Symmetry, entropy and nonlinear flows

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Outline

- **Symmetry breaking and linearization**
  - The critical Caffarelli-Kohn-Nirenberg inequality
  - Linearization and spectrum
  - A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities

- **Without weights: Gagliardo-Nirenberg inequalities and fast diffusion flows**
  - The Bakry-Emery method on the sphere
  - Rényi entropy powers
  - Self-similar variables and relative entropies
  - The role of the spectral gap

- **With weights: Caffarelli-Kohn-Nirenberg inequalities and weighted nonlinear flows**
  - Large time asymptotics and spectral gaps
  - A discussion of optimality cases
Collaborations

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratorri (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large
time asymptotics for the evolution problem)
M. del Pino, G. Toscani (nonlinear flows and entropy methods)
A. Blanchet, G. Grillo, J.L. Vázquez (large time asymptotics and
linearization for the evolution equations)

...and also

S. Filippas, A. Tertikas, G. Tarantello, M. Kowalczyk ...
Background references (partial)


- Entropy methods in PDEs
  - Rényi entropy powers (information theory) (Savaré, Toscani, 2014), (Dolbeault, Toscani)
Recent related papers

Symmetry and symmetry breaking results

- The critical Caffarelli-Kohn-Nirenberg inequality
- Linearization and spectrum
- A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities
Critical Caffarelli-Kohn-Nirenberg inequality

Let \( \mathcal{D}_{a,b} := \left\{ v \in L^p (\mathbb{R}^d, |x|^{-b} \, dx) : |x|^{-a} |\nabla v| \in L^2 (\mathbb{R}^d, dx) \right\} \)

\[
\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^b} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} \, dx \quad \forall v \in \mathcal{D}_{a,b}
\]

holds under conditions on \( a \) and \( b \)

\[
p = \frac{2 d}{d - 2 + 2 (b - a)} \quad \text{(critical case)}
\]

\( \triangleright \) An optimal function among radial functions:

\[
v_\ast(x) = \left( 1 + |x|^{(p-2)(a_c-a)} \right)^{-\frac{2}{p-2}} \quad \text{and} \quad C_{a,b}^\ast = \frac{\| |x|^{-b} v_\ast \|_p^2}{\| |x|^{-a} \nabla v_\ast \|_2^2}
\]

**Question:** \( C_{a,b} = C_{a,b}^\ast \) (symmetry) or \( C_{a,b} > C_{a,b}^\ast \) (symmetry breaking)?
Critical CKN: range of the parameters

Figure: \( d = 3 \)

\[
\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|
abla v|^2}{|x|^{2a}} \, dx
\]

\[ a \leq b \leq a + 1 \text{ if } d \geq 3 \]
\[ a < b \leq a + 1 \text{ if } d = 2, \quad a + 1/2 < b \leq a + 1 \text{ if } d = 1 \]
and \( a < a_c := (d - 2)/2 \)

\[ p = \frac{2d}{d - 2 + 2(b - a)} \]

(Glaser, Martin, Grosse, Thirring (1976))
(Caffarelli, Kohn, Nirenberg (1984))
[F. Catrina, Z.-Q. Wang (2001)]
Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{FS}(a) := \frac{d(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$

[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]

The functional

$$C_{a,b}^\star \int_{\mathbb{R}^d} \frac{|
abla v|^2}{|x|^{2a}} \, dx - \left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b p}} \, dx \right)^{2/p}$$

is linearly instable at $v = v_\star$. 
Symmetry versus symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (Inventiones 2016)]

**Theorem**

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{FS}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric.
The Emden-Fowler transformation and the cylinder

With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder

\[ v(r, \omega) = r^{a_c - a} \varphi(s, \omega) \quad \text{with} \quad r = |x|, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r} \]

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the subcritical interpolation inequality

\[ \| \partial_s \varphi \|_{L^2(C)}^2 + \| \nabla \omega \varphi \|_{L^2(C)}^2 + \Lambda \| \varphi \|_{L^2(C)}^2 \geq \mu(\Lambda) \| \varphi \|_{L^p(C)}^2 \quad \forall \varphi \in H^1(C) \]

where \( \Lambda := (a_c - a)^2 \), \( C = \mathbb{R} \times S^{d-1} \) and the optimal constant \( \mu(\Lambda) \) is

\[ \mu(\Lambda) = \frac{1}{C_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda} \]
Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_*(s, \omega) = (\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$H^1(C) \ni \varphi \mapsto \|\partial_s \varphi\|_{L^2(C)}^2 + \|\nabla_\omega \varphi\|_{L^2(C)}^2 + \Lambda \|\varphi\|_{L^2(C)}^2$$

under a constraint on $\|\varphi\|_{L^p(C)}^2$

$\varphi_*$ is not optimal for (CKN) if the Pöschl-Teller operator

$$-\partial_s^2 - \Delta_\omega + \Lambda - \varphi_*^{p-2} = -\partial_s^2 - \Delta_\omega + \Lambda - \frac{1}{(\cosh s)^2}$$

