 \cap L^p(\mathbb{R}^6))$ *for all* $p < \infty$*. As* $\varepsilon \to 0$*,* f^ε *weakly converges to a local Gibbs state* f^0 *given by*

$$f^0(x,v,t) = \gamma\left(\frac{1}{2}\,|v|^2 - \bar{\mu}(\rho(x,t))\right)$$

where ρ *is a solution of the nonlinear diffusion equation*

$$\partial_t \rho = \nabla_x \cdot (\nabla_x\, \nu(\rho) + \rho\, \nabla_x V(x))$$

with initial data $\rho(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v)\, dv$

$$\nu(\rho) = \int_0^\rho s\, \bar{\mu}'(s)\, ds$$

Outline

▷ **Symmetry breaking and linearization**

- The critical Caffarelli-Kohn-Nirenberg inequality
- A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities
- Linearization and spectrum

▷ **Without weights: Gagliardo-Nirenberg inequalities and fast diffusion flows**

- Rényi entropy powers
- Self-similar variables and relative entropies
- The role of the spectral gap

▷ **With weights: Caffarelli-Kohn-Nirenberg inequalities and weighted nonlinear flows**

- Towards a parabolic proof
- Large time asymptotics and spectral gaps
- A discussion of optimality cases

Collaborations

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratori (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large time asymptotics for the evolution problem)
M. del Pino, G. Toscani (nonlinear flows and entropy methods)
A. Blanchet, G. Grillo, J.L. Vázquez (large time asymptotics and linearization for the evolution equations)

...and also

S. Filippas, A. Tertikas, G. Tarantello, M. Kowalczyk ...

Background references (partial)

- Rigidity methods, uniqueness in nonlinear elliptic PDE's: [B. Gidas, J. Spruck, 1981], [M.-F. Bidaut-Véron, L. Véron, 1991]
- Probabilistic methods (Markov processes), semi-group theory and *carré du champ* methods (Γ_2 theory): [D. Bakry, M. Emery, 1984], [Bakry, Ledoux, 1996], [Demange, 2008], [JD, Esteban, Loss, 2014 & 2015] → *D. Bakry, I. Gentil, and M. Ledoux. Analysis and geometry of Markov diffusion operators (2014)*
- Entropy methods in PDEs

  ▷ Entropy-entropy production inequalities: Arnold, Carrillo, Desvillettes, JD, Jüngel, Lederman, Markowich, Toscani, Unterreiter, Villani..., [del Pino, JD, 2001], [Blanchet, Bonforte, JD, Grillo, Vázquez] → *A. Jüngel, Entropy Methods for Diffusive Partial Differential Equations (2016)*

  ▷ Mass transportation: [Otto] → *C. Villani, Optimal transport. Old and new (2009)*

  ▷ Rényi entropy powers (information theory) [Savaré, Toscani, 2014], [Dolbeault, Toscani]

Symmetry and symmetry breaking results

▷ The critical Caffarelli-Kohn-Nirenberg inequality

▷ A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities

▷ Linearization and spectrum

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

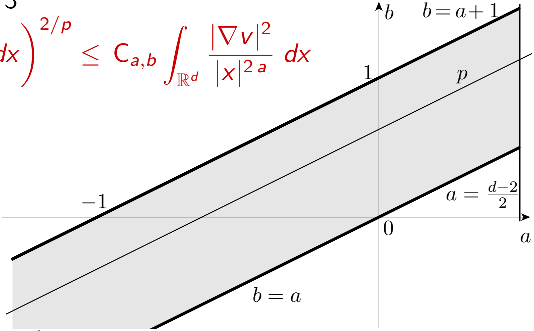
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d = 3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a \le b \le a+1$ if $d \ge 3$

$a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$

and $a < a_c := (d-2)/2$

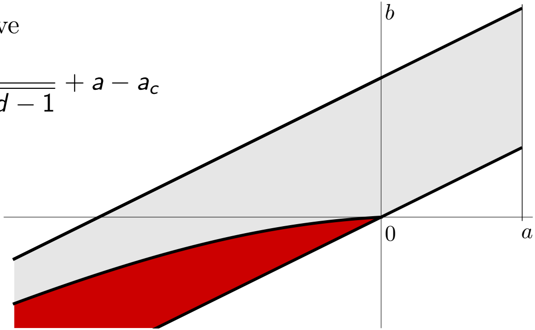
$$p = \frac{2\,d}{d - 2 + 2\,(b-a)}$$

[Glaser, Martin, Grosse, Thirring (1976)]
[Caffarelli, Kohn, Nirenberg (1984)]
[F. Catrina, Z.-Q. Wang (2001)]

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\rm FS}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$



[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]

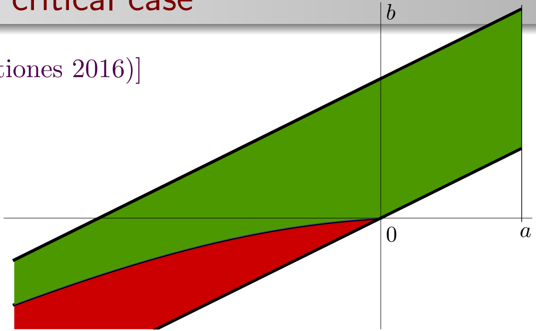
The functional

$$C^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v = v$

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (Inventiones 2016)]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\rm FS}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega)=r^{a-a_c}\,\varphi(s,\omega)\quad\text{with}\quad r=|x|\,,\quad s=-\log r\quad\text{and}\quad\omega=\frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the *subcritical* interpolation inequality

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}\ge\mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}\quad\forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda:=(a_c-a)^2$, $\mathcal{C}=\mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda)=\frac{1}{\mathsf{C}_{a,b}}\quad\text{with}\quad a=a_c\pm\sqrt{\Lambda}\quad\text{and}\quad b=\frac{d}{p}\pm\sqrt{\Lambda}$$

Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_\star(s,\omega) = (\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$\mathrm{H}^1(\mathcal{C}) \ni \varphi \mapsto \|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}$$

under a constraint on $\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}$

$\varphi_\star$ *is not* optimal for (CKN) if the Pöschl-Teller operator

$$-\partial_s^2 - \Delta_\omega + \Lambda - \varphi_\star^{p-2} = -\partial_s^2 - \Delta_\omega + \Lambda - \frac{1}{(\cosh s)^2}$$

has *a negative eigenvalue*

Subcritical Caffarelli-Kohn-Nirenberg inequalities

Norms: $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q}$, $\|w\|_{\mathrm{L}^{q}(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$
(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p}\,\|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}\,\|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \qquad \text{(CKN)}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \ge 2\,,\quad \gamma-2 < \beta < \frac{d-2}{d}\,\gamma\,,\quad \gamma \in (-\infty,d)\,,\quad p \in (1,p_\star] \quad \text{with } p_\star := \frac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$$

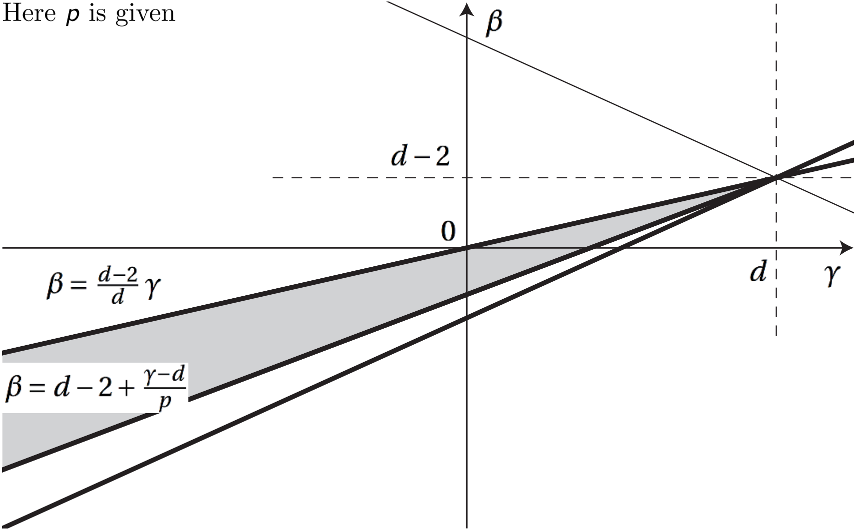
- Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\,x \in \mathbb{R}^d \quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions?

Range of the parameters

Here p is given



Symmetry and symmetry breaking

[M. Bonforte, J.D., M. Muratori and B. Nazaret, 2016] Let us define $\beta_{\rm FS}(\gamma) := d-2-\sqrt{(d-\gamma)^2-4\,(d-1)}$

Theorem

Symmetry breaking holds in (CKN) if

$$\gamma<0 \quad \textit{and} \quad \beta_{\rm FS}(\gamma)<\beta<\frac{d-2}{d}\,\gamma$$

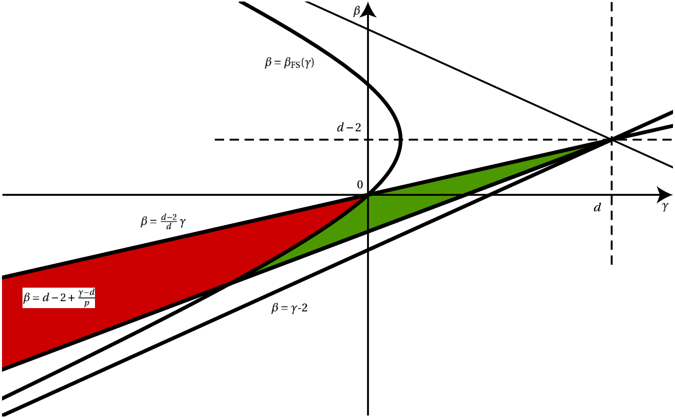
In the range $\beta_{\rm FS}(\gamma)<\beta<\frac{d-2}{d}\,\gamma$, $w_\star(x)=\left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$ *is not* optimal

[JD, Esteban, Loss, Muratori, 2016]

Theorem

Symmetry holds in (CKN) if

$$\gamma\ge0\,, \quad \textit{or} \quad \gamma\le0 \quad \textit{and} \quad \gamma-2\le\beta\le\beta_{\rm FS}(\gamma)$$



The green area is the region of symmetry, while the red area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$$

A useful change of variables

With

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta + 2 - \gamma}\,,$$

(CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|^{\vartheta}_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\|v\|^{1-\vartheta}_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$. Parameters are in the range

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,, \quad p_\star := \frac{n}{n-2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha < \alpha_{\rm FS} \quad \text{with} \quad \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$$

The second variation

$$\mathcal{J}[v] := \vartheta \log\left(\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\right) + (1-\vartheta) \log\left(\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}\right) + \log \mathsf{K}_{\alpha,n,p} - \log\left(\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\right)$$

Let us define $d\mu_\delta := \mu_\delta(x)\,dx$, where $\mu_\delta(x) := (1+|x|^2)^{-\delta}$. Since $v_\star$ is a critical point of $\mathcal{J}$, a Taylor expansion at order ε^2 shows that

$$\|\mathfrak{D}_\alpha v_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\, \mathcal{J}\left[v_\star + \varepsilon\, \mu_{\delta/2}\, f\right] = \frac{1}{2}\, \varepsilon^2\, \vartheta\, \mathcal{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2\,p}{p-1}$ and

$$\mathcal{Q}[f] = \int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\, |x|^{n-d}\, d\mu_\delta - \frac{4\,p\,\alpha^2}{p-1} \int_{\mathbb{R}^d} |f|^2\, |x|^{n-d}\, d\mu_{\delta+1}$$

