Une approche hypocoercive L^2 pour l'équation de Vlasov-Fokker-Planck

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Outline

- The goal is to understand the *rate of relaxation* of the solutions of a kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \mathsf{L} f$$

towards a global equilibrium when the collision term acts only on the velocity space. Here $f = f(t, x, v)$ is the distribution function. It can be seen as a probability distribution on the phase space, where x is the position and v the velocity. However, since we are in a linear framework, the fact that f has a constant sign plays no role.

- A key feature of our approach [J.D., Mouhot, Schmeiser] is that it distinguishes the mechanisms of relaxation at *microscopic level* (convergence towards a local equilibrium, in velocity space) and *macroscopic level* (convergence of the spatial density to a steady state), where the rate is given by a spectral gap which has to do with the underlying diffusion equation for the spatial density

A very brief review of the literature

- Non constructive decay results: [Ukai (1974)] [Desvillettes (1990)]
- Explicit $t^{-\infty}$-decay, no spectral gap: [Desvillettes, Villani (2001-05)], [Fellner, Miljanovic, Neumann, Schmeiser (2004)], [Cáceres, Carrillo, Goudon (2003)]
- *hypoelliptic theory*:
  [Hérau, Nier (2004)]: spectral analysis of the Vlasov-Fokker-Planck equation
  [Hérau (2006)]: linear Boltzmann relaxation operator
- Hypoelliptic theory vs. *hypocoercivity* (Gallay) approach and generalized entropies:
  [Mouhot, Neumann (2006)], [Villani (2007, 2008)]
- Other related approaches: non-linear Boltzmann and Landau equations:
  micro-macro decomposition: [Guo]
  hydrodynamic limits (fluid-kinetic decomposition): [Yu]

A toy problem

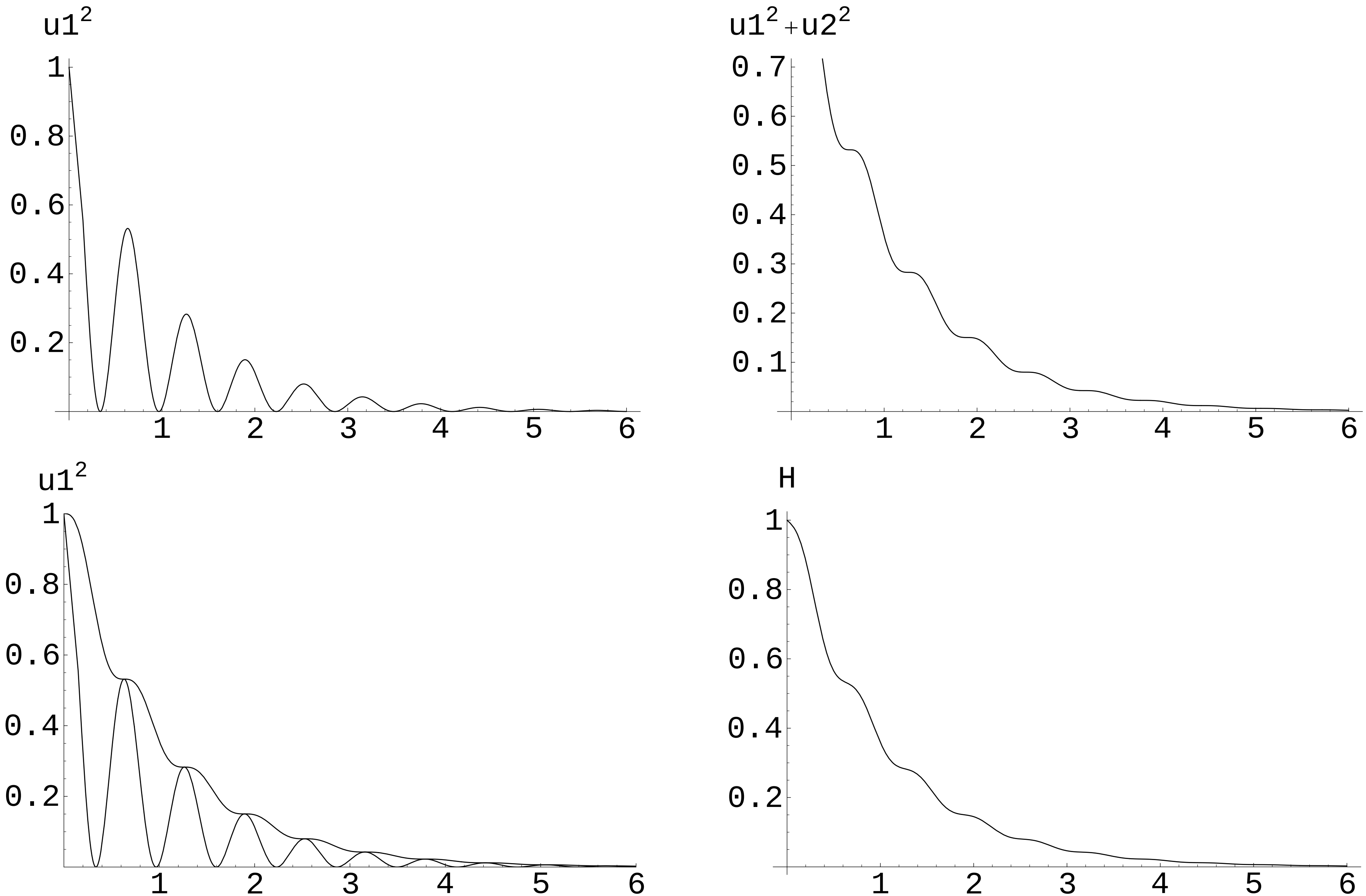
$$\frac{du}{dt} = (L - T)\,u\,, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad k^2 \geq \Lambda > 0$$

