A review of some old and more recent results on Keller-Segel and related problems

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Everything You Always Wanted to Know About Keller-Segel and Aggregation-Diffusion

Outline

Introduction: old basic facts

1. Asymptotic regimes of the Keller-Segel model
    - The super-critical range: life after blow-up
    - The subcritical range
    - Self-similar variables and an early convergence result
2. Functional framework and sharp asymptotics
    - Stationary solutions and linearization
    - Scalar product and spectrum
    - Rates of convergence for the nonlinear model
3. The critical mass case
    - A stable blow-up result
    - A global stability result

Large time asymptotics and / or blow up

The parabolic-elliptic Keller – Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \; t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \; t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t,x) = -\frac{1}{2\,\pi} \int_{\mathbb{R}^2} \log|x-y| \, u(t,y) \, dy$$

and observe that

$$\nabla v(t,x) = -\frac{1}{2\,\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \, u(t,y) \, dy$$

Mass conservation: $\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^2} u(t,x) \, dx = 0$

Blow-up: the virial computation

Collapse (S. Childress, J.K. Percus 81) $M=\int_{\mathbb{R}^2} n_0\,dx>8\,\pi$ and $\int_{\mathbb{R}^2}|x|^2\,n_0\,dx<\infty$: blow-up in finite time
A solution u of

$$\frac{\partial u}{\partial t}=\nabla\cdot(\nabla u-u\,\nabla v)$$

satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2\,u(t,x)\,dx$$

$$=-\underbrace{\int_{\mathbb{R}^2}2\,x\cdot\nabla u\,dx}_{-4\,M}+\frac{1}{2\,\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\underbrace{\tfrac{2x\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\,dx\,dy}_{\tfrac{(x-y)\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\,dx\,dy}$$

$$=4\,M-\frac{M^2}{2\,\pi}<0\quad\text{if}\quad M>8\,\pi$$

Asymptotic regimes of the Keller-Segel model

▷ Literature is huge
▷ Physics can be addressed in various ways: gravitation (Smoluchowski-Poisson) and statistics of gravitating systems, aggregation dynamics (sticky systems), biology (Patlak, Keller-Segel)
▷ Standard techniques have been reinvented many times: virial estimates, cumulated mass densities, matched asymptotics

- do not specialize to radial solutions
- put emphasis on functional analysis
- insist on nonlinear evolution

The super-critical range: regularization & life after blow-up

Regularize the Poisson kernel

$$(-\Delta)^{-1}_{\varepsilon} * \rho\,(x) = -\frac{1}{2\,\pi}\int_{\mathbb{R}^2} \log(|x-y|+\varepsilon)\,\rho(y)\,dy$$

[F. Poupaud, Diagonal defect measures, adhesion dynamics and Euler equations, Meth. Appl. Anal. **9** (2002), pp. 533–561]

Proposition (JD, C. Schmeiser 2009)

For every $\varepsilon > 0$, the regularized problem has a global solution satisfying

$$\|\rho^{\varepsilon}(\cdot,t)\|_{L^1(\mathbb{R}^2)} = \|\rho_0\|_{L^1(\mathbb{R}^2)} := M$$
$$\|\rho^{\varepsilon}(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le c\left(1+\frac{1}{\varepsilon^2}\right)$$

with an ε-independent constant c

The nonlinear term

$$m^\varepsilon(t,x) := \int_{\mathbb{R}^2} \mathcal{K}^\varepsilon(x-y)\,\rho^\varepsilon(t,x)\,\rho^\varepsilon(t,y)\,dy \text{ with } \mathcal{K}^\varepsilon(x) = \frac{x^{\otimes 2}}{|x|\,(|x|+\varepsilon)}$$

Lemma (Poupaud)

The families $\{\rho^\varepsilon(t)\}_{\varepsilon>0}$ and $\{m^\varepsilon(t)\}_{\varepsilon>0}$ are tightly bounded locally uniformly in t, and $\{\rho^\varepsilon(t)\}_{\varepsilon>0}$ is tightly equicontinuous in t

Tight boundedness and equicontinuity of $\rho^\varepsilon(t) \Longrightarrow$ compactness

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(x,y)\,\rho^\varepsilon(t,x)\,\rho^\varepsilon(t,y)\,dx\,dy \to \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(x,y)\,\rho(t,x)\,\rho(t,y)\,dx\,dy$$

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \varphi(t,x)\,m^\varepsilon(t,x)\,dx\,dt \to \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \varphi(t,x)\,m(t,x)\,dx\,dt$$

$$\text{for all } \varphi \in C_b([t_1,t_2]\times\mathbb{R}^2)$$

Defect measure

$$\nu(t,x) = m(t,x) - \int_{\mathbb{R}^2} \mathcal{K}(x-y)\,\rho(t,x)\,\rho(t,y)\,dy\,, \quad \mathcal{K}(x) = \frac{x^{\otimes 2}}{|x|^2}$$

Atomic support

The limit is characterized by the pair (ρ, ν)
The atomic support of ρ is an at most countable set

Lemma (Poupaud 2002)

ν is symmetric, nonnegative, and satisfies

$$\mathrm{tr}(\nu(t,x)) \leq \sum_{a \in S_{at}(\rho(t))} \Big(\rho(t)\big(\{a\}\big)\Big)^2 \delta(x-a)$$

$\mathcal{M}$: Radon measures, $\mathcal{M}_1^+$: nonnegative bounded measures

$$\begin{aligned}\mathcal{DM}^+(I;\mathbb{R}^2) \;=\; \Big\{ &(\rho,\nu):\; \rho(t) \in \mathcal{M}_1^+(\mathbb{R}^2)\ \forall\, t \in I,\ \nu \in \mathcal{M}(I \times \mathbb{R}^2)^{2\times 2} \\ &\rho \text{ is tightly continuous with respect to } t \\ &\nu \text{ is a nonnegative, symmetric, matrix valued measure} \\ &\mathrm{tr}(\nu(t,x)) \leq \sum_{a \in S_{at}(\rho(t))} \Big(\rho(t)\big(\{a\}\big)\Big)^2 \delta(x-a) \Big\}\end{aligned}$$