We assume that $\int_{\mathbb{R}^d} f\, |x|^{n-d}\, d\mu_{\delta+1} = 0$ (mass conservation)

Symmetry breaking: the proof

Proposition (Hardy-Poincaré inequality)

Let $d \ge 2$, $\alpha \in (0,+\infty)$, $n > d$ and $\delta \ge n$. If f has 0 average, then

$$\int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2 \, |x|^{n-d} \, d\mu_\delta \ge \Lambda \int_{\mathbb{R}^d} |f|^2 \, |x|^{n-d} \, d\mu_{\delta+1}$$

with optimal constant $\Lambda = \min\{2\,\alpha^2\,(2\,\delta - n), 2\,\alpha^2\,\delta\,\eta\}$ where η is the unique positive solution to $\eta\,(\eta + n - 2) = (d-1)/\alpha^2$. The corresponding eigenfunction is not radially symmetric if $\alpha^2 > \dfrac{(d-1)\,\delta^2}{n\,(2\,\delta - n)\,(\delta - 1)}$

$\mathcal{Q} \ge 0$ iff $\frac{4\,p\,\alpha^2}{p-1} \le \Lambda$ and symmetry breaking occurs in (CKN) if

$$2\,\alpha^2\,\delta\,\eta < \frac{4\,p\,\alpha^2}{p-1} \quad \Longleftrightarrow \quad \eta < 1$$

$$\Longleftrightarrow \quad \frac{d-1}{\alpha^2} = \eta\,(\eta + n - 2) < n - 1 \quad \Longleftrightarrow \quad \alpha > \alpha_{\rm FS}$$

Inequalities without weights and fast diffusion equations

▷ [The Bakry-Emery method on the sphere and its extension by non-linear diffusion flows]

▷ Rényi entropy powers

▷ Self-similar variables and relative entropies

▷ The role of the spectral gap

Rényi entropy powers and fast diffusion

▷ Rényi entropy powers, the entropy approach without rescaling: [Savaré, Toscani]: scalings, nonlinearity and a concavity property inspired by information theory

▷ Faster rates of convergence: [Carrillo, Toscani], [JD, Toscani]

The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial v}{\partial t} = \Delta v^m$$

with initial datum $v(x, t = 0) = v_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} v_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, v_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t, x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathcal{B}_\star\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, v^{m-1}$$

If v solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1) \, \mathsf{p} \, \Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m} \left(\frac{1}{d} + m - 1 \right) = \frac{2}{d} \, \frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The concavity property

Theorem

[Toscani-Savaré] *Assume that $m \geq 1 - \frac{1}{d}$ if $d > 1$ and $m > 0$ if $d = 1$. Then $F(t)$ is increasing, $(1-m)\,\mathsf{F}''(t) \leq 0$ and*

$$\lim_{t \to +\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t \to +\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

[Dolbeault-Toscani] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \geq \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

if $1 - \frac{1}{d} \leq m < 1$. Hint: $v^{m-1/2} = \frac{w}{\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}}$, $q = \frac{1}{2\,m-1}$

The proof

Lemma

If v solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $\frac{1}{d} \le m < 1$, then

$$\mathsf{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} v\,|\nabla \mathsf{p}|^2\,dx = -\,2\int_{\mathbb{R}^d} v^m \left(\|\mathrm{D}^2\mathsf{p}\|^2 + (m-1)\,(\Delta \mathsf{p})^2\right)dx$$

Explicit arithmetic geometric inequality

$$\|\mathrm{D}^2\mathsf{p}\|^2 - \frac{1}{d}\,(\Delta \mathsf{p})^2 = \left\|\,\mathrm{D}^2\mathsf{p} - \frac{1}{d}\,\Delta \mathsf{p}\,\mathrm{Id}\,\right\|^2$$

There are no boundary terms in the integrations by parts

Remainder terms

$\mathsf{F}'' = -\,\sigma\,(1-m)\,\mathsf{R}[v]$. The *pressure variable* is $\mathsf{P} = \frac{m}{1-m}\,v^{m-1}$

$$\mathsf{R}[v] := (\sigma-1)\,(1-m)\,\mathsf{E}^{\sigma-1}\int_{\mathbb{R}^d} v^m\,\left|\Delta\mathsf{P} - \frac{\int_{\mathbb{R}^d} v\,|\nabla\mathsf{P}|^2\,dx}{\int_{\mathbb{R}^d} v^m\,dx}\right|^2 dx$$
$$+\,2\,\mathsf{E}^{\sigma-1}\int_{\mathbb{R}^d} v^m\,\left\|\,\mathrm{D}^2\mathsf{P} - \tfrac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\,\right\|^2 dx$$

Let

$$\mathsf{G}[v] := \frac{\mathsf{F}[v]}{\sigma\,(1-m)} = \left(\int_{\mathbb{R}^d} v^m\,dx\right)^{\sigma-1}\int_{\mathbb{R}^d} v\,|\nabla\mathsf{P}|^2\,dx$$

The Gagliardo-Nirenberg inequality is equivalent to $\mathsf{G}[v_0] \ge \mathsf{G}[v_\star]$

Proposition

$$\mathsf{G}[v_0] \ge \mathsf{G}[v_\star] + \int_0^\infty \mathsf{R}[v(t,\cdot)]\,dt$$

