How far can we push entropy methods ?

Jean DOLBEAULT

Ceremade, Université Paris IX-Dauphine,
Place de Lattre de Tassigny,
75775 Paris Cédex 16, France

E-mail: dolbeaul@ceremade.dauphine.fr

http://www.ceremade.dauphine.fr/~dolbeaul/

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I — Entropy methods for linear diffusions
The logarithmic Sobolev inequality
Convex Sobolev inequalities

- *logarithmic Sobolev inequality:* [Gross], [Weissler], [Coulhon],...
- *probability theory:* [Bakry], [Emery], [Ledoux], [Coulhon],...
- *linear diffusions:* [Toscani], [Arnold, Markowich, Toscani, Unterreiter], [Otto, Kinderlehrer, Jordan], [Arnold, J.D.]

I-A. Intermediate asymptotics: heat equation

Heat equation: $$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n\,,\ t \in \mathbb{R}^+ \\ u(\cdot, t=0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0 \, dx = 1 \end{cases} \qquad (1)$$

As $t \to +\infty$, $u(x,t) \sim \mathcal{U}(x,t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}$.

Optimal rate of convergence of $\|u(\cdot,t) - \mathcal{U}(\cdot,t)\|_{L^1(\mathbb{R}^n)}$?

The time dependent rescaling

$$u(x,t) = \frac{1}{R^n(t)}\, v\left(\xi = \frac{x}{R(t)}, \tau = \log R(t) + \tau(0)\right)$$

allows to transform this question into that of the convergence to the stationary solution $v_\infty(\xi) = (2\pi)^{n/2}\, e^{-|\xi|^2/2}$.

- Ansatz: $\frac{dR}{dt} = \frac{1}{R} \qquad R(0) = 1 \qquad \tau(0) = 0$:

$$R(t) = \sqrt{1 + 2\,t}\,, \qquad \tau(t) = \log R(t)$$

As a consequence: $v(\tau = 0) = u_0$.

- Fokker-Planck equation:

$$\begin{cases} v_\tau = \Delta v + \nabla(\xi\, v) & \xi \in \mathbb{R}^n\,,\ \tau \in \mathbb{R}^+ \\ v(\cdot, \tau = 0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0\, dx = 1 \end{cases} \tag{2}$$

Entropy (relative to the stationary solution v_∞):

$$\Sigma[v] := \int_{\mathbf{R}^n} v \log\left(\frac{v}{v_\infty}\right) dx = \int_{\mathbf{R}^n} \left(v \log v + \frac{1}{2}|x|^2 v\right) dx + Const$$

If v is a solution of (2), then (I is the Fisher information)

$$\frac{d}{d\tau}\Sigma[v(\cdot,\tau)] = -\int_{\mathbf{R}^n} v \left|\nabla \log\left(\frac{v}{v_\infty}\right)\right|^2 dx =: -I[v(\cdot,\tau)]$$

- Euclidean logarithmic Sobolev inequality: If $\|v\|_{L^1} = 1$, then

$$\int_{\mathbf{R}^n} v \log v \, dx + n\left(1 + \frac{1}{2}\log(2\pi)\right) \le \frac{1}{2}\int_{\mathbf{R}^n} \frac{|\nabla v|^2}{v} dx$$

Equality: $v(\xi) = v_\infty(\xi) = (2\pi)^{-n/2} e^{-|\xi|^2/2}$

$$\Longrightarrow \Sigma[v(\cdot,\tau)] \le \tfrac{1}{2} I[v(\cdot,\tau)]$$

$$\Sigma[v(\cdot,\tau)] \le e^{-2\tau}\Sigma[u_0] = e^{-2\tau}\int_{\mathbf{R}^n} u_0 \log\left(\frac{u_0}{v_\infty}\right) dx$$

- Csiszár-Kullback inequality: Consider $v \geq 0$, $\bar{v} \geq 0$ such that $\int_{\mathbf{R}^n} v\,dx = \int_{\mathbf{R}^n} \bar{v}\,dx =: M > 0$

$$\int_{\mathbf{R}^n} v \log\left(\frac{v}{\bar{v}}\right)\,dx \geq \frac{1}{4\,M}\|v-\bar{v}\|^2_{L^1(\mathbf{R}^n)}$$

$$\Longrightarrow \|v - v_\infty\|^2_{L^1(\mathbf{R}^n)} \leq 4M\mathbf{\Sigma}[u_0]\,e^{-2\tau}$$

$\tau(t) = \log\sqrt{1+2t}$

$$\|u(\cdot,t) - u_\infty(\cdot,t)\|^2_{L^1(\mathbf{R}^n)} \leq \frac{4}{1+2t}\mathbf{\Sigma}[u_0]$$

$$u_\infty(x,t) = \frac{1}{R^n(t)}\,v_\infty\left(\frac{x}{R(t)},\tau(t)\right)$$

Proof of the Csiszár-Kullback inequality: Taylor development at second order.