Limiting problem

The limiting flux: definition

$$\forall\,\varphi\in C^1_b((0,T)\,,\times\mathbb{R}^2)\quad \int_0^T\int_{\mathbb{R}^2}\varphi(t,x)\,j[\rho,\nu](t,x)\,dx\,dt$$
$$=-\tfrac{1}{4\,\pi}\int_0^T\int_{\mathbb{R}^4}(\varphi(t,x)-\varphi(t,y))\,K(x-y)\,\rho(t,x)\,\rho(t,y)\,dx\,dy\,dt$$
$$-\tfrac{1}{4\,\pi}\int_0^T\int_{\mathbb{R}^2}\nu(t,x)\,\nabla\varphi(t,x)\,dx\,dt$$

Theorem (JD, C. Schmeiser 2009)

For every $T>0$, ρ^ε converges tightly and uniformly in time to $\rho(t)$ and there exists $\nu(t)$ such that $(\rho,\nu)\in\mathcal{DM}^+((0,T);\mathbb{R}^2)$ is a generalized solution of

$$\partial_t\rho+\nabla\cdot(j[\rho,\nu]-\nabla\rho)=0$$

$\rho(t=0)=\rho_0$ holds in the sense of tight continuity

Strong formulation (formal) : an *ansatz*

- $\rho = \overline{\rho} + \hat{\rho}$, $\hat{\rho}(t,x) = \sum_{n\in N} M_n(t)\,\delta_n(t,x)$, $\delta_n(t,x) = \delta(x - x_n(t))$
- $(\rho,\nu) \in \mathcal{DM}^+((0,T);\mathbb{R}^2)$

$$\Longrightarrow \nu(t,x) = \textstyle\sum_{n\in N} \nu_n(t)\,\delta_n(t,x),\ \ \mathrm{tr}(\nu_n) \le M_n^2$$

$$j[\rho,\nu] = \overline{\rho}\,\nabla S_0[\overline{\rho}+\hat{\rho}] + \sum_n M_n\,\delta_n\,\nabla S_0\left[\overline{\rho} + \sum_{m\neq n} M_m\,\delta_m\right] + \frac{1}{4\,\pi}\sum_n M_n\,\nu_n\,\nabla\delta_n$$

$$\begin{aligned}\partial_t\overline{\rho} + \nabla\cdot(\overline{\rho}\,\nabla S_0[\overline{\rho}] - \nabla\overline{\rho}) + \nabla\overline{\rho}\cdot\nabla S_0[\hat{\rho}]&\\ + \sum_n \delta_n(\dot{M}_n - \overline{\rho}\,M_n)&\\ - \sum_n M_n\,\nabla\delta_n\left(\dot{x}_n - \nabla S_0\left[\overline{\rho} + \textstyle\sum_{m\neq n} M_m\,\delta_m\right]\right)&\\ + \sum_n\left(\tfrac{1}{4\,\pi}\nu_n : \nabla^2\delta_n - M_n\,\Delta\delta_n\right) &= 0\end{aligned}$$

$$\nu_n = 4\,\pi M_n\,\mathrm{id}$$

As a consequence of $\mathrm{tr}(\nu_n) = 8\,\pi\,M_n \le M_n^2$, point masses have to be at least $8\,\pi$ (there is only a finite number of them)

$$\partial_t\overline{\rho} + \nabla\cdot(\overline{\rho}\,\nabla S_0[\overline{\rho}] - \nabla\overline{\rho}) - \frac{1}{2\,\pi}\nabla\overline{\rho}\cdot\sum_n M_n\,\frac{x-x_n}{|x-x_n|^2} = 0$$

$$\dot{M}_n = \overline{\rho}(x=x_n)\,M_n$$

$$\dot{x}_n = \nabla S_0[\overline{\rho}](x=x_n) - \frac{1}{2\,\pi}\sum_{m\neq n} M_m\,\frac{x_n-x_m}{|x_n-x_m|^2}$$

Note that $\frac{d}{dt}\left(\int_{\mathbb{R}^2}\overline{\rho}\,dx + \sum_n M_n\right) = 0$

... Comparison with Velázquez' results

Large time behaviour

Assume again

$$\nu(t,x) = 4\,\pi\,\mathrm{id} \sum_{a \in S_{at}(\rho(t))} \rho(t)(\{a\})\,\delta(x-a)$$

and

$$\int_{\mathbb{R}^2} |x|^2\,\rho_{\mathrm{in}}\;dx < \infty$$

With $\hat{M} = \sum_{a \in S_{at}(\rho(t))} \rho(t)(\{a\})$ and $\overline{M} = M - \hat{M}$

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\,\rho\,dx = 4\,M - \frac{1}{2\,\pi}\int_{\mathbb{R}^4} (1-\chi_D)\rho \otimes \rho\,dy\,dx - \frac{1}{2\,\pi}\int_{\mathbb{R}^2} \mathrm{tr}(\nu)\,dx$$

$$= \overline{M}\left(4 - \frac{M}{2\,\pi} - \frac{\hat{M}}{2\,\pi}\right) - \frac{1}{2\,\pi}\sum_{a \neq b,\ a,b \in S_{at}(\rho(t))} \rho(t)(\{a\})\,\rho(t)(\{b\})$$

... compatible with Wasserstein's framework
(Haškovec, Schmeiser 2009)

Local density profiles

For fixed t and $a \in S_{at}(\rho(t))$, let $\varepsilon\,\xi = x - a$ and $\varepsilon^2\,\rho^\varepsilon = R^\varepsilon$

$$\varepsilon^2\,\partial_t R^\varepsilon + \nabla_\xi \cdot \left(R^\varepsilon\,\nabla_\xi S_1[R^\varepsilon] - \nabla_\xi R^\varepsilon\right) = 0$$