Self-similar variables and relative entropies

The large time behavior of the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ is governed by the source-type *Barenblatt solutions*

$$v_\star(t,x) := \frac{1}{\kappa^d (\mu\, t)^{d/\mu}}\, \mathcal{B}_\star\!\left(\frac{x}{\kappa\,(\mu\, t)^{1/\mu}}\right) \quad \text{where} \quad \mu := 2 + d\,(m-1)$$

where $\mathcal{B}_\star$ is the Barenblatt profile (with appropriate mass)

$$\mathcal{B}_\star(x) := \left(1 + |x|^2\right)^{1/(m-1)}$$

A time-dependent rescaling: self-similar variables

$$v(t,x) = \frac{1}{\kappa^d\, R^d}\, u\!\left(\tau, \frac{x}{\kappa\, R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}\,, \quad \tau(t) := \tfrac{1}{2}\, \log\left(\frac{R(t)}{R_0}\right)$$

Then the function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \Big[u\, \big(\nabla u^{m-1} - 2\, x\big) \Big] = 0$$

Free energy and Fisher information

- The function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \Big[u \left(\nabla u^{m-1} - 2\,x \right) \Big] = 0$$

- [Ralston, Newman, 1984] Lyapunov functional:
Generalized entropy or *Free energy*

$$\mathcal{E}[u] := \int_{\mathbb{R}^d} \left(-\frac{u^m}{m} + |x|^2 u \right) dx - \mathcal{E}_0$$

- Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{E}[u] = -\mathcal{I}[u]\,, \quad \mathcal{I}[u] := \int_{\mathbb{R}^d} u \left| \nabla u^{m-1} + 2\,x \right|^2 \, dx$$

Without weights: relative entropy, entropy production

- *Stationary solution:* choose C such that $\|u_\infty\|_{\mathrm{L}^1} = \|u\|_{\mathrm{L}^1} = M > 0$

$$u_\infty(x) := \left(C + |x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{E}_0$ so that $\mathcal{E}[u_\infty] = 0$

- *Entropy – entropy production inequality* [del Pino, J.D.]

Theorem

$d \ge 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[u] \ge 4\,\mathcal{E}[u]$$

Corollary

[del Pino, J.D.] *A solution u with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies*

$$\mathcal{E}[u(t,\cdot)] \le \mathcal{E}[u_0]\,e^{-4\,t}$$

A computation on a large ball, with boundary terms

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \Big[u \left(\nabla u^{m-1} - 2\,x \right) \Big] = 0 \quad \tau > 0\,, \quad x \in B_R$$

where B_R is a centered ball in $\mathbb{R}^d$ with radius $R > 0$, and assume that u satisfies zero-flux boundary conditions

$$\left(\nabla u^{m-1} - 2\,x \right) \cdot \frac{x}{|x|} = 0 \quad \tau > 0\,, \quad x \in \partial B_R\,.$$

With $z(\tau,x) := \nabla Q(\tau,x) := \nabla u^{m-1} - 2\,x$, the *relative Fisher information* is such that

$$\frac{d}{d\tau} \int_{B_R} u\,|z|^2\,dx + 4 \int_{B_R} u\,|z|^2\,dx$$
$$+ 2\,\tfrac{1-m}{m} \int_{B_R} u^m \left(\left\| \mathrm{D}^2 Q \right\|^2 - (1-m)\,(\Delta Q)^2 \right) dx$$
$$= \int_{\partial B_R} u^m \left(\omega \cdot \nabla |z|^2 \right) d\sigma \le 0 \text{ (by Grisvard's lemma)}$$

Another improvement of the GN inequalities

Let us define the *relative entropy*

$$\mathcal{E}[u] := -\frac{1}{m}\int_{\mathbb{R}^d}\left(u^m - \mathcal{B}_\star^m - m\,\mathcal{B}_\star^{m-1}\,(u - \mathcal{B}_\star)\right)dx$$

the *relative Fisher information*

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u\,|z|^2\,dx = \int_{\mathbb{R}^d} u\,\left|\nabla u^{m-1} - 2\,x\right|^2 dx$$

and $\quad \mathcal{R}[u] := 2\,\frac{1-m}{m}\int_{\mathbb{R}^d} u^m\left(\left\|\mathrm{D}^2\mathrm{Q}\right\|^2 - (1-m)\,(\Delta\mathrm{Q})^2\right)dx$

Proposition

If $1 - 1/d \le m < 1$ *and* $d \ge 2$*, then*

$$\mathcal{I}[u_0] - 4\,\mathcal{E}[u_0] \ge \int_0^\infty \mathcal{R}[u(\tau,\cdot)]\,d\tau$$

Entropy – entropy production, Gagliardo-Nirenberg ineq.

$$4\,\mathcal{E}[u] \le \mathcal{I}[u]$$

Rewrite it with $p = \frac{1}{2m-1}$, $u = w^{2p}$, $u^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \ge 0$$

- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} u\,dx = \int_{\mathbb{R}^d} w^{2p}\,dx\right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal

Theorem

[Del Pino, J.D.] *With* $1 < p \le \frac{d}{d-2}$ *(fast diffusion case) and* $d \ge 3$

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le \mathcal{C}^{\mathrm{GN}}_{p,d}\, \|\nabla w\|^{\theta}_{L^2(\mathbb{R}^d)}\, \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

$$\mathcal{C}^{\mathrm{GN}}_{p,d} = \left(\frac{y\,(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}},\ \theta = \frac{d(p-1)}{p(d+2-(d-2)p)},\ y = \frac{p+1}{p-1}$$

Sharp asymptotic rates of convergence

Assumptions on the initial datum v_0

(H1) *$V_{D_0} \le v_0 \le V_{D_1}$ for some $D_0 > D_1 > 0$*

(H2) *if $d \ge 3$ and $m \le m_*$, $(v_0 - V_D)$ is integrable for a suitable $D \in [D_1, D_0]$*

Theorem

[Blanchet, Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if $m < 1$ and $m \neq m_* := \frac{d-4}{d-2}$, the entropy decays according to*