has a negative eigenvalue, i.e., for $\Lambda > \Lambda_1$ (explicit)
The variational problem on the cylinder

\[ \Lambda \mapsto \mu(\Lambda) := \min_{\varphi \in H^1(C)} \frac{\|\partial_s \varphi\|_{L^2(C)}^2 + \|\nabla \omega \varphi\|_{L^2(C)}^2 + \Lambda \|\varphi\|_{L^2(C)}^2}{\|\varphi\|_{L^p(C)}^2} \]

is a concave increasing function

Restricted to symmetric functions, the variational problem becomes

\[ \mu_*(\Lambda) := \min_{\varphi \in H^1(\mathbb{R})} \frac{\|\partial_s \varphi\|_{L^2(\mathbb{R}^d)}^2 + \Lambda \|\varphi\|_{L^2(\mathbb{R}^d)}^2}{\|\varphi\|_{L^p(\mathbb{R}^d)}^2} = \mu_*(1) \Lambda^\alpha \]

Symmetry means \( \mu(\Lambda) = \mu_*(\Lambda) \)
Symmetry breaking means \( \mu(\Lambda) < \mu_*(\Lambda) \)
Numerical results

Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$.
Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point $\Lambda_1$ computed by V. Felli and M. Schneider. The branch behaves for large values of $\Lambda$ as predicted by F. Catrina and Z.-Q. Wang
what we want to discard...

When the local criterion (linear stability) differs from global results in a larger family of inequalities (center, right)...

J. Dolbeault

Symmetry, entropy and nonlinear flows
Subcritical Caffarelli-Kohn-Nirenberg inequalities

Norms: \( \| w \|_{L^{q,\gamma}(\mathbb{R}^d)} := (\int_{\mathbb{R}^d} |w|^q |x|^{-\gamma} \, dx)^{1/q} \), \( \| w \|_{L^q(\mathbb{R}^d)} := \| w \|_{L^{q,0}(\mathbb{R}^d)} \)


\[
\| w \|_{L^{2p,\gamma}(\mathbb{R}^d)} \leq C_{\beta,\gamma,p} \| \nabla w \|_{L^{2,\beta}(\mathbb{R}^d)}^{\vartheta} \| w \|_{L^{p+1,\gamma}(\mathbb{R}^d)}^{1-\vartheta},
\]

Here \( C_{\beta,\gamma,p} \) denotes the optimal constant, the parameters satisfy

\[
d \geq 2, \quad \gamma - 2 < \beta < \frac{d-2}{d} \gamma, \quad \gamma \in (-\infty, d), \quad p \in (1, p_*) \quad \text{with} \quad p_* := \frac{d-\gamma}{d-\beta-2}
\]

and the exponent \( \vartheta \) is determined by the scaling invariance, \( i.e.\),

\[
\vartheta = \frac{(d-\gamma)(p-1)}{p \left( d+\beta+2-2 \gamma - p(d-\beta-2) \right)}
\]

Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

\[
w_*(x) = \left( 1 + |x|^{2+\beta-\gamma} \right)^{-1/(p-1)} \quad \forall \, x \in \mathbb{R}^d
\]

Do we know \( (\text{symmetry}) \) that the equality case is achieved among radial functions?
Here $p$ is given

\[ \beta = \frac{d-2}{d} \gamma \]

\[ \beta = d - 2 + \frac{\gamma - d}{p} \]
Symmetry and symmetry breaking

(M. Bonforte, J.D., M. Muratori and B. Nazaret, 2016) Let us define

\[ \beta_{FS}(\gamma) := d - 2 - \sqrt{(d - \gamma)^2 - 4 (d - 1)} \]

**Theorem**

*Symmetry breaking holds in (CKN) if*

\[ \gamma < 0 \quad \text{and} \quad \beta_{FS}(\gamma) < \beta < \frac{d - 2}{d} \gamma \]

In the range \( \beta_{FS}(\gamma) < \beta < \frac{d-2}{d} \gamma \), \( w_\star(x) = (1 + |x|^{2+\beta-\gamma})^{-1/(p-1)} \) is not optimal

(JD, Esteban, Loss, Muratori, 2016)

**Theorem**

*Symmetry holds in (CKN) if*

\[ \gamma \geq 0, \quad \text{or} \quad \gamma \leq 0 \quad \text{and} \quad \gamma - 2 \leq \beta \leq \beta_{FS}(\gamma) \]
The green area is the region of symmetry, while the red area is the region of symmetry breaking. The threshold is determined by the hyperbola

\[(d - \gamma)^2 - (\beta - d + 2)^2 - 4(d - 1) = 0\]
Inequalities without weights and fast diffusion equations