R^ε is uniformly bounded, implying compactness of $\nabla_\xi S_1[R^\varepsilon]$. The L^∞-weak* limit R of R^ε (take subsequences, formal) satisfies

$$\nabla_\xi \cdot \left(R\,\nabla_\xi S_1[R] - \nabla_\xi R\right) = 0$$

Observe that

$$\int_{\mathbb{R}^2} R(\xi)\,d\xi = \frac{1}{8\,\pi}\int_{\mathbb{R}^4} \frac{|\xi-\eta|}{|\xi-\eta|+1}\,R(\xi)\,R(\eta)\,d\eta\,d\xi \le \frac{1}{8\,\pi}\left(\int_{\mathbb{R}^2} R(\xi)\,d\xi\right)^2$$

This shows that either R vanishes or its mass is not smaller than $8\,\pi$

Free energy (1/2)

$$F_\varepsilon[\rho] := \int_{\mathbb{R}^2} \left(\rho \log \rho - \frac{1}{2} \rho\, S_\varepsilon[\rho] \right) dx$$
$$= \int_{\mathbb{R}^2} \rho \log \rho \, dx + \frac{1}{4\,\pi} \int_{\mathbb{R}^4} \log(|x-y| + \varepsilon)\, \rho(x)\, \rho(y)\, dy\, dx$$

and

$$\frac{d}{dt} F_\varepsilon[\rho^\varepsilon] = - \int_{\mathbb{R}^2} \rho^\varepsilon \left| \nabla \left(\log \rho^\varepsilon - S_\varepsilon[\rho^\varepsilon] \right) \right|^2 dx$$

With an arbitrary $a \in \mathbb{R}^2$ and $R(\xi) = \varepsilon^2\, \rho(a + \varepsilon\, \xi)$ we have

$$F_\varepsilon[\rho] = \left(2\,M - \frac{M^2}{4\,\pi} \right) \log \frac{1}{\varepsilon} + F_1[R]$$

Free energy (2/2)

Lemma

Let $R \in L^1_+(\mathbb{R}^2)$ *be radial,* $\int_{\mathbb{R}^2} \log(1+|x|)\, R(x)\, dx < \infty$, $M = \int_{\mathbb{R}^2} R\, dx$

$$\frac{1}{4\,\pi} \int_{\mathbb{R}^2} \log(1+|x-y|)\, R(y)\, dy \geq \frac{M}{4\,\pi} \log|x| \quad \forall\, x \in \mathbb{R}^2$$

$L^1_{+,M} := \{R \in L^1_+(\mathbb{R}^2) : \ \int_{\mathbb{R}^2} R\, d\xi = M\}$, $J_M := \inf_{R \in L^1_{+,M}} F_1[R] \geq -\infty$

Theorem

$J_M = -\infty$ *for* $M > 8\,\pi$, *and* $J_M > -\infty$ *for* $M \leq 8\,\pi$. *If* $M < 8\,\pi$, *there exists a radial nonincreasing minimizer*

Keller-Segel model: the subcritical range

Existence and free energy

$M = \int_{\mathbb{R}^2} n_0\,dx \le 8\,\pi$: global existence (W. Jäger, S. Luckhaus 1992), (JD, B. Perthame 2004), (A. Blanchet, JD, B. Perthame 2006)

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot \left[u \left(\nabla \left(\log u \right) - \nabla v \right) \right]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u \; dx - \frac{1}{2} \int_{\mathbb{R}^2} u\, v \; dx$$

satisfies

$$\frac{d}{dt} F[u(t,\cdot)] = - \int_{\mathbb{R}^2} u \left| \nabla \left(\log u \right) - \nabla v \right|^2 \, dx$$

(log HLS) inequality (E. Carlen, M. Loss 1992):
F is bounded from below if $M \le 8\,\pi$

... $M = 8\,\pi$ the critical case (A. Blanchet, J.A. Carrillo, N. Masmoudi 2008), (A. Blanchet et al.)

The existence setting for the subcritical regime

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \; t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \; t > 0 \\ u(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\, dx) \,, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx) \,, \quad M := \int_{\mathbb{R}^2} n_0(x)\, dx < 8\,\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\, dx \; \forall \, t \geq 0$

$v = -\frac{1}{2\,\pi} \, \log |\cdot| * u$

Time-dependent rescaling

$$u(x,t)=\frac{1}{R^2(t)}\,n\left(\frac{x}{R(t)},\tau(t)\right)\quad\text{and}\quad v(x,t)=c\left(\frac{x}{R(t)},\tau(t)\right)$$

with $R(t)=\sqrt{1+2t}$ and $\tau(t)=\log R(t)$

$$\begin{cases}\dfrac{\partial n}{\partial t}=\Delta n-\nabla\cdot(n\,(\nabla c-x)) & x\in\mathbb{R}^2\,,\;\; t>0\\ c=-\dfrac{1}{2\,\pi}\,\log|\cdot|*n & x\in\mathbb{R}^2\,,\;\; t>0\\ n(\cdot,t=0)=n_0\geq 0 & x\in\mathbb{R}^2\end{cases}$$

(A. Blanchet, JD, B. Perthame) Convergence in self-similar variables

$$\lim_{t\to\infty}\|n(\cdot,\cdot+t)-n_\infty\|_{\mathrm{L}^1(\mathbb{R}^2)}=0\quad\text{and}\quad\lim_{t\to\infty}\|\nabla c(\cdot,\cdot+t)-\nabla c_\infty\|_{\mathrm{L}^2(\mathbb{R}^2)}=0$$

means *intermediate asymptotics* in original variables:

$$\|u(x,t)-\tfrac{1}{R^2(t)}\,n_\infty\left(\tfrac{x}{R(t)},\tau(t)\right)\|_{L^1(\mathbb{R}^2)}\searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M \, \frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\,\pi}\log|\cdot| * n_\infty$$

Radial symmetry (Y. Naito)

Uniqueness (P. Biler, G. Karch, P. Laurençot, T. Nadzieja)

As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\varepsilon)|x|^2/2}$ for any $\varepsilon \in (0,1)$ (A. Blanchet, JD, B. Perthame)