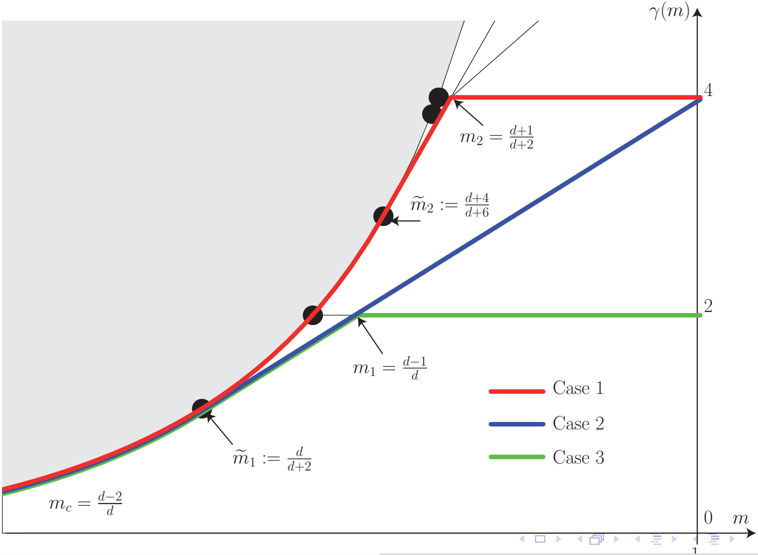
$$\mathcal{E}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\, t \ge 0$$

where $\Lambda_{\alpha,d} > 0$ is the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

Spectral gaps and best constants



Comments

- The spectral gap corresponding to the red curves relies on a refined notion of relative entropy with respect to *best matching Barenblatt profiles* [J.D., Toscani]
- A result by [Denzler, Koch, McCann] *Higher order time asymptotics of fast diffusion in Euclidean space: a dynamical systems approach*
- The constant C in

$$\mathcal{E}[v(t,\cdot)] \le C\, e^{-2\,\gamma(m)\,t} \quad \forall\, t \ge 0$$

can be made explicit, under additional restrictions on the initial data [Bonforte, J.D., Grillo, Vázquez]

Weighted nonlinear flows: Caffarelli-Kohn-Nirenberg inequalities

▷ Entropy and Caffarelli-Kohn-Nirenberg inequalities

▷ Large time asymptotics and spectral gaps

▷ Optimality cases

CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\frac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{E}[v]\le\mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}$. Here the *free energy* and the *relative Fisher information* are defined by

$$\mathcal{E}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\frac{dx}{|x|^\beta}$$

If v solves the *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0 \qquad \text{(WFDE-FP)}$$

then

$$\frac{d}{dt}\mathcal{E}[v(t,\cdot)]=-\frac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$$

Proposition

Let $m=\frac{p+1}{2\,p}$ and consider a solution to (WFDE-FP) with nonnegative initial datum $u_0\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$ such that $\|u_0^m\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}$ and $\int_{\mathbb{R}^d}u_0\,|x|^{2+\beta-2\gamma}\,dx$ are finite. Then

$$\mathcal{E}[v(t,\cdot)]\le\mathcal{E}[u_0]\,e^{-(2+\beta-\gamma)^2t}\quad\forall\,t\ge0$$

if one of the following two conditions is satisfied:
(i) either u_0 is a.e. radially symmetric
(ii) or symmetry holds in (CKN)

The strategy of the proof (1/3: changing the dimension)

We rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$v(|x|^{\alpha-1}\,x) = w(x)\,, \quad \alpha = 1 + \frac{\beta-\gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta+2-\gamma}\,,$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta} \quad \forall\,v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$ and

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,.$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1+|x|^2\right)^{-1/(p-1)} \quad \forall\,x \in \mathbb{R}^d$$

The strategy of the proof (2/3: Rényi entropy)

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathfrak{D}_\alpha \mathsf{P}|^2 \, d\mu$$

where $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d} \, dx$ and the pressure is

$$\mathsf{P} := \frac{m}{1-m} \, u^{m-1}$$

With $\mathcal{L}_\alpha = -\mathfrak{D}_\alpha^*\,\mathfrak{D}_\alpha = \alpha^2\left(u'' + \frac{n-1}{s}\,u'\right) + \frac{1}{s^2}\,\Delta_\omega\,u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{L}_\alpha u^m$$

in the subcritical range $1 - 1/n < m < 1$. The key computation is the proof that

$$\begin{aligned}
&-\tfrac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\textstyle\int_{\mathbb{R}^d} u^m\,d\mu\right)^{1-\sigma}\\
&\geq (1-m)\,(\sigma-1)\int_{\mathbb{R}^d} u^m\left|\mathcal{L}_\alpha \mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathfrak{D}_\alpha \mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d} u^m\,d\mu}\right|^2 d\mu\\
&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\tfrac{1}{n}\right)\left|\mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega \mathsf{P}' - \frac{\nabla_\omega \mathsf{P}}{s}\right|^2\right) u^m\,d\mu\\
&+2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\rm FS}^2 - \alpha^2\right)|\nabla_\omega \mathsf{P}|^2 + c(n,m,d)\,\frac{|\nabla_\omega \mathsf{P}|^4}{\mathsf{P}^2}\right) u^m\,d\mu =: \mathcal{H}[u]
\end{aligned}$$

for some numerical constant $c(n,m,d) > 0$. Hence if $\alpha \leq \alpha_{\rm FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result.

(3/3: elliptic regularity, boundary terms) [...]