Nonmonotone decay, reminiscent of [Filbet, Mouhot, Pareschi (2006)]

- H-theorem: $\frac{d}{dt}|u|^2 = -2\,u_2^2$
- macroscopic limit: $\frac{du_1}{dt} = -k^2\,u_1$
- generalized entropy: $H(u) = |u|^2 - \frac{\varepsilon\,k}{1+k^2}\,u_1\,u_2$

$$\begin{aligned} \frac{dH}{dt} &= -\left(2 - \frac{\varepsilon\,k^2}{1+k^2}\right)u_2^2 - \frac{\varepsilon\,k^2}{1+k^2}\,u_1^2 + \frac{\varepsilon\,k}{1+k^2}\,u_1\,u_2 \\ &\leq -(2-\varepsilon)\,u_2^2 - \frac{\varepsilon\Lambda}{1+\Lambda}\,u_1^2 + \frac{\varepsilon}{2}\,u_1 u_2 \end{aligned}$$

Plots for the toy problem



. . . compared to plots for the Boltzmann equation

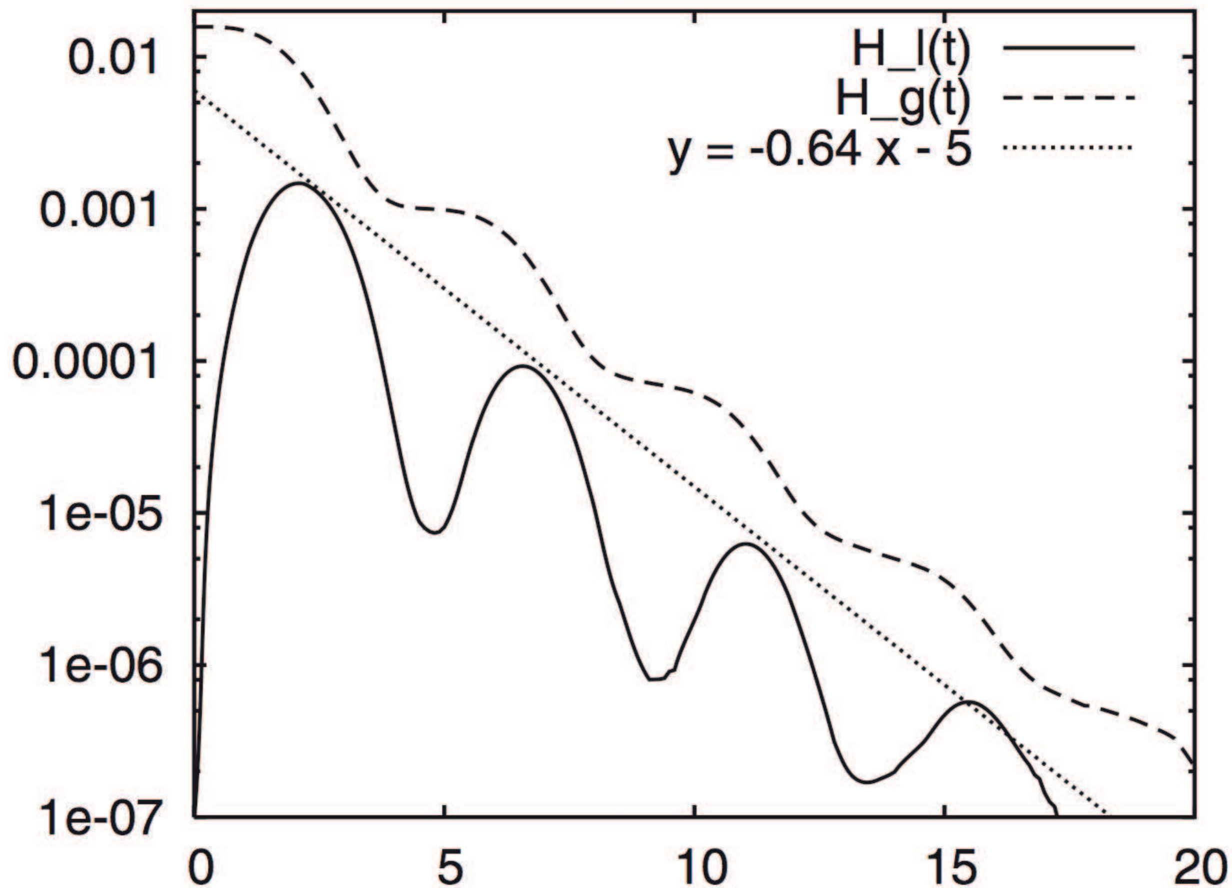


Figure 1: [Filbet, Mouhot, Pareschi (2006)]

The kinetic equation

$$\partial_t f + \mathsf{T} f = \mathsf{L} f\,, \quad f = f(t,x,v)\,,\ t>0,\ \ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d \tag{1}$$

- L is a linear collision operator
- V is a given *external potential* on $\mathbb{R}^d$, $d \geq 1$
- $\mathsf{T} := v\cdot\nabla_x - \nabla_x V\cdot\nabla_v$ is a transport operator

There exists a scalar product $\langle\cdot,\cdot\rangle$, such that L is symmetric and T is antisymmetric

$$\frac{d}{dt}\,\|f-F\|^2 = -2\,\|\mathsf{L}\,f\|^2$$

... seems to imply that the decay stops when $f\in\mathcal{N}(L)$
but we expect $f\to F$ as $t\to\infty$ since F generates $\mathcal{N}(L)\cap\mathcal{N}(T)$
Hypocoercivity: prove an H-theorem for a generalized entropy

$$H(f) := \frac{1}{2}\,\|f\|^2 + \varepsilon\,\langle \mathsf{A}\,f, f\rangle$$

Examples

- L is a linear relaxation operator L

$$\mathsf{L}\, f = \Pi\, f - f\,, \quad \Pi\, f := \frac{\rho}{\rho_F}\, F(x,v)$$

$$\rho = \rho_f := \int_{\mathbb{R}^d} f\, dv$$

  - Maxwellian case: $F(x,v) := M(v)\, e^{-V(x)}$ with $M(v) := (2\pi)^{-d/2}\, e^{-|v|^2/2} \implies \Pi f = \rho_f\, M(v)$