Bifurcation diagram of $\|n_\infty\|_{\mathrm{L}^\infty(\mathbb{R}^2)}$ as a function of M

$$\lim_{M \to 0_+} \|n_\infty\|_{\mathrm{L}^\infty(\mathbb{R}^2)} = 0$$

(D.D. Joseph, T.S. Lundgren) (JD, R. Stańczy)
(The bifurcation diagram will be shown later)

The stationary solution when mass varies

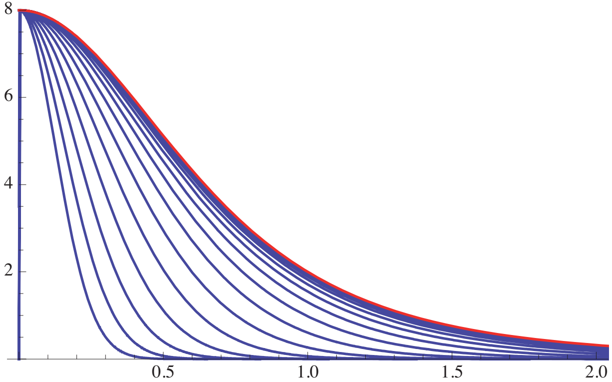

Figure: Representation of the solution appropriately scaled so that the $8\,\pi$ case appears as a limit (in red)

Self-similar variables and an early convergence result

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla\Big[n\,(\log n - x + \nabla c)\Big]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \, dx + \int_{\mathbb{R}^2} \frac{1}{2}\,|x|^2\, n\, dx - \frac{1}{2}\int_{\mathbb{R}^2} n\, c\, dx$$

satisfies

$$\frac{d}{dt}F[n(t,\cdot)] = -\int_{\mathbb{R}^2} n\,\big|\nabla\,(\log n) + x - \nabla c\big|^2\, dx$$

Yet another remark on $8\,\pi$ and scalings: $n^\lambda(x) = \lambda^2\, n(\lambda\, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2)\, dx + \int_{\mathbb{R}^2} \tfrac{\lambda^{-2}-1}{2}\,|x|^2\, n\, dx + \frac{1}{4\,\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\, n(y)\,\log\frac{1}{\lambda}\, dx\, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2\,M - \frac{M^2}{4\,\pi}\right)}_{>0 \text{ if } M<8\,\pi}\log\lambda + \frac{\lambda^{-2}-1}{2}\int_{\mathbb{R}^2}|x|^2\, n\, dx$$

Keller-Segel with subcritical mass in self-similar variables

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\; t > 0 \\ c = -\dfrac{1}{2\,\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\; t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

$$\lim_{t\to\infty} \|n(\cdot, \cdot + t) - n_\infty\|_{\mathrm{L}^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{\mathrm{L}^2(\mathbb{R}^2)} = 0$$

$$n_\infty = M\,\frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\;, \qquad c_\infty = -\frac{1}{2\,\pi}\log|\cdot| * n_\infty$$

First result: small mass case

Theorem (A. Blanchet, JD, M. Escobedo, J. Fernández)

There exists a positive constant M^* *such that, for any initial data* $n_0 \in L^2(n_\infty^{-1}\,dx)$ *of mass* $M < M^*$ *satisfying the above assumptions, there is a unique solution* $n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau,\infty)\times\mathbb{R}^2)$ *for any* $\tau > 0$

Moreover, there are two positive constants, C *and* δ*, such that*

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \, \frac{dx}{n_\infty} \le C\, e^{-\,\delta\, t} \quad \forall\; t > 0$$

As a function of M*,* δ *is such that* $\lim_{M\to 0_+} \delta(M) = 1$

Four steps proof

The condition $M \leq 8\,\pi$ is necessary and sufficient for the global existence of the solutions, but there are two extra smallness conditions in our proof:

- Uniform estimate: the *method of the trap*
- L^p and H^1 estimates in the self-similar variables
- *Spectral gap* of a linearized operator $\mathcal{L}$
- Duhamel formula and nonlinear estimates

Keller-Segel model: functional framework and sharp asymptotics

bifurcation diagrams

spectrum of the linearized operator

symmetrization

nonlinear estimates

rates of convergence for subcritical masses

... some preliminaries are needed

Stationary solutions and linearization

A parametrization of the stationary solutions

(J. Campos, JD)

$$-\Delta c = M \, \frac{e^{-\frac{1}{2}\,|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}\,|x|^2+c}\,dx}$$

Solve

$$-\varphi'' - \frac{1}{r}\,\varphi' = e^{-\frac{1}{2}\,r^2+\varphi}\,, \quad r>0$$

with initial conditions $\varphi(0)=a$, $\varphi'(0)=0$ and get with $r=|x|$

$$M(a) := 2\,\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2}\,r^2+\varphi_a}\,dx$$

$$n_a(x) = M(a)\,\frac{e^{-\frac{1}{2}\,r^2+\varphi_a(r)}}{2\,\pi\int_{\mathbb{R}^2} r\,e^{-\frac{1}{2}\,r^2+\varphi_a}\,dx} = e^{-\frac{1}{2}\,r^2+\varphi_a(r)}$$

Mass

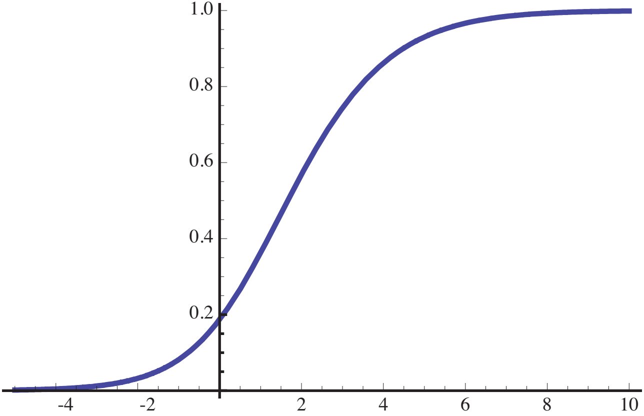

Figure: *The mass can be computed as* $M(a) = 2\,\pi \int_0^\infty n_a(r)\, r\, dr$
Plot of $a \mapsto M(a)/8\,\pi$