This method has a hidden difficulty: integrations by parts ! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if u solves the Euler-Lagrange equation, we test by $\mathcal{L}_\alpha u^m$

$$0=\int_{\mathbb{R}^d} \mathrm{d}\mathcal{G}[u]\cdot\mathcal{L}_\alpha u^m\,d\mu \ge \mathcal{H}[u] \ge 0$$

$\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by $|x|^\gamma \operatorname{div}\left(|x|^{-\beta}\,\nabla w^{1+p}\right)$ the equation

$$\frac{(p-1)^2}{\cdots}\,w^{1-3p}\operatorname{div}\left(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\right)+|\nabla w^{1-p}|^2+|x|^{-\gamma}\left(c_1\,w^{1-p}-c_2\right)=0$$

Towards a parabolic proof

For any $\alpha \geq 1$, let $\mathsf{D}_\alpha W = \left(\alpha\,\partial_r W,\, r^{-1}\,\nabla_\omega W\right)$ so that

$$\mathsf{D}_\alpha = \nabla + (\alpha-1)\,\frac{x}{|x|^2}\,(x\cdot\nabla) = \nabla + (\alpha-1)\,\omega\,\partial_r$$

and define the diffusion operator L_α by

$$\mathsf{L}_\alpha = -\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha = \alpha^2\left(\partial_r^2 + \frac{n-1}{r}\,\partial_r\right) + \frac{\Delta_\omega}{r^2}$$

where Δ_ω denotes the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$
$\frac{\partial g}{\partial t} = \mathsf{L}_\alpha g^m$ is changed into

$$\frac{\partial u}{\partial \tau} = \mathsf{D}_\alpha^*\,(u\,z)\,, \quad z := \mathsf{D}_\alpha\,\mathsf{q}\,, \quad \mathsf{q} := u^{m-1} - \mathcal{B}_\alpha^{m-1}\,, \quad \mathcal{B}_\alpha(x) := \left(1+\frac{|x|^2}{\alpha^2}\right)^{\frac{1}{m-1}}$$

by the change of variables

$$g(t,x) = \frac{1}{\kappa^n\,R^n}\,u\Big(\tau,\frac{x}{\kappa\,R}\Big) \quad \text{where} \quad \left\{\begin{array}{l} \frac{dR}{dt} = R^{1-\mu}\,, \quad R(0) = R_0 \\ \tau(t) = \frac12\,\log\left(\frac{R(t)}{R_0}\right) \end{array}\right.$$

[...]

If the weight does not introduce any singularity at $x = 0$...

$$\frac{m}{1-m}\,\frac{d}{d\tau}\int_{B_R} u\,|z|^2\,d\mu_n$$

$$= \int_{\partial B_R} u^m\,\left(\omega\cdot \mathsf{D}_\alpha\,|z|^2\right)|x|^{n-d}\,d\sigma \quad (\leq 0 \text{ by Grisvard's lemma})$$

$$-\,2\,\tfrac{1-m}{m}\left(m-1+\tfrac{1}{n}\right)\int_{B_R} u^m\,|\mathsf{L}_\alpha q|^2\,d\mu_n$$

$$-\int_{B_R} u^m\left(\alpha^4\,m_1\left|\mathsf{q}''-\tfrac{\mathsf{q}'}{r}-\tfrac{\Delta_\omega \mathsf{q}}{\alpha^2\,(n-1)\,r^2}\right|^2+\tfrac{2\,\alpha^2}{r^2}\left|\nabla_\omega \mathsf{q}'-\tfrac{\nabla_\omega \mathsf{q}}{r}\right|^2\right)d\mu_n$$

$$-\,(n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)\int_{B_R}\frac{|\nabla_\omega \mathsf{q}|^2}{r^4}\,d\mu_n$$

A formal computation that still needs to be justified
(singularity at $x = 0$?)

- Other potential application: the computation of Bakry, Gentil and Ledoux (chapter 6) for non-integer dimensions; weights on manifolds

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \nabla \cdot \Big[|x|^{-\beta} v \nabla \big(v^{m-1} - |x|^{2+\beta-\gamma} \big) \Big] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

Relative uniform convergence

$$\zeta := 1 - \left(1 - \tfrac{2-m}{(1-m)\,q}\right)\left(1 - \tfrac{2-m}{1-m}\,\theta\right)$$
$\theta := \frac{(1-m)\,(2+\beta-\gamma)}{(1-m)\,(2+\beta)+2+\beta-\gamma}$ is in the range $0 < \theta < \frac{1-m}{2-m} < 1$

Theorem

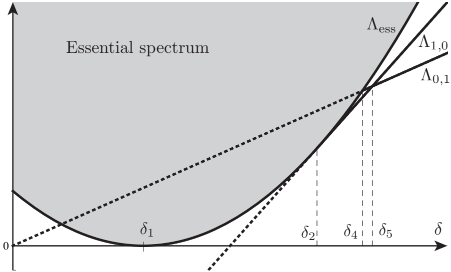
For "good" initial data, there exist positive constants $\mathcal{K}$ and t_0 such that, for all $q \in \left[\frac{2-m}{1-m}, \infty\right]$, the function $w = v/\mathfrak{B}$ satisfies

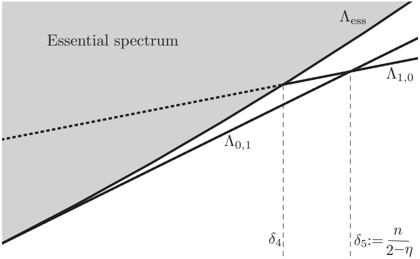
$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,\zeta\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \in (0,d)$, and

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \le 0$





The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\rm ess}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions

Main steps of the proof:

- Existence of weak solutions, $\mathrm{L}^{1,\gamma}$ contraction, Comparison Principle, conservation of relative mass
- Self-similar variables and the Ornstein-Uhlenbeck equation in relative variables: the ratio $w(t,x) := v(t,x)/\mathfrak{B}(x)$ solves

$$\begin{cases} |x|^{-\gamma}\,w_t = -\,\frac{1}{\mathfrak{B}}\,\nabla\cdot\Big(|x|^{-\beta}\,\mathfrak{B}\,w\,\nabla\big((w^{m-1}-1)\,\mathfrak{B}^{m-1}\big)\Big) & \text{in } \mathbb{R}^+\times\mathbb{R}^d \\ w(0,\cdot) = w_0 := v_0/\mathfrak{B} & \text{in } \mathbb{R}^d \end{cases}$$