▷ The Bakry-Emery method on the sphere

▷ Euclidean space: self-similar variables and relative entropies

▷ The role of the spectral gap
The Bakry-Emery method on the sphere

Entropy functional

\[ E_p[\rho] := \frac{1}{p-2} \left[ \int_{S^d} \rho^{\frac{2}{p}} \, d\mu - \left( \int_{S^d} \rho \, d\mu \right)^{\frac{2}{p}} \right] \quad \text{if} \quad p \neq 2 \]

\[ E_2[\rho] := \int_{S^d} \rho \log \left( \frac{\rho}{\|\rho\|_{L^1(S^d)}} \right) \, d\mu \]

Fisher information functional

\[ I_p[\rho] := \int_{S^d} |\nabla \rho|_p^2 \, d\mu \]

Bakry-Emery (carré du champ) method: use the heat flow

\[ \frac{\partial \rho}{\partial t} = \Delta \rho \]

and compute \( \frac{d}{dt} E_p[\rho] = -I_p[\rho] \) and \( \frac{d}{dt} I_p[\rho] \leq -d I_p[\rho] \) to get

\[ \frac{d}{dt} \left( I_p[\rho] - d E_p[\rho] \right) \leq 0 \quad \Rightarrow \quad I_p[\rho] \geq d E_p[\rho] \]

with \( \rho = |u|^p \), if \( p \leq 2^\# := \frac{2d^2+1}{(d-1)^2} \)
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^*$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m. \quad (1)$$

(Demange), (JD, Esteban, Kowalczyk, Loss): for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt} \left( I_p[\rho] - d E_p[\rho] \right) \leq 0$$

$p, m\) admissible region, $d = 5$
Rényi entropy powers and fast diffusion

- The Euclidean space without weights

▷ Rényi entropy powers, the entropy approach without rescaling: (Savaré, Toscani): scalings, nonlinearity and a concavity property inspired by information theory

▷ Faster rates of convergence: (Carrillo, Toscani), (JD, Toscani)
The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial \nu}{\partial t} = \Delta \nu^m$$

with initial datum $\nu(x, t = 0) = \nu_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} \nu_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \nu_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_*(t, x) := \frac{1}{(\kappa \, t^{1/\mu})^d} \, \mathcal{B}_* \left( \frac{x}{\kappa \, t^{1/\mu}} \right)$$

where

$$\mu := 2 + d \,(m - 1), \quad \kappa := \left| \frac{2 \, \mu \, m}{m - 1} \right|^{1/\mu}$$

and $\mathcal{B}_*$ is the Barenblatt profile

$$\mathcal{B}_*(x) := \begin{cases} 
(C_* - |x|^2)^{1/(m-1)} & \text{if } m > 1 \\
(C_* + |x|^2)^{1/(m-1)} & \text{if } m < 1 
\end{cases}$$
The Rényi entropy power $F$

The entropy is defined by

$$E := \int_{\mathbb{R}^d} \nu^m \, dx$$

and the Fisher information by

$$I := \int_{\mathbb{R}^d} \nu |\nabla p|^2 \, dx \quad \text{with} \quad p = \frac{m}{m-1} \nu^{m-1}$$

If $\nu$ solves the fast diffusion equation, then

$$E' = (1 - m) I$$

To compute $I'$, we will use the fact that

$$\frac{\partial p}{\partial t} = (m - 1) p \Delta p + |\nabla p|^2$$

$F := E^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d \left(1 - m\right)} = 1 + \frac{2}{1 - m} \left(\frac{1}{d} + m - 1\right) = \frac{2}{d} \frac{1}{1 - m} - 1$

has a linear growth asymptotically as $t \to +\infty$. 
The variation of the Fisher information

Lemma

If $v$ solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $\frac{1}{d} \leq m < 1$, then

$$I' = \frac{d}{dt} \int_{\mathbb{R}^d} v |\nabla p|^2 \, dx = -2 \int_{\mathbb{R}^d} v^m \left( \|D^2 p\|^2 + (m - 1)(\Delta p)^2 \right) \, dx$$

Explicit arithmetic geometric inequality

$$\|D^2 p\|^2 - \frac{1}{d} (\Delta p)^2 = \left\| D^2 p - \frac{1}{d} \Delta p \operatorname{Id} \right\|^2$$

There are no boundary terms in the integrations by parts.
The concavity property

**Theorem**

[Toscani-Savare] Assume that $m \geq 1 - \frac{1}{d}$ if $d > 1$ and $m > 0$ if $d = 1$. Then $F(t)$ is increasing, $(1 - m) F''(t) \leq 0$ and

$$\lim_{t \to +\infty} \frac{1}{t} F(t) = (1 - m) \sigma \lim_{t \to +\infty} E^{\sigma-1} l = (1 - m) \sigma E_{\star}^{\sigma-1} I_{\star}$$