Bifurcation diagram

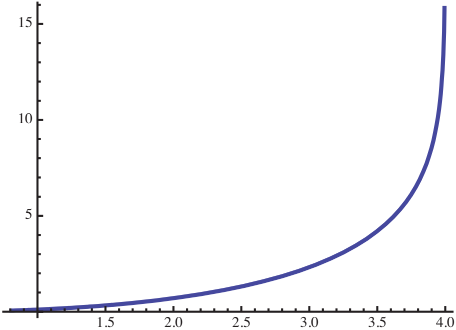

Figure: *The bifurcation diagram can be parametrized by*
$a \mapsto \left(\frac{1}{2\pi} M(a), \|c_a\|_\infty\right)$ *with* $\|c_a\|_\infty = c_a(0)$
(cf. Keller-Segel system in a ball with no flux boundary conditions)

Linearization

We can introduce two functions f and g such that

$$n = n_\infty\,(1+f) \quad \text{and} \quad c = c_\infty(1+g)$$

and rewrite the Keller-Segel model as

$$\frac{\partial f}{\partial t} = \mathcal{L}\,f + \frac{1}{n_\infty}\,\nabla(f\,n_\infty\,\nabla(c_\infty\,g))$$

where the linearized operator is

$$\mathcal{L}\,f = \frac{1}{n_\infty}\nabla\cdot\big(n_\infty\nabla(f - c_\infty\,g)\big)$$

and

$$-\Delta(c_\infty\,g) = n_\infty\,f$$

Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

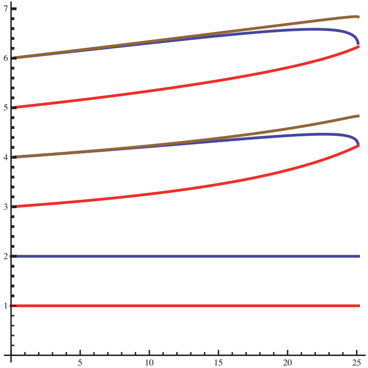

Figure: The lowest eigenvalues of $-\mathcal{L} = (-\Delta)^{-1}(n_a\, f)$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

(A. Blanchet, JD, M. Escobedo, J. Fernández), (J. Campos, JD), (V. Calvez, J.A. Carrillo), (J. Bedrossian, N. Masmoudi)

Spectral analysis in the functional framework determined by the relative entropy method

Simple eigenfunctions

Kernel Let $f_0 = \frac{\partial}{\partial M} c_\infty$ be the solution of

$$-\Delta\, f_0 = n_\infty\, f_0$$

and observe that $g_0 = f_0/c_\infty$ is such that

$$\frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f_0 - c_\infty\, g_0)\big) =: \mathcal{L}\, f_0 = 0$$

Lowest non-zero eigenvalues $f_1 := \frac{1}{n_\infty}\, \frac{\partial n_\infty}{\partial x_1}$ associated with $g_1 = \frac{1}{c_\infty}\, \frac{\partial c_\infty}{\partial x_1}$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L}\, f_1 = f_1$

With $D := x \cdot \nabla$, let $f_2 = 1 + \frac{1}{2}\, D\, \log n_\infty = 1 + \frac{1}{2\, n_\infty}\, D\, n_\infty$. Then

$$-\Delta\, (D\, c_\infty) + 2\, \Delta c_\infty = D\, n_\infty = 2\, (f_2 - 1)\, n_\infty$$

and so $g_2 := \frac{1}{c_\infty}\, (-\Delta)^{-1} (n_\infty\, f_2)$ is such that $-\mathcal{L}\, f_2 = 2\, f_2$

Functional setting...

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality (A. Blanchet, JD, B. Perthame)

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)\,(c - c_\infty)\, dx \geq 0$$

achieves its minimum for $n = n_\infty$

$$\mathsf{Q}_1[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\, F[n_\infty(1 + \varepsilon\, f)] \geq 0$$

if $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$. Notice that f_0 generates the kernel of Q_1

Lemma (J. Campos, JD)

Poincaré type inequality. For any $f \in H^1(\mathbb{R}^2, n_\infty\, dx)$ such that $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$, we have

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f\, n_\infty)|^2\, n_\infty\, dx = \int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, n_\infty\, dx \leq \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

… and eigenvalues

With g such that $-\Delta(g\,c_\infty) = f\,n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1\,f_2\,n_\infty\,dx - \int_{\mathbb{R}^2} f_1\,n_\infty\,(g_2\,c_\infty)\,dx$$

on the orthogonal space to f_0 in $L^2(n_\infty\,dx)$

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\,c_\infty)|^2\,n_\infty\,dx \quad \text{with} \quad g = -\frac{1}{c_\infty}\,\frac{1}{2\,\pi}\,\log|\cdot| * (f\,n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L}\,f \rangle = \mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma (J. Campos, JD)

$\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is 1

Linearized Keller-Segel theory

$$\mathcal{L}\,f = \frac{1}{n_\infty}\nabla\cdot\Big(n_\infty\nabla(f - c_\infty\,g)\Big)$$

Corollary (J. Campos, JD)

$$\langle f, f\rangle \le \langle \mathcal{L}\,f, f\rangle$$

The linearized problem takes the form

$$\frac{\partial f}{\partial t} = \mathcal{L}\,f$$

where $\mathcal{L}$ is a self-adjoint operator on the orthogonal of f_0 equipped with $\langle\cdot,\cdot\rangle$. A solution of

$$\frac{d}{dt}\,\langle f, f\rangle = -\,2\,\langle \mathcal{L}\,f, f\rangle$$

has therefore exponential decay

More on functional inequalities

A subcritical logarithmic HLS inequality

Recall that

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality (A. Blanchet, JD, B. Perthame)

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)(c - c_\infty)\, dx \geq 0$$

achieves its minimum for $n = n_\infty$

Lemma (J. Campos, JD)

Poincaré type inequality For any $f \in H^1(\mathbb{R}^2, n_\infty\, dx)$ *such that* $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$, *we have*

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f\, n_\infty)|^2\, n_\infty\, dx = \int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, n_\infty\, dx \leq \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