Euclidean logarithmic Sobolev inequality: other formulations

1) independent from the dimension [Gross, 75]

$$\int_{\mathbb{R}^n} w \log w \, d\mu(x) \le \frac{1}{2} \int_{\mathbb{R}^n} w \, |\nabla \log w|^2 \, d\mu(x)$$

with $w = \frac{v}{M\, v_\infty}$, $\|v\|_{L^1} = M$, $d\mu(x) = v_\infty(x)\, dx$.

2) invariant under scaling [Weissler, 78]

$$\int_{\mathbb{R}^n} w^2 \log w^2 \, dx \le \frac{n}{2} \log \left(\frac{2}{\pi\, n\, e} \int_{\mathbb{R}^n} |\nabla w|^2 \, dx \right)$$

for any $w \in H^1(\mathbb{R}^n)$ such that $\int w^2 \, dx = 1$

Proof: take $w = \sqrt{\frac{v}{M\, v_\infty}}$ and optimize on λ for $w_\lambda(x) = \lambda^{n/2} w(\lambda\, x)$

$$\int_{\mathbb{R}^n} |\nabla w_\lambda|^2\, dx - \int_{\mathbb{R}^n} w_\lambda^2 \log w_\lambda^2\, dx$$
$$= \lambda^2 \int_{\mathbb{R}^n} |\nabla w|^2\, dx - \int_{\mathbb{R}^n} w^2 \log w^2\, dx - n \log \lambda \int_{\mathbb{R}^n} w^2\, dx$$

Entropy-entropy production method

A method to prove the Euclidean logarithmic Sobolev inequality:

$$\frac{d}{d\tau}\left(I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)]\right) = -C \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \left| w_{ij} + a\, \frac{w_i w_j}{w} + b\, w\, \delta_{ij} \right|^2 dx < 0$$

for some $C > 0$, a, $b \in \mathbb{R}$. Here $w = \sqrt{v}$.

$I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)] \searrow I[v_\infty] - 2\Sigma[v_\infty] = 0$

$\Longrightarrow \quad \forall\, u_0\,, \quad I[u_0] - 2\Sigma[u_0] \geq I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)] \geq 0$ for $\tau > 0$

I-B. Entropy-entropy production method: improved convex Sobolev inequalities

Goal: large time behavior of parabolic equations:

$$\begin{cases} v_t = \mathrm{div}_x[D(x)\,(\nabla_x v + v\nabla_x A(x))] = \mathrm{div}[D(x)\,e^{-A}\nabla(ve^A)] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in\mathbb{R}^n \\ v(x,t=0) = v_0(x) \in L^1_+(\mathbb{R}^n) \end{cases} \qquad (3)$$

$A(x)\ldots$ given 'potential'
$v_\infty(x) = e^{-A(x)} \in L^1 \ldots$ (unique) steady state
mass conservation: $\int_{\mathbb{R}^d} v(t)\,dx = \int_{\mathbb{R}^d} v_\infty\,dx = 1$

questions: exponential rate ? connection to logarithmic Sobolev inequalities ?

Entropy-entropy production method

[Bakry, Emery, 84]

[Toscani '96], [Arnold, Markowich, Toscani, Unterreiter, 01]

Relative entropy of $v(x)$ w.r.t. $v_\infty(x)$:

$$\Sigma[v|v_\infty] := \int_{\mathbf{R}^d} \psi\left(\frac{v}{v_\infty}\right) v_\infty \, dx \geq 0$$

with

$$\begin{aligned} \psi(w) &\geq 0 \text{ for } w \geq 0, \quad \text{convex} \\ \psi(1) &= \psi'(1) = 0 \end{aligned}$$

Admissibility condtion: $(\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV}$

Examples:

$\psi_1 = w \ln w - w + 1$, $\Sigma_1(v|v_\infty) = \int v \ln\left(\frac{v}{v_\infty}\right) dx \ldots$ physical entropy

$\psi_p = w^p - p(w-1) - 1$, $1 < p \leq 2$, $\quad \Sigma_2(v|v_\infty) = \int_{\mathbf{R}^d} (v - v_\infty)^2 v_\infty^{-1} dx$

Exponential decay of entropy production

$$I(v(t)|v_\infty) := \frac{d}{dt}\Sigma[v(t)|v_\infty] = -\int \psi''\left(\frac{v}{v_\infty}\right) |\underbrace{\nabla\left(\frac{v}{v_\infty}\right)}_{=:u}|^2 v_\infty\, dx \leq 0$$

Assume: $D \equiv 1$, $\underbrace{\frac{\partial^2 A}{\partial x^2}}_{\text{Hessian}} \geq \lambda_1 Id$, $\lambda_1 > 0$ $\quad$ ($A(x) \ldots$ unif. convex)

Entropy production rate:

$$\begin{aligned} I' &= 2\int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u\, v_\infty\, dx + \underbrace{2\int \mathrm{Tr}\,(XY)\, v_\infty\, dx}_{\geq 0} \\ &\geq -2\lambda_1 I \end{aligned}$$

Positivity of Tr (XY) ?