  - Linearized fast diffusion case: $F(x,v) := \omega\left(\frac{1}{2}\,|v|^2 + V(x)\right)^{-(k+1)}$
- L is a Fokker-Planck operator: $\mathsf{L}\, f = \Delta_v f + \nabla \cdot (v\, f)$
- L is a linear scattering operator (including the case of non-elastic collisions)

Some conventions. Cauchy problem

- F is a positive probability distribution
- Measure: $d\mu(x,v) = F(x,v)^{-1}\,dx\,dv$ on $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v)$
- Scalar product and norm $\langle f,g\rangle = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f\,g\,d\mu$ and $\|f\|^2 = \langle f,f\rangle$

The equation

$$\partial_t f + \mathsf{T}\,f = \mathsf{L}\,f$$

with initial condition $f(t=0,\cdot,\cdot) = f_0 \in L^2(d\mu)$ such that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f_0\,dx\,dv = 1$$

has a unique global solution (under additional technical assumptions): [Poupaud], [JD, Markowich,Ölz, Schmeiser]. The solution preserves mass

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\,dx\,dv = 1 \quad \forall\, t \geq 0$$

Maxwellian case: Assumptions

We assume that $F(x,v) := M(v)\,e^{-V(x)}$ with $M(v) := (2\pi)^{-d/2}\,e^{-|v|^2/2}$ where V satisfies the following assumptions

(H1) *Regularity:* $V \in W^{2,\infty}_{\rm loc}(\mathbb{R}^d)$

(H2) *Normalization:* $\int_{\mathbb{R}^d} e^{-V}\,dx = 1$

(H3) *Spectral gap condition: there exists a positive constant* Λ *such that*

$$\int_{\mathbb{R}^d} |u|^2\,e^{-V}\,dx \le \Lambda \int_{\mathbb{R}^d} |\nabla_x u|^2\,e^{-V}\,dx$$

for any $u \in H^1(e^{-V}\,dx)$ *such that* $\int_{\mathbb{R}^d} u\,e^{-V}\,dx = 0$

(H4) *Pointwise condition 1: there exists* $c_0 > 0$ *and* $\theta \in (0,1)$ *such that* $\Delta V \le \frac{\theta}{2}\,|\nabla_x V(x)|^2 + c_0\ \forall\,x \in \mathbb{R}^d$

(H5) *Pointwise condition 2: there exists* $c_1 > 0$ *such that* $|\nabla_x^2 V(x)| \le c_1\,(1 + |\nabla_x V(x)|)\ \forall\,x \in \mathbb{R}^d$