... Legendre duality

An Onofri type inequality

Theorem (J. Campos, JD)

For any $M \in (0, 8\,\pi)$, if $n_\infty = M\,\frac{e^{c_\infty - \frac{1}{2}\,|x|^2}}{\int_{\mathbb{R}^2} e^{c_\infty - \frac{1}{2}\,|x|^2}\,dx}$ with $c_\infty = (-\Delta)^{-1}\,n_\infty$, $d\mu_M = \frac{1}{M}\,n_\infty\,dx$, we have the inequality

$$\log\left(\int_{\mathbb{R}^2} e^{\varphi}\,d\mu_M\right) - \int_{\mathbb{R}^2} \varphi\,d\mu_M \le \frac{1}{2\,M}\int_{\mathbb{R}^2} |\nabla\varphi|^2\,dx \quad \forall\,\varphi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$$

Corollary (J. Campos, JD)

The following Poincaré inequality holds

$$\int_{\mathbb{R}^2} \left|\psi - \overline{\psi}\right|^2 n_\infty\,dx \le \int_{\mathbb{R}^2} |\nabla\psi|^2\,dx \quad \text{where} \quad \overline{\psi} = \int_{\mathbb{R}^2} \psi\,d\mu_M$$

An improved interpolation inequality (coercivity estimate)

Lemma (J. Campos, JD)

For any $f \in L^2(\mathbb{R}^2, n_\infty\, dx)$ *such that* $\int_{\mathbb{R}^2} f\, f_0\, n_\infty\, dx = 0$ *holds, we have*

$$-\frac{1}{2\,\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)\, n_\infty(x)\, \log|x-y|\, f(y)\, n_\infty(y)\, dx\, dy \le {\color{red}(1-\varepsilon)} \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

for some $\varepsilon > 0$*, where* $g\, c_\infty = G_2 * (f\, n_\infty)$ *and, if* $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$ *holds,*

$$\int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, dx \le {\color{red}(1-\varepsilon)} \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

Equivalence of the norms

As a consequence

$$\langle f, f \rangle := \int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx - \int_{\mathbb{R}^2} f \, n_\infty \, (g \, c_\infty) \, dx$$

is equivalent to

$$\int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx$$

under the condition that $\int_{\mathbb{R}^2} f \, f_0 \, n_\infty \, dx = 0$

A similar result is true in the critical case:
(J. Bedrossian, N. Masmoudi), (P. Raphaël, R. Schweyer)

A spectral gap estimate

Theorem (J. Campos, JD)

For any function $f \in \mathcal{D}(\mathsf{L}_2)$, *we have*

$$\langle f, f \rangle = \mathsf{Q}_1[f] \le \mathsf{Q}_2[f] = \langle f, \mathcal{L}\, f \rangle$$

The nonlinear Keller-Segel model, a functional analysis approach

Exponential convergence for any mass $M \leq 8\pi$

If $n_{0,*}(\sigma)$ stands for the symmetrized function associated to n_0, assume that for any $s \geq 0$

$$(H) \quad \exists\, \varepsilon \in (0, 8\pi - M) \quad \text{such that} \quad \int_0^s n_{0,*}(\sigma)\, d\sigma \leq \int_{B\left(0,\sqrt{s/\pi}\right)} n_{\infty,M+\varepsilon}(x)\, dx$$

Theorem (J. Campos, JD)

Under the above assumption, if $n_0 \in L^2_+(n_\infty^{-1}\, dx)$ and $M := \int_{\mathbb{R}^2} n_0\, dx < 8\pi$, then any solution with initial datum n_0 is such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \frac{dx}{n_\infty(x)} \leq C\, e^{-2t} \quad \forall\, t \geq 0$$

for some positive constant C, where n_∞ is the unique stationary solution with mass M

Sketch of the proof

- (J. Campos, JD) Uniform convergence of $n(t,\cdot)$ to n_∞ can be established for any $M\in(0,8\,\pi)$ by an adaptation of the symmetrization techniques of (J.I. Díaz, T. Nagai, J.M. Rakotoson)
- Uniform estimates (with no rates) easily follow
- Estimates based on Duhammel's formula inspired by (M. Escobedo, E. Zuazua) allow to prove uniform convergence
- Spectral estimates can be incorporated to the relative entropy approach
- Exponential convergence of the relative entropy follows

Step 1: symmetrization (1/2)

To any measurable function $u : \mathbb{R}^2 \mapsto [0,+\infty)$,
we associate the distribution function defined by $\mu(t,\tau) := |\{u > \tau\}|$
and its decreasing rearrangement given by

$$u_* : [0,+\infty) \;\to\; [0,+\infty] \,, \quad s \;\mapsto\; u_*(s) = \inf\{\tau \geq 0 \;:\; \mu(t,\tau) \leq s\} \,.$$

- For every measurable function $F : \mathbb{R}^+ \mapsto \mathbb{R}^+$, we have

$$\int_{\mathbb{R}^2} F(u)\,dx = \int_{\mathbb{R}^+} F(u_*)\,ds$$

- If $u \in W^{1,q}(0,T;L^p(\mathbb{R}^N))$ is a nonnegative function, with $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then $u_* \in W^{1,q}(0,T;L^p(0,\infty))$ and the formula

$$\int_0^{\mu(t,\tau)} \frac{\partial u_*}{\partial t}(t,\sigma)\,d\sigma = \int_{\{u(t,\cdot)>\tau\}} \frac{\partial u}{\partial t}(t,x)\,dx$$

holds for almost every $t \in (0,T)$ (J.I. Díaz, T. Nagai, J.M. Rakotoson)

Step 1: symmetrization (2/2)