[Dolbeault-Toscani] The inequality

$$E^{\sigma-1} l \geq E_{\star}^{\sigma-1} I_{\star}$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\| \nabla w \|_{L^2(\mathbb{R}^d)}^{\theta} \| w \|_{L^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq C_{GN} \| w \|_{L^q(\mathbb{R}^d)}$$

if $1 - \frac{1}{d} \leq m < 1$. Hint: $v^{m-1/2} = \frac{w}{\| w \|_{L^2(\mathbb{R}^d)}}, \quad q = \frac{1}{2 m-1}$
Euclidean space: self-similar variables and relative entropies

The large time behavior of the solution of \( \frac{\partial v}{\partial t} = \Delta v^m \) is governed by the source-type Barenblatt solutions

\[
v_\star(t, x) := \frac{1}{\kappa^d (\mu t)^d / \mu} \mathcal{B}_\star \left( \frac{x}{\kappa (\mu t)^{1/\mu}} \right) \quad \text{where} \quad \mu := 2 + d (m - 1)
\]

where \( \mathcal{B}_\star \) is the Barenblatt profile (with appropriate mass)

\[
\mathcal{B}_\star(x) := (1 + |x|^2)^{1/(m-1)}
\]

A time-dependent rescaling: self-similar variables

\[
v(t, x) = \frac{1}{\kappa^d R^d} u(\tau, \frac{x}{\kappa R}) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \frac{1}{2} \log \left( \frac{R(t)}{R_0} \right)
\]

Then the function \( u \) solves a Fokker-Planck type equation

\[
\frac{\partial u}{\partial \tau} + \nabla \cdot \left[ u \left( \nabla u^{m-1} - 2 x \right) \right] = 0
\]
Free energy and Fisher information

- The function $u$ solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[ u \left( \nabla u^{m-1} - 2x \right) \right] = 0$$

- (Ralston, Newman, 1984) Lyapunov functional:

Generalized entropy or Free energy

$$\mathcal{E}[u] := \int_{\mathbb{R}^d} \left( -\frac{u^m}{m} + |x|^2 u \right) \, dx - \mathcal{E}_0$$

- Entropy production is measured by the Generalized Fisher information

$$\frac{d}{dt} \mathcal{E}[u] = -\mathcal{I}[u] \, , \quad \mathcal{I}[u] := \int_{\mathbb{R}^d} u \, |\nabla u^{m-1} + 2x|^2 \, dx$$
Without weights: relative entropy, entropy production

Stationary solution: choose $C$ such that $\|u_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$ u_\infty(x) := (C + |x|^2)^{-1/(1-m)} $$

Relative entropy: Fix $E_0$ so that $E[u_\infty] = 0$

Entropy – entropy production inequality (del Pino, J.D.)

Theorem

$d \geq 3$, $m \in \left[\frac{d-1}{d}, +\infty\right)$, $m > \frac{1}{2}$, $m \neq 1$

$$ \mathcal{I}[u] \geq 4 \mathcal{E}[u] $$

Corollary

(del Pino, J.D.) A solution $u$ with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that

$$ |x|^2u_0 \in L^1(\mathbb{R}^d), \quad u_0^m \in L^1(\mathbb{R}^d) $$

satisfies

$$ \mathcal{E}[u(t, \cdot)] \leq \mathcal{E}[u_0] e^{-4t} $$
A computation on a large ball, with boundary terms

\[ \frac{\partial u}{\partial \tau} + \nabla \cdot \left[ u \left( \nabla u^{m-1} - 2x \right) \right] = 0 \quad \tau > 0, \quad x \in B_R \]

where \( B_R \) is a centered ball in \( \mathbb{R}^d \) with radius \( R > 0 \), and assume that \( u \) satisfies zero-flux boundary conditions

\[ (\nabla u^{m-1} - 2x) \cdot \frac{x}{|x|} = 0 \quad \tau > 0, \quad x \in \partial B_R. \]

With \( z(\tau, x) := \nabla Q(\tau, x) := \nabla u^{m-1} - 2x \), the relative Fisher information is such that

\[
\frac{d}{d\tau} \int_{B_R} u |z|^2 \, dx + 4 \int_{B_R} u |z|^2 \, dx + 2 \frac{1-m}{m} \int_{B_R} u^m \left( \| \nabla^2 Q \|^2 - (1-m)(\Delta Q)^2 \right) \, dx
\]

\[ = \int_{\partial B_R} u^m (\omega \cdot \nabla |z|^2) \, d\sigma \leq 0 \quad \text{(by Grisvard’s lemma)} \]
Entropy – entropy production, Gagliardo-Nirenberg ineq.

\[ 4 \mathcal{E}[u] \leq \mathcal{I}[u] \]

Rewrite it with \( p = \frac{1}{2m-1} \), \( u = w^{2p} \), \( u^m = w^{p+1} \) as

\[
\frac{1}{2} \left( \frac{2m}{2m-1} \right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \left( \frac{1}{1-m} - d \right) \int_{\mathbb{R}^d} |w|^{1+p} \, dx - K \geq 0
\]