(H6) *Growth condition:* $\int_{\mathbb{R}^d} |\nabla_x V|^2\,e^{-V}\,dx < \infty$

Maxwellian case: Main result

Theorem 1. *If $\partial_t f + \mathsf{T}\, f = \mathsf{L}\, f$, for $\varepsilon > 0$, small enough, there exists an explicit, positive constant $\lambda = \lambda(\varepsilon)$ such that*

$$\|f(t) - F\| \leq (1+\varepsilon)\,\|f_0 - F\|\, e^{-\lambda t} \quad \forall\, t \geq 0$$

- The operator L has no regularization property: hypo-coercivity fundamentally differs from hypo-ellipticity
- Coercivity due to L is only on velocity variables

$$\frac{d}{dt}\|f(t) - F\|^2 = -\|(1-\Pi)f\|^2 = -\iint_{\mathbb{R}^d\times\mathbb{R}^d} |f - \rho_f\, M(v)|^2\, dv\, dx$$

- T and L do not commute: coercivity in v is transferred to the x variable. In the diffusion limit, ρ solves a Fokker-Planck equation

$$\partial_t \rho = \Delta\rho + \nabla\cdot(\rho\,\nabla V) \quad t > 0\,, \quad x \in \mathbb{R}^d$$

the goal of the hypo-coercivity theory is to quantify the interaction of T and L and build a norm which controls $\|\cdot\|$ and decays exponentially

The operators. A Lyapunov functional

On $L^2(d\mu)$, define

$$\mathsf{b}f := \Pi\,(v\,f)\,,\quad \mathsf{a}f := \mathsf{b}\,(\mathsf{T}\,f)\,,\quad \hat{\mathsf{a}}\,f := -\Pi\,(\nabla_x f)\,,\quad \mathsf{A} := (1+\hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi)^{-1}\,\hat{\mathsf{a}}\cdot\mathsf{b}$$

$$\mathsf{b}\,f = \frac{F}{\rho_F}\int_{\mathbb{R}^d} v\,f\,dv = \frac{F}{\rho_F}\,j_f \quad \text{with} \quad j_f := \int_{\mathbb{R}^d} v\,f\,dv$$

$$\mathsf{a}\,f = \frac{F}{\rho_F}\left(\nabla_x\cdot\int_{\mathbb{R}^d} v\otimes v\,f\,dv + \rho_f\,\nabla_x V\right),\quad \hat{\mathsf{a}}\,f = -\frac{F}{\rho_F}\,\nabla_x\rho_f$$

$$\mathsf{A}\,\mathsf{T} = (1+\hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi)^{-1}\,\hat{\mathsf{a}}\cdot\mathsf{a}$$

Define the Lyapunov functional (generalized entropy)

$$H(f) := \frac{1}{2}\,\|f\|^2 + \varepsilon\,\langle\mathsf{A}\,f, f\rangle$$

. . . a Lyapunov functional (continued): positivity, equivalence

$$\langle \hat{\mathsf{a}} \cdot \mathsf{a}\,\Pi\, f, f \rangle = \frac{1}{d} \int_{\mathbb{R}^d} \left| \nabla_x \left(\frac{\rho_f}{\rho_F} \right) \right|^2 m_F \, dx$$

with $m_F := \int_{\mathbb{R}^d} |v|^2 \, F(\cdot, v) \, dv$. Let $g := \mathsf{A}\, f$, $u = \rho_g / \rho_F$, $j_f := \int_{\mathbb{R}^d} v\, f \, dv$

$$(1 + \hat{\mathsf{a}} \cdot \mathsf{a}\,\Pi)\, g = \hat{\mathsf{a}} \cdot \mathsf{b} f \iff g - \frac{1}{d} \nabla_x \left(m_F \, \nabla_x u \right) \, \frac{F}{\rho_F} = -\frac{F}{\rho_F} \nabla_x \, j_f$$

$$\rho_F \, u - \frac{1}{d} \nabla_x \left(m_F \, \nabla_x u \right) = -\nabla_x \, j_f$$

$\|\mathsf{A}\, f\|^2 = \int_{\mathbb{R}^d} |u|^2 \, \rho_F \, dx$ and $\|\mathsf{T}\,\mathsf{A}\, f\|^2 = \frac{1}{d} \int_{\mathbb{R}^d} |\nabla_x u|^2 \, m_F \, dx$ are such that

$$2\, \|\mathsf{A}\, f\|^2 + \|\mathsf{T}\,\mathsf{A}\, f\|^2 \le \|(1 - \Pi)\, f\|^2$$