Lemma

If n is a solution, then the function

$$k(t,s) := \int_0^s n_*(t,\sigma)\,d\sigma$$

satisfies $k \in L^\infty\left([0,+\infty)\times(0,+\infty)\right) \cap H^1\left([0,+\infty); W^{1,p}_{\rm loc}(0,+\infty)\right) \cap L^2\left([0,+\infty); W^{2,p}_{\rm loc}(0,+\infty)\right)$ *and*

$$\begin{cases} \frac{\partial k}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 k}{\partial s^2} - (k+2\,s)\,\frac{\partial k}{\partial s} \le 0 & \textit{a.e. in } (0,+\infty)\times(0,+\infty) \\ k(t,0)=0\,, \quad k(t,+\infty) = \int_{\mathbb{R}^2} n_0\,dx & \textit{for } t \in (0,+\infty) \\ k(0,s) = \int_0^s (n_0)_*\,d\sigma & \textit{for } s \ge 0 \end{cases}$$

Step 2: Uniform estimates

Proposition (J.I. Díaz, T. Nagai, J.M. Rakotoson)

Let f, g be two continuous functions on $Q = \mathbb{R}^+ \times (0,+\infty)$ such that ...

$$\begin{cases} \frac{\partial f}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 f}{\partial s^2} - (f + 2\,s)\,\frac{\partial f}{\partial s} \le \frac{\partial g}{\partial t} - 4\,\pi\,s\,\frac{\partial^2 g}{\partial s^2} - (g + 2\,s)\,\frac{\partial g}{\partial s} \;\; a.e. \;\; in\ Q \\ f(t,0) = 0 = g(t,0) \quad and \quad f(t,+\infty) \le g(t,+\infty) \text{ for any } t \in (0,+\infty) \\ f(0,s) \le g(0,s) \text{ for } s \ge 0\,, \text{ and } g(t,s) \ge 0 \text{ in } Q \end{cases}$$

then $f \le g$ on Q

Corollary

Assume that $n_0 \in L^2_+(n_\infty^{-1}\,dx)$ satisfies (H) and $M := \int_{\mathbb{R}^2} n_0\,dx < 8\,\pi$. Then there exist positive constants $C_1 = C_1(M,p)$ and $C_2 = C_2(M,p)$ such that

$$\|n\|_{L^p(\mathbb{R}^2)} \le C_1 \quad and \quad \|\nabla c\|_{L^\infty(\mathbb{R}^2)} \le C_2$$

Step 3: Estimates based on Duhammel's formula

Consider the kernel associated to the Fokker-Planck equation

$$K(t,x,y) := \frac{1}{2\,\pi\,(1-e^{-2\,t})}\,e^{-\frac{1}{2}\,\frac{|x-e^{-t}y|^2}{1-e^{-2\,t}}} \quad x \in \mathbb{R}^2\,, \quad y \in \mathbb{R}^2\,, \quad t>0$$

If n is a solution, then

$$n(t,x) = \int_{\mathbb{R}^2} K(t,x,y)\,n_0(y)\;dy + \int_0^t \int_{\mathbb{R}^2} \nabla_x K(t-s,x,y)\cdot n(s,y)\,\nabla c(s,y)\;dy\;ds$$

Corollary

Assume that n is a solution. Then

$$\lim_{t\to\infty} \|n(t,\cdot)-n_\infty\|_p = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|\nabla c(t,\cdot)-\nabla c_\infty\|_q = 0$$

for any $p \in [1,\infty]$ and any $q \in [2,\infty]$

Step 4: Spectral estimates can be incorporated

With $\mathsf{Q}_1[f] = \langle f, f \rangle$ and $\mathsf{Q}_2[f] = \langle f, \mathcal{L} f \rangle$

- For any function f in the orthogonal of the kernel of $\mathcal{L}$, we have

$$\mathsf{Q}_1[f] \leq \mathsf{Q}_2[f]$$

- For any radial function $f \in \mathcal{D}(\mathsf{L}_2)$, we have

$$2\,\mathsf{Q}_1[f] \leq \mathsf{Q}_2[f]$$

Cf. (V. Calvez, J.A. Carrillo) in the radial case

Step 5: Exponential convergence of the relative entropy

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f - \frac{1}{n_\infty}\,\nabla\left[n_\infty\, f\, \nabla(g\, c_\infty)\right]$$

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] = -\,2\,\mathsf{Q}_2[f(t,\cdot)] + \int_{\mathbb{R}^2} \nabla(f - g\, c_\infty)\, f\, n_\infty \cdot \nabla(g\, c_\infty)\; dx$$

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] \le -\,2\,\mathsf{Q}_2[f(t,\cdot)] + \delta(t,\varepsilon)\,\sqrt{\mathsf{Q}_1[f(t,\cdot)]\,\mathsf{Q}_2[f(t,\cdot)]}$$

$$\mathsf{Q}_1[f(t,\cdot)] \le \mathcal{Q} \quad \forall\; t \ge 0$$

$$\frac{d}{dt}\,\mathsf{Q}_1[f(t,\cdot)] \le -\,\mathsf{Q}_1[f(t,\cdot)]\left[2 - \,\delta(t,\varepsilon)\left(\left(\mathsf{Q}_1[f(t,\cdot)]\right)^{\frac{1-\varepsilon}{2-\varepsilon}} + \mathsf{Q}_1[f(t,\cdot)]\right)^{\frac{1}{2+\varepsilon}}\right)\right]$$

As a consequence, we finally get that

$$\limsup_{t\to\infty} e^{2\,t}\,\mathsf{Q}_1[f(t,\cdot)] < \infty$$

Some key ideas

- Lyapunov / Entropy functionals and functional inequalities
- Linearization and best constants
- Functional framework for linearized operators can be deduced from the entropy functional

- (G. Egaña Fernández, S. Mischler, 2016)

- weak notion of solution (based on free energy estimates)
- uniqueness, smoothing
- linearized and nonlinear stability in rescaled variables and exponential convergence under weaker assumptions
- sharp rates in $\mathrm{L}^{4/3}(\mathbb{R}^2)$

The critical mass case: infinite time blow up

The critical mass case: basic facts

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\, \nabla v) & \text{in } \mathbb{R}^2 \times (0,\infty) \\ v = (-\Delta)^{-1} u := \dfrac{1}{2\,\pi} \displaystyle\int_{\mathbb{R}^2} \log \frac{1}{|x-z|}\, u(z,t)\, dz \end{cases}$$