• for some \( \gamma \), \( K = K_0 \left( \int_{\mathbb{R}^d} u \, dx = \int_{\mathbb{R}^d} w^{2p} \, dx \right)^\gamma \)
• \( w = w_\infty = v_\infty^{1/2p} \) is optimal

Theorem

[Del Pino, J.D.] With \( 1 < p \leq \frac{d}{d-2} \) (fast diffusion case) and \( d \geq 3 \)

\[
\|w\|_{L^{2p}(\mathbb{R}^d)} \leq C_{p,d}^{GN} \|\nabla w\|_{L^2(\mathbb{R}^d)}^\theta \|w\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta}
\]

\[
C_{p,d}^{GN} = \left( \frac{y(p-1)^2}{2\pi d^2} \right)^{\frac{\theta}{2}} \left( \frac{2y-d}{2y} \right)^{\frac{1}{2p}} \left( \frac{\Gamma(y)}{\Gamma(y-d/2)} \right)^{\frac{\theta}{d}}, \quad \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \quad y = \frac{p+1}{p-1}
\]
Spectral gap: sharp asymptotic rates of convergence

Assumptions on the initial datum \( v_0 \)

**(H1)** \( V_{D_0} \leq v_0 \leq V_{D_1} \) for some \( D_0 > D_1 > 0 \)

**(H2)** if \( d \geq 3 \) and \( m \leq m_* \), \( (v_0 - V_D) \) is integrable for a suitable \( D \in [D_1, D_0] \)

**Theorem**

(Blanchet, Bonforte, J.D., Grillo, Vázquez) Under Assumptions (H1)-(H2), if \( m < 1 \) and \( m \neq m_* := \frac{d-4}{d-2} \), the entropy decays according to

\[
\mathcal{E}[v(t, \cdot)] \leq C e^{-2(1-m) \Lambda_{\alpha, d} t} \quad \forall \ t \geq 0
\]

where \( \Lambda_{\alpha, d} > 0 \) is the best constant in the Hardy–Poincaré inequality

\[
\Lambda_{\alpha, d} \int_{\mathbb{R}^d} |f|^2 \, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_{\alpha} \quad \forall \ f \in H^1(d\mu_{\alpha})
\]

with \( \alpha := 1/(m - 1) < 0 \), \( d\mu_{\alpha} := h_{\alpha} \, dx \), \( h_{\alpha}(x) := (1 + |x|^2)^{\alpha} \)
Spectral gap and best constants

\[ m_c = \frac{d-2}{d} \]
\[ m_1 = \frac{d-1}{d} \]
\[ \tilde{m}_1 = \frac{d}{d+2} \]
\[ \tilde{m}_2 := \frac{d+4}{d+6} \]
\[ m_2 = \frac{d+1}{d+2} \]
Weighted nonlinear flows: Caffarelli-Kohn-Nirenberg inequalities

- Entropy and Caffarelli-Kohn-Nirenberg inequalities
- Large time asymptotics and spectral gaps
- Optimality cases
When symmetry holds, (CKN) can be written as an entropy – entropy production inequality

\[
\frac{1-m}{m} (2 + \beta - \gamma)^2 \mathcal{E}[v] \leq \mathcal{I}[v]
\]

and equality is achieved by \( \mathcal{B}_{\beta,\gamma}(x) := (1 + |x|^{2+\beta-\gamma})^{\frac{1}{m-1}} \).

Here the free energy and the relative Fisher information are defined by

\[
\mathcal{E}[v] := \frac{1}{m - 1} \int_{\mathbb{R}^d} \left( v^m - \mathcal{B}_{\beta,\gamma}^m - m \mathcal{B}_{\beta,\gamma}^{m-1} (v - \mathcal{B}_{\beta,\gamma}) \right) \frac{dx}{|x|^\gamma}
\]

\[
\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} - \nabla \mathcal{B}_{\beta,\gamma}^{m-1} \right|^2 \frac{dx}{|x|^\beta}
\]

If \( v \) solves the Fokker-Planck type equation

\[
v_t + |x|^{\gamma} \nabla \cdot \left( |x|^{-\beta} v \nabla (v^{m-1} - |x|^{2+\beta-\gamma}) \right) = 0 \quad \text{(WFDE-FP)}
\]

then

\[
\frac{d}{dt} \mathcal{E}[v(t, \cdot)] = - \frac{m}{1 - m} \mathcal{I}[v(t, \cdot)]
\]
Proof of symmetry (1/3: changing the dimension)