As a consequence

$$(1 - \varepsilon) \|f\|^2 \le 2\, H(f) \le (1 + \varepsilon) \|f\|^2$$

...a Lyapunov functional (continued): decay term

$$H(f) := \frac{1}{2}\,\|f\|^2 + \varepsilon\,\langle \mathsf{A}\,f, f\rangle$$

The operator T is skew-symmetric on $L^2(d\mu)$. If f is a solution, then

$$\frac{d}{dt} H(f-F) = D(f-F)$$

$$D(f) := \ldots$$

$$\ldots \underbrace{\langle f, \mathsf{L}\,f\rangle}_{\text{micro}} - \varepsilon\,\underbrace{\langle \mathsf{A}\,\mathsf{T}\,\Pi\,f, f\rangle}_{\text{macro}} - \varepsilon\,\langle \mathsf{A}\,\mathsf{T}\,(1-\Pi)\,f, f\rangle + \varepsilon\,\langle \mathsf{T}\,\mathsf{A}\,f, f\rangle + \varepsilon\,\langle \mathsf{L}\,f, (\mathsf{A}+\mathsf{A}^*)\,f\rangle$$

Lemma 2. *For some $\varepsilon > 0$ small enough, there exists an explicit constant $\lambda > 0$ such that*

$$D(f-F) + \lambda\,H(f-F) \le 0$$

Preliminary computations (1/2)

- Maxwellian case: $\Pi f = \rho_f \, M(v)$ with $\rho_f := \int_{\mathbb{R}^d} f \, dv$
- Replace f by $f - F$: $0 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f \, dx \, dv = \langle f, F \rangle$, $\int_{\mathbb{R}^d} (\Pi \, f - f) \, dv = 0$
- L is a linear relaxation operator: $\mathsf{L} \, f = \Pi \, f - f$

$$\langle L f, f \rangle \leq -\| (1 - \Pi) \, f \|^2$$

- The other terms: for any c,

$$-\varepsilon \, \langle \mathsf{A} \, \mathsf{T} \, (1 - \Pi) \, f, f \rangle = -\varepsilon \, \langle \mathsf{A} \, \mathsf{T} \, (1 - \Pi) \, f, \Pi \, f \rangle$$

$$\leq \frac{c}{2} \, \| \mathsf{A} \, \mathsf{T} \, (1 - \Pi) \, f \|^2 + \frac{\varepsilon^2}{2 \, c} \, \| \Pi \, f \|^2$$

$$\varepsilon \, \langle \mathsf{T} \, \mathsf{A} \, f, f \rangle = \varepsilon \, \langle \mathsf{T} \, \mathsf{A} \, f, \Pi \, f \rangle \leq \frac{c}{2} \, \| \mathsf{T} \, A \, f \|^2 + \frac{\varepsilon^2}{2 \, c} \, \| \Pi \, f \|^2$$

$$\varepsilon \, \langle (\mathsf{A} + \mathsf{A}^*) \, \mathsf{L} \, f, f \rangle \leq \varepsilon \, \| (1 - \Pi) \, f \|^2 + \varepsilon \, \| \mathsf{A} \, f \|^2$$

Reminder: A and $\mathsf{T\,A}$ are bounded operators

$\mathsf{A\,T}\,(1-\Pi) = \Pi\,\mathsf{A\,T}\,(1-\Pi)$ and $\mathsf{T\,A} = \Pi\,\mathsf{T\,A}$. With $m_F := \int_{\mathbb{R}^d} |v|^2\,F\,dv$

$$\langle \hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi\,f, f\rangle = \frac{1}{d}\int_{\mathbb{R}^d}\left|\nabla_x\left(\tfrac{\rho_f}{\rho_F}\right)\right|^2 m_F\,dx$$

Recall that one ca compute $\mathsf{A}\,f := (1+\hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi)^{-1}\,\hat{\mathsf{a}}\cdot\mathsf{b}\,f$ as follows: let

$$g := \mathsf{A}\,f \qquad u := \rho_g/\rho_F$$

By definition of A, $\hat{\mathsf{a}}\cdot\mathsf{b}f = (1+\hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi)\,g$ means

$$-\nabla_x\cdot\int_{\mathbb{R}^d} v\,f\,dv =: -\nabla_x\cdot j_f = \rho_F\,u - \frac{1}{d}\,\nabla_x\cdot(m_F\,\nabla_x u)$$

$\|\mathsf{A}\,f\|^2 = \int_{\mathbb{R}^d}|u|^2\,\rho_F\,dx$ and $\|\mathsf{T\,A}\,f\|^2 = \frac{1}{d}\int_{\mathbb{R}^d}|\nabla_x u|^2\,m_F\,dx$ are such that

$$2\,\|\mathsf{A}\,f\|^2 + \|\mathsf{T\,A}\,f\|^2 \le \|(1-\Pi)\,f\|^2$$

Preliminary computations (2/2)

$$D(f) \le \underbrace{-\left(1-\frac{c}{2}-2\,\varepsilon\right)\,\|(1-\Pi)\,f\|^2}_{\text{micro: } \le 0} - \varepsilon \underbrace{\langle \mathsf{A}\,\mathsf{T}\,\Pi\,f,\,f\rangle}_{\text{macro: “first estimate”}}$$

$$+\,\frac{c}{2}\,\underbrace{\|\mathsf{A}\,\mathsf{T}\,(1-\Pi)\,f\|^2}_{\text{“second estimate...”}} + \frac{\varepsilon^2}{c}\,\|\Pi\,f\|^2$$