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} \geq 0$$

$$Y = \begin{pmatrix} \sum_{ij}(\frac{\partial u_i}{\partial x_j})^2 & u^T \cdot \frac{\partial u}{\partial x} \cdot u \\ u^T \cdot \frac{\partial u}{\partial x} \cdot u & |u|^4 \end{pmatrix} \geq 0$$

$$\Rightarrow \quad |I(t)| \leq e^{-2\lambda_1 t}\,|I(t=0)| \qquad t > 0$$

$$\forall v_0 \text{ with } |I(v_0|v_\infty)| < \infty$$

Exponential decay of relative entropy

Known: $I' \geq -2\lambda_1 \underbrace{I}_{=\Sigma'}$

$\int_t^\infty \dots dt$ $\Rightarrow \Sigma' = I \leq -2\lambda_1 \Sigma$ (4)

Theorem 1 [Bakry, Emery], [Arnold, Markowich, Toscani, Unterreiter]

$\frac{\partial^2 A}{\partial x^2} \geq \lambda_1 Id$ (*"Bakry–Emery condition"*), $\Sigma[v_0|v_\infty]<\infty$

$$\Rightarrow \Sigma[v(t)|v_\infty] \leq \Sigma[v_0|v_\infty]\, e^{-2\lambda_1 t}, \qquad t > 0$$

$\|v(t) - v_\infty\|_{L^1}^2 \leq C\, \Sigma[v(t)|v_\infty] \dots$ Csiszár-Kullback

Convex Sobolev inequalities

Entropy–entropy production estimate (4) for $A(x) = -\ln v_\infty$ (uniformly convex):

$$\Sigma[v|v_\infty] \le \frac{1}{2\lambda_1} |I(v|v_\infty)|$$

Example 1: logarithmic entropy $\psi_1(w) = w \ln w - w + 1$

$$\int v \ln\left(\frac{v}{v_\infty}\right) dx \le \frac{1}{2\lambda_1} \int v \left|\nabla \ln\left(\frac{v}{v_\infty}\right)\right|^2 dx$$

$$\forall v,\ v_\infty \in L^1_+(\mathbb{R}^n),\ \int v\, dx = \int v_\infty\, dx = 1$$

logarithmic Sobolev inequality – “entropy version”

Logarithmic Sobolev inequality–dv_∞ measure version [Gross '75]

$$f^2 = \frac{v}{v_\infty} \Rightarrow \int f^2 \ln f^2 dv_\infty \le \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

$$\forall f \in L^2(\mathbb{R}^n, dv_\infty), \ \int f^2 dv_\infty = 1$$

Example 2: non-logarithmic entropies:

$\psi_p(w) = w^p - p(w-1) - 1, \qquad 1 < p \le 2$

$$(B_p) \ \frac{p}{p-1}\left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^p\right] \le \frac{2}{\lambda_1}\int |\nabla f|^2 dv_\infty$$

from (4) with $\frac{v}{v_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}} dv_\infty}$ $\qquad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, v_\infty dx)$

Poincaré-type inequality [Beckner '89], $(B_p) \Rightarrow (B_2), \quad 1 < p \le 2$

Refined convex Sobolev inequalities

Estimate of entropy production rate / entropy production:

$$I' = 2\int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u v_\infty dx + \underbrace{2\int \mathrm{Tr}\,(XY) v_\infty dx}_{\geq 0}$$

$$\geq -2\lambda_1 I$$

[Arnold, J.D.]: Observe that $\psi_p(w) = w^p - p(w-1) - 1$, $1 < p < 2$:

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} > 0$$

- Assume $\frac{\partial A^2}{\partial x^2} \geq \lambda_1 Id \Rightarrow \Sigma'' \geq -2\lambda_1 \Sigma' + \kappa \frac{|\Sigma'|^2}{1+\Sigma}$, $\qquad \kappa = \frac{2-p}{p} < 1$

$$\Rightarrow \boxed{k(\Sigma[v|v_\infty]) \leq \frac{1}{2\lambda_1}|\Sigma'|} = \frac{1}{2\lambda_1}\int \psi''\left(\frac{v}{v_\infty}\right)|\nabla \frac{v}{v_\infty}|^2 dv_\infty$$