We rephrase our problem in a space of higher, artificial dimension $n > d$ (here $n$ is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$
\nu(|x|^{\alpha-1}x) = w(x), \quad \alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2 \frac{d - \gamma}{\beta + 2 - \gamma},
$$

we claim that Inequality (CKN) can be rewritten for a function $\nu(|x|^{\alpha-1}x) = w(x)$ as

$$
\|\nu\|_{L^{2p,d-n}(\mathbb{R}^d)} \leq K_{\alpha,n,p} \|D_\alpha \nu\|_{L^{2,d-n}(\mathbb{R}^d)} \|\nu\|_{L^{p+1,d-n}(\mathbb{R}^d)}^{1-\theta} \quad \forall \nu \in H^{p}_{d-n,d-n}(\mathbb{R}^d)
$$

with the notations $s = |x|$, $D_\alpha \nu = (\alpha \frac{\partial \nu}{\partial s}, \frac{1}{s} \nabla \omega \nu)$ and

$$
d \geq 2, \quad \alpha > 0, \quad n > d \quad \text{and} \quad p \in (1, p_\star].
$$

By our change of variables, $w_\star$ is changed into

$$
\nu_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall x \in \mathbb{R}^d
$$
The strategy of the proof (2/3: Rényi entropy)

The derivative of the generalized Rényi entropy power functional is

$$G[u] := \left( \int_{\mathbb{R}^d} u^m \, d\mu \right)^{\sigma^{-1}} \int_{\mathbb{R}^d} u |D\alpha P|^2 \, d\mu$$

where $$\sigma = \frac{2}{d} \frac{1}{1-m} - 1$$. Here $$d\mu = |x|^{n-d} \, dx$$ and the pressure is

$$P := \frac{m}{1 - m} u^{m-1}$$

Looking for an optimal function in (CKN) is equivalent to minimize $$G$$ under a mass constraint.
With \( L_\alpha = -D_\alpha^* D_\alpha = \alpha^2 \left( u'' + \frac{n-1}{s} u' \right) + \frac{1}{s^2} \Delta_\omega u \), we consider the fast diffusion equation

\[
\frac{\partial u}{\partial t} = L_\alpha u^m
\]

in the subcritical range \( 1 - 1/n < m < 1 \). The key computation is the proof that

\[
- \frac{d}{dt} G[u(t, \cdot)] \left( \int_{\mathbb{R}^d} u^m \, d\mu \right)^{1-\sigma} \\
\geq (1 - m) (\sigma - 1) \int_{\mathbb{R}^d} u^m \left| L_\alpha P - \frac{\int_{\mathbb{R}^d} u |D_\alpha P|^2 \, d\mu}{\int_{\mathbb{R}^d} u^m \, d\mu} \right|^2 \, d\mu \\
+ 2 \int_{\mathbb{R}^d} \left( \alpha^4 \left( 1 - \frac{1}{n} \right) \left| P'' - \frac{P'}{s} - \frac{\Delta_\omega P}{\alpha^2 (n-1) s^2} \right|^2 + \frac{2 \alpha^2}{s^2} \left| \nabla_\omega P' - \frac{\nabla_\omega P}{s} \right|^2 \right) u^m \, d\mu \\
+ 2 \int_{\mathbb{R}^d} \left( (n - 2) \left( \alpha_{FS}^2 - \alpha^2 \right) |\nabla_\omega P|^2 + c(n, m, d) \frac{|\nabla_\omega P|^4}{P^2} \right) u^m \, d\mu =: \mathcal{H}[u]
\]

for some numerical constant \( c(n, m, d) > 0 \). Hence if \( \alpha \leq \alpha_{FS} \), the r.h.s. \( \mathcal{H}[u] \) vanishes if and only if \( P \) is an affine function of \( |x|^2 \), which proves the symmetry result. A quantifier elimination problem (Tarski, 1951)?
This method has a hidden difficulty: integrations by parts! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if $u$ solves the Euler-Lagrange equation, we test by $L_\alpha u^m$

$$0 = \int_{\mathbb{R}^d} dG[u] \cdot L_\alpha u^m d\mu \geq \mathcal{H}[u] \geq 0$$

$\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by $|x|^{\gamma} \text{div} \left(|x|^{-\beta} \nabla w^{1+p}\right)$ the equation

$$\frac{(p - 1)^2}{p(p + 1)} w^{1-3p} \text{div} \left(|x|^{-\beta} w^{2p} \nabla w^{1-p}\right) + |\nabla w^{1-p}|^2 + |x|^{-\gamma} (c_1 w^{1-p} - c_2) = 0$$
Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here $v$ solves the *Fokker-Planck type equation*

$$v_t + |x|^{\gamma} \nabla \cdot \left[ |x|^{-\beta} v \nabla (v^{m-1} - |x|^{2+\beta-\gamma}) \right] = 0 \quad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret
Relative uniform convergence

\[ \zeta := 1 - \left(1 - \frac{2-m}{(1-m)q}\right) \left(1 - \frac{2-m}{1-m} \theta\right) \]
\[ \theta := \frac{(1-m)(2+\beta-\gamma)}{(1-m)(2+\beta)+2+\beta-\gamma} \] is in the range \(0 < \theta < \frac{1-m}{2-m} < 1\)