- *Regularity*: [Chiarenza, Serapioni], Harnack inequalities; relative uniform convergence (without rates) and asymptotic rates (linearization)
- The relative free energy and the relative Fisher information: linearized free energy and linearized Fisher information
- A Duhamel formula and a bootstrap

Asymptotic rates of convergence

Corollary

Assume that $m \in (0,1)$, *with* $m \neq m_* := \frac{n-4}{n-2}$. *Under the relative mass condition, for any "good solution"* v *there exists a positive constant* $\mathcal{C}$ *such that*

$$\mathcal{E}[v(t)] \le \mathcal{C}\, e^{-\,2\,(1-m)\,\Lambda\,t} \quad \forall\, t \ge 0\,.$$

- With Csiszár-Kullback-Pinsker inequalities, these estimates provide a rate of convergence in $\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$
- Improved estimates can be obtained using "best matching techniques"

From asymptotic to global estimates

When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{E}[v]\le\frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{E}[v(t)]\le\mathcal{E}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t}\quad\forall\,t\ge0\quad\text{with}\quad\Lambda_\star:=\frac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{E}[v]\le\mathcal{I}[v]\quad\forall\,v\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d)\quad\text{such that}\quad\|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}=M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M):=\frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{E}[v(t)]\le\mathcal{E}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t}\quad\forall\,t\ge0$$

Symmetry breaking and global entropy – entropy production inequalities

Proposition

- *In the symmetry breaking range of (CKN), for any $M > 0$, we have*
$$0 < \mathcal{K}(M) \le \tfrac{2}{m}\,(1-m)^2\,\Lambda_{0,1}$$
- *If symmetry holds in (CKN) then*
$$\mathcal{K}(M) \ge \tfrac{1-m}{m}\,(2+\beta-\gamma)^2$$

Corollary

Assume that $m \in [m_1, 1)$

(i) For any $M > 0$, if $\Lambda(M) = \Lambda_\star$ then $\beta = \beta_{\rm FS}(\gamma)$

(ii) If $\beta > \beta_{\rm FS}(\gamma)$ then $\Lambda_{0,1} < \Lambda_\star$ and $\Lambda(M) \in (0, \Lambda_{0,1}]$ for any $M > 0$

(iii) For any $M > 0$, if $\beta < \beta_{\rm FS}(\gamma)$ and if symmetry holds in (CKN), then
$$\Lambda(M) > \Lambda_\star$$

Linearization and optimality

Joint work with M.J. Esteban and M. Loss

Linearization and scalar products

With u_ε such that

$$u_\varepsilon = \mathcal{B}_\star \left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\varepsilon\, dx = M_\star$$

at first order in $\varepsilon \to 0$ we obtain that f solves

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f \quad \text{where} \quad \mathcal{L}\, f := (1-m)\, \mathcal{B}_\star^{m-2}\, |x|^\gamma\, \mathsf{D}_\alpha^* \left(|x|^{-\beta}\, \mathcal{B}_\star\, \mathsf{D}_\alpha f\right)$$

Using the scalar products

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1\, f_2\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx \quad \text{and} \quad \langle\!\langle f_1, f_2 \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D}_\alpha f_1 \cdot \mathsf{D}_\alpha f_2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx$$

we compute

$$\frac{1}{2}\, \frac{d}{dt} \langle f, f \rangle = \langle f, \mathcal{L}\, f \rangle = \int_{\mathbb{R}^d} f\, (\mathcal{L}\, f)\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx = -\int_{\mathbb{R}^d} |\mathsf{D}_\alpha f|^2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx$$

for any f smooth enough:

$$\frac{1}{2}\, \frac{d}{dt} \langle\!\langle f, f \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D}_\alpha f \cdot \mathsf{D}_\alpha (\mathcal{L}\, f)\, \mathcal{B}_\star\, |x|^{-\beta}\, dx = -\langle\!\langle f, \mathcal{L}\, f \rangle\!\rangle$$

Linearization of the flow, eigenvalues and spectral gap

Now let us consider an eigenfunction associated with the smallest positive eigenvalue λ_1 of $\mathcal{L}$

$$-\,\mathcal{L}\,f_1 = \lambda_1\,f_1$$

so that f_1 realizes the equality case in the *Hardy-Poincaré inequality*

$$\langle\!\langle g, g\rangle\!\rangle = -\,\langle g, \mathcal{L}\,g\rangle \ge \lambda_1\,\|g-\bar{g}\|^2\,,\quad \bar{g} := \langle g, 1\rangle\,/\,\langle 1, 1\rangle$$

$$-\,\langle\!\langle g, \mathcal{L}\,g\rangle\!\rangle \ge \lambda_1\,\langle\!\langle g, g\rangle\!\rangle$$

Proof: expansion of the square :
$-\,\langle\!\langle (g-\bar{g}), \mathcal{L}\,(g-\bar{g})\rangle\!\rangle = \langle \mathcal{L}\,(g-\bar{g}), \mathcal{L}\,(g-\bar{g})\rangle = \|\mathcal{L}\,(g-\bar{g})\|^2$

- Key observation:

$$\lambda_1 \ge 4 \quad\Longleftrightarrow\quad \alpha \le \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$$