...$(\mathsf{A}\,\mathsf{T}\,(1-\Pi))^* f = (\hat{\mathsf{a}}\cdot\mathsf{a}\,(1-\Pi))^* g$ with $g = (1+\hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi)^{-1} f$ means

$$\rho_f = \rho_F\,u - \frac{1}{d}\,\nabla_x\,(m_F\,\nabla_x u)$$

where $u = \rho_g/\rho_F$. Let $q_F := \int_{\mathbb{R}^d} |v_1|^4\,F\,dv$, $u_{ij} := \partial^2 u/\partial x_i \partial x_j$

$$\|(\mathsf{A}\,\mathsf{T}\,(1-\Pi))^* f\|^2 = \sum_{i,\,j=1}^{d} \int_{\mathbb{R}^d} \left[\left(\tfrac{2\,\delta_{ij}+1}{3}\,q_F - \tfrac{m_F^2\,\delta_{ij}}{d^2\,\rho_F}\right) u_{ii}\,u_{jj} + \tfrac{2(1-\delta_{ij})}{3}\,q_F\,u_{ij}^2\right] dx$$

Maxwellian case

With $\rho_F = e^{-V} = \frac{1}{d}\, m_F = q_F$, $\mathsf{B} = \hat{\mathsf{a}} \cdot \mathsf{a}\,\Pi$, $g = (1 + \mathsf{B})^{-1} f$ means

$$\rho_f = u\, e^{-V} - \nabla_x \left(e^{-V}\, \nabla_x u\right) \quad \text{if} \quad u = \frac{\rho_g}{\rho_F}$$

Spectral gap condition: there exists a positive constant Λ such that

$$\int_{\mathbb{R}^d} |u|^2\, e^{-V}\, dx \le \Lambda \int_{\mathbb{R}^d} |\nabla_x u|^2\, e^{-V}\, dx$$

Since $\mathsf{A}\,\mathsf{T}\,\Pi = (1 + \mathsf{B})^{-1}\,\mathsf{B}$, we get the "first estimate"

$$\underbrace{\langle \mathsf{A}\,\mathsf{T}\,\Pi\, f, f \rangle}_{\text{macro}} \ge \frac{\Lambda}{1 + \Lambda}\, \|\Pi\, f\|^2$$

Notice that $\mathsf{B} = \hat{\mathsf{a}} \cdot \mathsf{a}\,\Pi = (\mathsf{T}\,\Pi)^*(\mathsf{T}\,\Pi)$ so that $\langle \mathsf{B}\, f, f \rangle = \|\mathsf{T}\,\Pi\, f\|^2$

Second estimate (1/3)

We have to bound (H^2 estimate)

$$\|(\mathsf{A}\,\mathsf{T}\,(1-\Pi))^* f\|^2 \le 2 \sum_{i,\,j=1}^{d} \int_{\mathbb{R}^d} |u_{ij}|^2\, e^{-V} dx$$

Let $\|u\|_0^2 := \int_{\mathbb{R}^d} |u|^2\, e^{-V} dx$. Multiply $\rho_f = u\, e^{-V} - \nabla_x \left(e^{-V}\, \nabla_x u\right)$ by $u\, e^{-V}$ to get

$$\|u\|_0^2 + \|\nabla_x u\|_0^2 \le \|\Pi\, f\|^2$$

By expanding the square in $|\nabla_x (u\, e^{-V/2})|^2$, with $\kappa = (1-\theta)/(2\,(2+\Lambda\, c_0))$, we obtain an improved Poincaré inequality

$$\kappa\, \|W\ u\|_0^2 \le \|\nabla_x u\|_0^2$$

for any $u \in H^1(e^{-V} dx)$ such that $\int_{\mathbb{R}^d} u\, e^{-V} dx = 0$

Here: $W := |\nabla_x V|$

Second estimate (2/3)

Multiply $\rho_f = u\,e^{-V} - \nabla_x\left(e^{-V}\,\nabla_x u\right)$ by $W^2\,u$ with $W := |\nabla_x V|$ and integrate by parts

$$\|W\,u\|_0^2 + \|W\,\nabla_x u\|_0^2 - 2\,c_1\left(\|\nabla_x u\|_0 + \|W\,\nabla_x u\|_0\right)\cdot\|W\,u\|_0 \le \frac{\kappa}{8}\,\|W^2 u\|_0^2 + \frac{2}{\kappa}\,\|\Pi\,f\|^2$$