**Theorem**

For “good” initial data, there exist positive constants \(K\) and \(t_0\) such that, for all \(q \in \left[\frac{2-m}{1-m}, \infty\right]\), the function \(w = v / \mathcal{B}\) satisfies

\[ \|w(t) - 1\|_{L^q, \gamma(\mathbb{R}^d)} \leq K e^{-\Lambda \zeta (t-t_0)} \quad \forall \ t \geq t_0 \]

in the case \(\gamma \in (0, d)\), and

\[ \|w(t) - 1\|_{L^q, \gamma(\mathbb{R}^d)} \leq K e^{-2 \left(\frac{1-m}{2-m}\right)^2 \Lambda (t-t_0)} \quad \forall \ t \geq t_0 \]

in the case \(\gamma \leq 0\)
The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\text{ess}}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions.
Global vs. asymptotic estimates

Estimates on the global rates. When symmetry holds (CKN) can be written as an entropy – entropy production inequality

\[(2 + \beta - \gamma)^2 \mathcal{E}[v] \leq \frac{m}{1 - m} \mathcal{I}[v]\]

so that

\[\mathcal{E}[v(t)] \leq \mathcal{E}[v(0)] e^{-2(1-m)\Lambda_* t} \quad \forall \ t \geq 0 \quad \text{with} \quad \Lambda_* := \frac{(2 + \beta - \gamma)^2}{2 (1 - m)}\]

Optimal global rates. Let us consider again the entropy – entropy production inequality

\[\mathcal{K}(M) \mathcal{E}[v] \leq \mathcal{I}[v] \quad \forall \ v \in L^{1,\gamma}(\mathbb{R}^d) \quad \text{such that} \quad \|v\|_{L^{1,\gamma}(\mathbb{R}^d)} = M,\]

where \(\mathcal{K}(M)\) is the best constant: with \(\Lambda(M) := \frac{m}{2} (1 - m)^{-2} \mathcal{K}(M)\)

\[\mathcal{E}[v(t)] \leq \mathcal{E}[v(0)] e^{-2(1-m)\Lambda(M) t} \quad \forall \ t \geq 0\]
Linearization and optimality

Joint work with M.J. Esteban and M. Loss
Linearization and scalar products

With $u_\varepsilon$ such that

$$u_\varepsilon = B_* \left( 1 + \varepsilon f B_*^{1-m} \right) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\varepsilon \, dx = M_*$$

at first order in $\varepsilon \to 0$ we obtain that $f$ solves

$$\frac{\partial f}{\partial t} = \mathcal{L} f \quad \text{where} \quad \mathcal{L} f := (1 - m) B_*^{m-2} |x|^{\gamma} D_\alpha^* \left( |x|^{-\beta} B_* D_\alpha f \right)$$

Using the scalar products

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1 f_2 B_*^{2-m} |x|^{-\gamma} \, dx \quad \text{and} \quad \llangle f_1, f_2 \rrangle = \int_{\mathbb{R}^d} D_\alpha f_1 \cdot D_\alpha f_2 B_* |x|^{-\beta} \, dx$$

we compute

$$\frac{1}{2} \frac{d}{dt} \langle f, f \rangle = \langle f, \mathcal{L} f \rangle = \int_{\mathbb{R}^d} f (\mathcal{L} f) B_*^{2-m} |x|^{-\gamma} \, dx = - \int_{\mathbb{R}^d} |D_\alpha f|^2 B_* |x|^{-\beta} \, dx$$

for any $f$ smooth enough: with $\langle f, \mathcal{L} f \rangle = - \llangle f, f \rrangle$

$$\frac{1}{2} \frac{d}{dt} \llangle f, f \rrangle = \int_{\mathbb{R}^d} D_\alpha f \cdot D_\alpha (\mathcal{L} f) B_* |x|^{-\beta} \, dx = - \llangle f, \mathcal{L} f \rrangle$$
Now let us consider an eigenfunction associated with the smallest positive eigenvalue $\lambda_1$ of $\mathcal{L}$

$$-\mathcal{L} f_1 = \lambda_1 f_1$$

so that $f_1$ realizes the equality case in the **Hardy-Poincaré inequality**

$$\langle g, g \rangle = -\langle g, \mathcal{L} g \rangle \geq \lambda_1 \| g - \bar{g} \|^2, \quad \bar{g} := \langle g, 1 \rangle / \langle 1, 1 \rangle$$

$$-\langle g, \mathcal{L} g \rangle \geq \lambda_1 \langle g, g \rangle$$

Proof: expansion of the square:

$$-\langle (g - \bar{g}), \mathcal{L} (g - \bar{g}) \rangle = \langle \mathcal{L} (g - \bar{g}), \mathcal{L} (g - \bar{g}) \rangle = \| \mathcal{L} (g - \bar{g}) \|^2$$

Key observation:

$$\lambda_1 \geq 4 \iff \alpha \leq \alpha_{FS} := \sqrt{\frac{d - 1}{n - 1}}$$
Why is this method optimal?