Symmetry breaking in CKN inequalities

- Symmetry holds in (CKN) if $\mathcal{J}[w] \ge \mathcal{J}[w_\star]$ with

$$\mathcal{J}[w] := \vartheta\,\log\left(\|\mathsf{D}_\alpha\,w\|_{\mathrm{L}^{2,\delta}(\mathbb{R}^d)}\right)+(1-\vartheta)\,\log\left(\|w\|_{\mathrm{L}^{p+1,\delta}(\mathbb{R}^d)}\right)-\log\left(\|w\|_{\mathrm{L}^{2p,\delta}(\mathbb{R}^d)}\right)$$

with $\delta := d-n$ and

$$\mathcal{J}[w_\star+\varepsilon\,g] = \varepsilon^2\,\mathcal{Q}[g]+o(\varepsilon^2)$$

where

$$\tfrac{2}{\vartheta}\,\|\mathsf{D}_\alpha\,w_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\mathcal{Q}[g]$$
$$=\|\mathsf{D}_\alpha\,g\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}+\frac{p\,(2+\beta-\gamma)}{(p-1)^2}\left[d-\gamma-p\,(d-2-\beta)\right]\int_{\mathbb{R}^d}|g|^2\,\frac{|x|^{n-d}}{1+|x|^2}\,dx$$
$$-p\,(2\,p-1)\,\frac{(2+\beta-\gamma)^2}{(p-1)^2}\int_{\mathbb{R}^d}|g|^2\,\frac{|x|^{n-d}}{(1+|x|^2)^2}\,dx$$

is a nonnegative quadratic form if and only if $\alpha \le \alpha_{\mathrm{FS}}$

- Symmetry breaking holds if $\alpha > \alpha_{\mathrm{FS}}$

Information – production of information inequality

Let $\mathcal{K}[u]$ be such that

$$\frac{d}{d\tau}\mathcal{I}[u(\tau,\cdot)] = -\,\mathcal{K}[u(\tau,\cdot)] = -\,\text{(sum of squares)}$$

If $\alpha \le \alpha_{\mathrm{FS}}$, then $\lambda_1 \ge 4$ and

$$u \mapsto \frac{\mathcal{K}[u]}{\mathcal{I}[u]} - 4$$

is a nonnegative functional
With $u_\varepsilon = \mathcal{B}_\star\,\big(1+\varepsilon\,f\,\mathcal{B}_\star^{1-m}\big)$, we observe that

$$4 \le \mathcal{C}_2 := \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} \le \liminf_{\varepsilon\to 0}\,\inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \inf_f \frac{\langle\!\langle f,\mathcal{L}\,f\rangle\!\rangle}{\langle\!\langle f,f\rangle\!\rangle} = \frac{\langle\!\langle f_1,\mathcal{L}\,f_1\rangle\!\rangle}{\langle\!\langle f_1,f_1\rangle\!\rangle} = \lambda_1$$

- if $\lambda_1 = 4$, that is, if $\alpha = \alpha_{\mathrm{FS}}$, then $\inf \mathcal{K}/\mathcal{I} = 4$ is achieved in the asymptotic regime as $u \to \mathcal{B}_\star$ and determined by the spectral gap of $\mathcal{L}$
- if $\lambda_1 > 4$, that is, if $\alpha < \alpha_{\mathrm{FS}}$, then $\mathcal{K}/\mathcal{I} > 4$

Symmetry in Caffarelli-Kohn-Nirenberg inequalities

If $\alpha \le \alpha_{\rm FS}$, the fact that $\mathcal{K}/\mathcal{I} \ge 4$ has an important consequence. Indeed we know that

$$\frac{d}{d\tau}\left(\mathcal{I}[u(\tau,\cdot)] - 4\,\mathcal{E}[u(\tau,\cdot)]\right) \le 0$$

so that

$$\mathcal{I}[u] - 4\,\mathcal{E}[u] \ge \mathcal{I}[\mathcal{B}_\star] - 4\,\mathcal{E}[\mathcal{B}_\star] = 0$$

This inequality is equivalent to $\mathcal{J}[w] \ge \mathcal{J}[w_\star]$, which establishes that optimality in (CKN) is achieved among symmetric functions. In other words, the linearized problem shows that for $\alpha \le \alpha_{\rm FS}$, the function

$$\tau \mapsto \mathcal{I}[u(\tau,\cdot)] - 4\,\mathcal{E}[u(\tau,\cdot)]$$

is monotone decreasing

- This explains why the method based on nonlinear flows provides the *optimal range for symmetry*

Entropy – production of entropy inequality

Using $\frac{d}{d\tau}\left(\mathcal{I}[u(\tau,\cdot)] - \mathcal{C}_2\,\mathcal{E}[u(\tau,\cdot)]\right) \le 0$, we know that

$$\mathcal{I}[u] - \mathcal{C}_2\,\mathcal{E}[u] \ge \mathcal{I}[\mathcal{B}_\star] - \mathcal{C}_2\,\mathcal{E}[\mathcal{B}_\star] = 0$$

As a consequence, we have that

$$\mathcal{C}_1 := \inf_u \frac{\mathcal{I}[u]}{\mathcal{E}[u]} \ge \mathcal{C}_2 = \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]}$$

With $u_\varepsilon = \mathcal{B}_\star\left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right)$, we observe that

$$\mathcal{C}_1 \le \lim_{\varepsilon\to0} \inf_f \frac{\mathcal{I}[u_\varepsilon]}{\mathcal{E}[u_\varepsilon]} = \inf_f \frac{\langle f, \mathcal{L}\, f\rangle}{\langle f, f\rangle} = \frac{\langle f_1, \mathcal{L}\, f_1\rangle}{\langle f_1, f\rangle_1} = \lambda_1 = \lim_{\varepsilon\to0} \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]}$$

- If $\lim_{\varepsilon\to0} \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \mathcal{C}_2$, then $\mathcal{C}_1 = \mathcal{C}_2 = \lambda_1$

This happens if $\alpha = \alpha_{\rm FS}$ and in particular in the case without weights (Gagliardo-Nirenberg inequalities)

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Thank you for your attention !