Improved Poincaré inequality applied to $W\,u - \int_{\mathbb{R}^d} W\,u\,e^{-V}\,dx$ gives

$$\kappa\,\|W^2 u\|_0^2 \le \int_{\mathbb{R}^d} |\nabla_x(W\,u)|^2\,e^{-V}\,dx + 2\,\kappa\int_{\mathbb{R}^d} W\,u\,e^{-V}\,dx\int_{\mathbb{R}^d} W^3\,u\,e^{-V}\,dx$$

(...) $\kappa\,\|W^2\,u\|_0^2 \le 4\,\|W\,\nabla_x u\|_0^2 + 8\,c_1^2\,(\|u\|_0^2 + \|W\,u\|_0^2) + 4\,\kappa\,\|W\|_0^4\,\|u\|_0^2$

$$\|W\,\nabla_x u\|_0 \le c_5\,\|\Pi\,f\|$$

Second estimate (3/3)

Multiply $\rho_f = u\,e^{-V} - \nabla_x\left(e^{-V}\,\nabla_x u\right)$ by Δu and integrating by parts, we get

$$\|\nabla_x^2 u\|_0^2 - \left(\|W\,\nabla_x u\|_0 + \|\Pi\,f\|\right)\|\nabla_x^2 u\|_0 \leq \|W\,u\|_0\,\|\nabla_x u\|_0$$

Altogether (...)

$$\|(\mathsf{A}\,\mathsf{T}\,(1-\Pi))\,f\|^2 \leq c_6\,\|\,(1-\Pi)\,f\|^2$$

Summarizing, with $\lambda_1 = 1 - \frac{c}{2}\,(1+c_6) - 2\,\varepsilon$ and $\lambda_2 = \frac{\Lambda\,\varepsilon}{1+\Lambda} - \frac{\varepsilon^2}{c}$

$$D(f) \leq -\lambda_1\,\|(1-\Pi)\,f\|^2 - \lambda_2\,\|\Pi\,f\|^2$$

The Vlasov-Fokker-Planck equation

$$\partial_t f + \mathsf{T}\, f = \mathsf{L}\, f \quad \text{with} \quad \mathsf{L}\, f = \Delta_v f + \nabla_v (v\, f)$$

Under the same assumptions as in the linear BGK model... same result !
A list of changes

- $\underbrace{\langle f, \mathsf{L}\, f \rangle}_{\text{micro}} = -\|\nabla_v f\|^2 \le -\|(1-\Pi)\, f\|^2$ by the Poincaré inequality
- Since $\Pi\, f = \rho_f\, M(v)$, where $M(v)$ is the gaussian function, and $F(x,v) = M(v)\, e^{-V(x)}$

$$\langle \mathsf{A}\, f, \mathsf{L}\, f \rangle = \iint \rho_{\mathsf{A}\, f}\, M(v)\, (\mathsf{L}\, f)\, \frac{dx\, dv}{F} = \iint \rho_{\mathsf{A}\, f}\, (\mathsf{L}\, f)\, e^V\, dx\, dv = 0$$

- $\mathsf{A}\, f = u\, F$ means $\rho_F\, u - \frac{1}{d}\, \nabla_x (m_F\, \nabla_x u) = -\nabla_x\, j_f$, $j_f := \int_{\mathbb{R}^d} v\, f\, dv$. Hence $j_{\mathsf{L}\, f} = -j_f$ gives

$$\langle \mathsf{A}\, \mathsf{L}\, f, f \rangle = -\langle \mathsf{A}\, f, f \rangle$$

Motivation: nonlinear diffusion as a diffusion limit

$$\varepsilon^2 \, \partial_t f + \varepsilon \Big[v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f \Big] = Q(f)$$

$$\text{with} \quad Q(f) = \gamma \left(\frac{1}{2} |v|^2 - \overline{\mu}(\rho_f) \right) - f$$

Local mass conservation determines $\overline{\mu}(\rho)$

Theorem [Dolbeault, Markowich, Oelz, CS, 2007] ρ_f converges as $\varepsilon \to 0$ to a solution of

$$\partial_t \rho = \nabla_x \cdot (\nabla_x \nu(\rho) + \rho \nabla_x V)$$

with $\nu'(\rho) = \rho \, \overline{\mu}'(\rho)$

$\gamma(s) = (-s)_+^k$, $\nu(\rho) = \rho^m$, $0 < m = m(k) < 5/3$ ($\mathbb{R}^3$ case)