- The condition $\lambda_1 < 4$ is sufficient for symmetry breaking.

- With $\lambda_1 \geq 4$, we prove that

$$\mathcal{H}[\nu] := \frac{m}{1 - m} \mathcal{I}[\nu] - (2 + \beta - \gamma)^2 \mathcal{E}[\nu]$$

is monotone decaying along the flow and that equality is achieved only on the stationary solution, which attracts all solutions. This has to do with the (formal) gradient flow structure of the problem.

- The condition $\lambda_1 \geq 4$ is enough to prove that
  - the Fisher information $\mathcal{I}[\nu]$ exponentially decays with rate $e^{-4t}$
  - the functional $\mathcal{H}[\nu]$ is decreasing

- The decay of the Fisher information $\mathcal{I}[\nu]$ “has to be given” by the condition $\lambda_1 \geq 4$ because the problem degenerates into a sharp spectral gap problem in the asymptotic regime.
Let $\mathcal{K}[u]$ be such that

$$\frac{d}{d\tau} \mathcal{I}[u(\tau, \cdot)] = -\mathcal{K}[u(\tau, \cdot)] = - \text{(sum of squares)}$$

If $\alpha \leq \alpha_{FS}$, then $\lambda_1 \geq 4$ and

$$u \mapsto \frac{\mathcal{K}[u]}{\mathcal{I}[u]} - 4$$

is a nonnegative functional

With $u_\varepsilon = B_\star (1 + \varepsilon f B_\star^{1-m})$ and $\alpha \leq \alpha_{FS}$, we observe that

$$4 \leq C_2 := \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} \leq \lim_{\varepsilon \to 0} \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \inf_f \frac{\langle f, \mathcal{L} f \rangle}{\langle f, f \rangle} = \frac{\langle f_1, \mathcal{L} f_1 \rangle}{\langle f_1, f_1 \rangle} = \lambda_1$$

- if $\lambda_1 = 4$, that is, if $\alpha = \alpha_{FS}$, then $\inf \mathcal{K}/\mathcal{I} = 4$ is achieved in the asymptotic regime as $u \to B_\star$ and determined by the spectral gap of $\mathcal{L}$
- if $\lambda_1 > 4$, that is, if $\alpha < \alpha_{FS}$, then $\mathcal{K}/\mathcal{I} > 4$
If $\alpha \leq \alpha_{FS}$, the fact that $\mathcal{K}/\mathcal{I} \geq 4$ has an important consequence. Indeed we know that

$$\frac{d}{d\tau} (\mathcal{I}[u(\tau, \cdot)] - 4 \mathcal{E}[u(\tau, \cdot)]) \leq 0$$

so that

$$\mathcal{I}[u] - 4 \mathcal{E}[u] \geq \mathcal{I}[B_{\ast}] - 4 \mathcal{E}[B_{\ast}] = 0$$

This inequality is equivalent to $\mathcal{J}[w] \geq \mathcal{J}[B_{\ast}]$, which establishes that optimality in (CKN) is achieved among symmetric functions. In other words, the linearized problem shows that for $\alpha \leq \alpha_{FS}$, the function

$$\tau \mapsto \mathcal{I}[u(\tau, \cdot)] - 4 \mathcal{E}[u(\tau, \cdot)]$$

is monotone decreasing.

This explains why the method based on nonlinear flows provides the optimal range for symmetry.
Entropy – production of entropy inequality

Using $\frac{d}{d\tau} (\mathcal{I}[u(\tau, \cdot)] - C_2 \mathcal{E}[u(\tau, \cdot)]) \leq 0$, we know that

$$\mathcal{I}[u] - C_2 \mathcal{E}[u] \geq \mathcal{I}[\mathcal{B}_\star] - C_2 \mathcal{E}[\mathcal{B}_\star] = 0$$

As a consequence, we have that

$$C_1 := \inf_u \frac{\mathcal{I}[u]}{\mathcal{E}[u]} \geq C_2 = \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]}$$

With $u_\varepsilon = \mathcal{B}_\star (1 + \varepsilon f \mathcal{B}_\star^{1-m})$, we observe that

$$C_1 \leq \lim_{\varepsilon \to 0} \inf_f \frac{\mathcal{I}[u_\varepsilon]}{\mathcal{E}[u_\varepsilon]} = \inf_f \frac{\langle f, \mathcal{L} f \rangle}{\langle f, f \rangle} = \frac{\langle f_1, \mathcal{L} f_1 \rangle}{\langle f_1, f_1 \rangle_1} = \lambda_1 = \lim_{\varepsilon \to 0} \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]}$$

If $\lim_{\varepsilon \to 0} \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = C_2$, then $C_1 = C_2 = \lambda_1$

This happens if $\alpha = \alpha_{FS}$ and in particular in the case without weights (Gagliardo-Nirenberg inequalities)
These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
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Thank you for your attention!