Mass and second moment of the solution

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2\, u(x,t)\, dx = 4\, M - \frac{M^2}{2\,\pi} = 0 \quad \text{if} \quad M = 8\,\pi$$

If the initial second moment is finite, there is *infinite time blow-up* for the solution, (Blanchet, Carrillo, Masmoudi, 2008)
Positive finite mass steady states (bubbles) with mass $8\,\pi$

$$U_{\lambda,\xi}(x) = \frac{1}{\lambda^2} U\left(\frac{x-\xi}{\lambda}\right), \quad U(y) = \frac{8}{(1+|y|^2)^2}\,, \quad \lambda > 0,\ \xi \in \mathbb{R}^2$$

are globally defined in time but have infinite second moment

The critical mass case: blow-up behaviour

- If a solution is attracted by the family $(U_{\lambda,\xi})$, then $M \geq 8\,\pi$ (Raphael, Schweyer, 2014)
- If $K_0 := \int_{\mathbb{R}^2} |x|^2\, u_0\, dx < \infty$, then blow-up occurs as $\lambda \to 0_+$
- If $M = 8\,\pi$ and $K_0 < \infty$, the infinite-time blow-up takes place as a bubble with $\lambda = \lambda(t) \to 0$: (Biler, Karch, Laurençot, Nadzieja, 2006) (Blanchet, Carrillo, Masmoudi, 2008)
- Formal rates: $\lambda(t) \sim \frac{c}{\sqrt{\log t}}$ as $t \to +\infty$ (Chavanis, Sire, 2006), (Campos Serrano, 2012)
- A radial solution with this rate and the stability within the radial class (Ghoul, Masmoudi, 2018)
- Spectral gap inequality known in the radial case only, but numerical evidence from (Campos Serrano, JD, 2014)
- Refined spectral estimates, variational methods and consequences: (del Pino, 2017), (Collot, Ghoul, Masmoudi, Nguyen, 2019-22) and (Davila, del Pino, JD, Musso, Wei, 2019-22)

... and much more: *Cf.* C. Collot (tomorrow)

A dynamical stability result

Theorem (Davila, del Pino, JD, Musso, Wei, 2022)

There exists a nonnegative, radially symmetric function u_* *with* $\int_{\mathbb{R}^2} u_*\,dx = 8\,\pi$ *and* $\int_{\mathbb{R}^2} |x|^2\,u_*\,dx < +\infty$ *such that every solution* $u(x,t)$ *with initial condition* u_0 *satisfies*

$$u(x,t) = \frac{1}{\lambda(t)^2}\,U\left(\frac{x-\xi(t)}{\lambda(t)}\right)(1+o(1)) \quad \text{with} \quad U(y) = \frac{8}{(1+|y|^2)^2}$$

uniformly on bounded sets of $\mathbb{R}^2$, *and for some* $c>0$ *and* $q \in \mathbb{R}^2$

$$\lambda(t) \sim \frac{c}{\sqrt{\log t}} \quad \text{and} \quad \xi(t) \to q \quad \text{as } t \to +\infty$$

under the condition that $\int_{\mathbb{R}^2} u_0\,dx = 8\,\pi$ *and* $\int_{\mathbb{R}^2} |x|^2\,u_0\,dx < +\infty$ *and* u_0 *is sufficiently close to* u_*

Details on the assumptions and strategy

- *Sufficiently close* for the perturbation $u_0(x) := u_*(x) + \varphi(x)$ is measured in the C^1-weighted norm

$$\|\varphi\|_{*\sigma} := \left\|(1+|\cdot|^{4+\sigma})\,\varphi\right\|_{\mathrm{L}^\infty(\mathbb{R}^2)} + \left\|(1+|\cdot|^{5+\sigma})\,\nabla\varphi(x)\right\|_{\mathrm{L}^\infty(\mathbb{R}^2)} < +\infty$$

with $\sigma > 1$ (the second moment of φ is finite)
and the perturbation φ must have zero mass

- *Uniformly on bounded sets*: for any bounded $K \subset \mathbb{R}^2$

$$\lim_{t\to\infty} \sup_{x\in K} \lambda(t)^2 U\left(\frac{x-\xi(t)}{\lambda(t)}\right)^{-1} \left|u(x,t) - \frac{1}{\lambda(t)^2} U\left(\frac{x-\xi(t)}{\lambda(t)}\right)\right| = 0$$

- The strategy

▷ Inner part: a truncated bubble
▷ Outer part: a heat equation in $\mathbb{R}^6$
▷ Inner-outer gluing, construction of an approximation (adjust parameters), control of the error along evolution

Log HLS stability and consequences, by E. Carlen

Logarithmic Hardy-Littlewood-Sobolev inequality

$$\mathcal{H}(\rho) := \int_{\mathbb{R}^2} \rho(x)\,\log \rho(x)\,dx + 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)\,\log(|x-y|)\rho(y)\,dx\,dy + 1 + \log \pi \geq 0$$

if $\rho \in \mathrm{L}^1(\mathbb{R}^2, \log(e+|x|^2)\,dx))$ is such that $\int_{\mathbb{R}^2} \rho\,dx = 1$
There is equality if and only if for some $x_0 \in \mathbb{R}^d$ and $s > 0$,

$$\rho(x) = \frac{1}{s^2}\,h\left(\frac{x}{s} - x_0\right) \quad \text{where} \quad h(x) = \frac{1}{\pi}\left(\frac{1}{1+|x|^2}\right)^2$$

(Carlen, Loss, 1992), (Figalli, Carlen, 2013), (Carlen, 2016), (Seuffert, 2017)

Theorem (Carlen, 2025)

$$\mathcal{H}(\rho) \geq \frac{1}{8}\inf_{g\in\mathcal{M}} \|\rho - g\|_1^2$$

The dynamical stability for the solution to Keller-Segel (critical mass case) improves to $\inf_{g\in\mathcal{M}} \|u(t) - 8\,\pi\,g\|_1 \leq C\,t^{-1/16}$, instead of the previously known bound: $O(Ct^{-1/160})$

These slides can be found at

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Thank you for your attention !