Linearized fast diffusion case

Consider a solution of $\partial_t f + \mathsf{T}\, f = \mathsf{L}\, f$ where $\mathsf{L}\, f = \Pi\, f - f$, $\Pi\, f := \frac{\rho}{\rho_F}$

$$F(x,v) := \omega \left(\frac{1}{2}\,|v|^2 + V(x)\right)^{-(k+1)}\,, \quad V(x) = \left(1 + |x|^2\right)^{\beta}$$

where ω is a normalization constant chosen such that $\iint_{\mathbb{R}^d\times\mathbb{R}^d} F\,dx\,dv = 1$ and $\rho_F = \omega_0\, V^{d/2-k-1}$ for some $\omega_0 > 0$

Theorem 3. *Let $d \geq 1$, $k > d/2+1$. There exists a constant $\beta_0 > 1$ such that, for any $\beta \in (\min\{1, (d-4)/(2k-d-2)\}, \beta_0)$, there are two positive, explicit constants C and λ for which the solution satisfies:*

$$\forall\, t \geq 0\,, \quad \|f(t) - F\|^2 \leq C\, \|f_0 - F\|^2\, e^{-\lambda t}.$$

Computations are the same as in the Maxwellian case except for the “first estimate" and the “second estimate"

Linearized fast diffusion case: first estimate

For $p = 0, 1, 2$, let $w_p^2 := \omega_0 \, V^{p-q}$, where $q = k + 1 - d/2$, $w_0^2 := \rho_F$

$$\|u\|_i^2 = \int_{\mathbb{R}^d} |u|^2 \, w_i^2 \, dx$$

Now $g = (1 + \hat{\mathsf{a}} \cdot \mathsf{a}\,\Pi)^{-1} f$ means

$$\rho_f = w_0^2 \, u - \frac{2}{2k - d} \nabla_x \left(w_1^2 \, \nabla_x u \right)$$

Hardy-Poincaré inequality [Blanchet, Bonforte, J.D., Grillo, Vázquez]

$$\|u\|_0^2 \leq \Lambda \, \|\nabla_x u\|_1^2$$

under the condition $\int_{\mathbb{R}^d} u \, w_0^2 \, dx = 0$ if $\beta \geq 1$. As a consequence

$$\langle \mathsf{A}\,\mathsf{T}\,\Pi\, f, f \rangle \geq \frac{\Lambda}{1 + \Lambda} \, \|\Pi\, f\|^2$$

Linearized fast diffusion case: second estimate

Observe that $\rho_f = w_0^2\, u - \frac{2}{2k-d}\,\nabla_x\left(w_1^2\,\nabla_x u\right)$ multiplied by u gives

$$\|u\|_0^2 + (q-1)^{-1}\,\|\nabla_x u\|_1^2 \le \|\Pi\, f\|^2$$

By the Hardy-Poincaré inequality (condition $\beta < \beta_0$)

$$\int_{\mathbb{R}^d} V^{\alpha+1-q-\frac{1}{\beta}}\,|u|^2\,dx - \frac{\left(\int_{\mathbb{R}^d} V^{\alpha+1-q-\frac{1}{\beta}}\,u\,dx\right)^2}{\int_{\mathbb{R}^d} V^{\alpha+1-q-\frac{1}{\beta}}\,dx}$$

$$\le \frac{1}{4\,(\beta_0-1)^2}\int_{\mathbb{R}^d} V^{\alpha+1-q}\,|\nabla_x u|^2\,dx$$

By multiplying $\rho_f = w_0^2\, u - \frac{2}{2k-d}\,\nabla_x\left(w_1^2\,\nabla_x u\right)$ by $V^\alpha\, u$ with $\alpha := 1-1/\beta$ or by $V\,\Delta u$ we find

$$\|\nabla_x^2 u\|_2^2 \le C\,\|\Pi\, f\|^2$$

Diffusion limits and hypocoercivity

The strategy of the method is to introduce at kinetic level the macroscopic quantities that arise by taking the diffusion limit

kinetic equation	diffusion equation	functional inequality (macroscopic)
Vlasov + BGK / Fokker-Planck	Fokker-Planck	Poincaré (gaussian weight)
linearized Vlasov-BGK	linearized porous media	Hardy-Poincaré
nonlinear Vlasov-BGK	porous media	Gagliardo-Nirenberg

- from kinetic to diffusive scales: parabolic scaling and diffusion limit
- heuristics: convergence of the macroscopic part at kinetic level is governed by the functional inequality at macroscopic level
- interplay between diffusion limits and hypocoercivity is still work in progress as well as the nonlinear case

Concluding remarks

- hypo-coercivity vs. hypoellipticity
- diffusion limits, a motivation for the “fast diffusion case”
- other collision kernels: scattering operators
- other functional spaces
- nonlinear kinetic models
- hydrodynamical models

Reference

J. Dolbeault, C. Mouhot, C. Schmeiser, Hypocoercivity for kinetic equations with linear relaxation terms, CRAS 347 (2009), pp. 511–516

`http://www.ceremade.dauphine.fr/∼dolbeaul/Conferences/`