Refined convex Sobolev inequality with $x \leq k(x) = \frac{1+x-(1+x)^\kappa}{1-\kappa}$

- Set $v/v_\infty = |f|^{\frac{2}{p}} / \int |f|^{\frac{2}{p}} dv_\infty \Rightarrow$

Theorem 2

$$\frac{1}{2}\left(\frac{p}{p-1}\right)^2 \left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^{2(p-1)} \left(\int f^2 dv_\infty\right)^{\frac{2-p}{p}}\right]$$
$$\leq \frac{2}{\lambda_1}\int |\nabla f|^2 dv_\infty \quad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, dv_\infty)$$

Refined Beckner inequality [Arnold, J.D. 2000, 2004]

$(rB_p) \Rightarrow (rB_2) = (B_2), \quad 1 < p \leq 2$

I-C. Applications...

- Homogeneous and non-homogenous collisional kinetic equations [L. Desvillettes, C. Villani, G. Toscani,...]
- Drift-diffusion-Poisson equations for semi-conductors [A. Arnold, P. Markowich, G. Toscani], [P. Biler, J.D., P. Markowich]
- Streater's models [P. Biler, J.D., M. Esteban, G. Karch]
- Heat equation with a source term [[J.D., G. Karch]
- The flashing ratchet [J.D., D. Kinderlehrer, M. Kowalczyk]
- Models for traffic flow [J.D., Reinhard Illner]
- Navier-Stokes in dimension 2 [T. Gallay, Wayne], [C. Villani], [J.D., A. Munnier]

... and questions under investigation

• Hierarchies of inequalities

• Derivation of entropy - entropy-production inequalities in non-standard frameworks:

- singular potentials: [JD, Nazaret, Otto]

- vanishing diffusion cooefficients: [Bartier, JD, Illner, Kowalczyk]

• Homogeneization and long time behaviour: [Allaire, JD, Kinderlehrer, Kowalczyk]

• Relaxation and diffusion properties on intermediate time scales, corrections to convex Sobolev inequalities: [Bartier, JD, Markowich]

II — Porous media / fast diffusion equation and generalizations

[coll. Manuel del Pino (Universidad de Chile)] $\Longrightarrow$ Relate entropy and entropy-production by Gagliardo-Nirenberg inequalities

Other approaches:
1) "entropy – entropy-production method"
2) mass transportation techniques
3) hypercontractivity for appropriate semi-groups

• *nonlinear diffusions:* [Carrillo, Toscani], [Del Pino, J.D.], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vazquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub]

II-A. Porous media / Fast diffusion equation

[Del Pino, JD]

$$
\begin{aligned}
u_t &= \Delta u^m \quad \text{in } \mathbb{R}^n \\
u_{|t=0} &= u_0 \geq 0 \\
u_0(1+|x|^2) \in L^1 \,, &\quad u_0^m \in L^1
\end{aligned}
\tag{5}
$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0 \, dx = 1$, the self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-n/(2-n(1-m))})$ as $t \to +\infty$, [Friedmann, Kamin, 1980]

$$\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-n/(2-n(1-m))})$$

Rescaling: Take $u(t,x) = R^{-n}(t)\, v\,(\tau(t), x/R(t))$ where

$$\dot{R} = R^{n(1-m)-1}\,, \quad R(0) = 1\,, \quad \tau = \log R$$

$$\boxed{v_\tau = \Delta v^m + \nabla \cdot (x\, v)\,, \quad v_{|\tau=0} = u_0}$$

[Ralston, Newman, 1984] Lyapunov functional: *Entropy*

$$\boxed{\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0}$$

$$\frac{d}{d\tau}\Sigma[v] = -I[v]\,, \quad I[v] = \int v \left| \frac{\nabla v^{m-1}}{v} + x \right|^2 dx$$

Stationary solution: C s.t. $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left(C + \frac{1-m}{2m}\,|x|^2\right)_+^{-1/(1-m)}$$

Fix Σ_0 so that $\Sigma[v_\infty] = 0$.

$$\Sigma[v] = \int \psi\left(\tfrac{v^m}{v_\infty^m}\right)\, v_\infty^{m-1}\,dx \quad \textit{with}\ \psi(t) = \tfrac{m t^{1/m} - 1}{1-m} + 1$$

Theorem 1 $m \in [\frac{n-1}{n}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\boxed{I[v] \geq 2\,\Sigma[v]}$$

An equivalent formulation

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) \, dx - \Sigma_0 \leq \frac{1}{2} \int v \left| \frac{\nabla v^{m-1}}{v} + x \right|^2 \, dx = \frac{1}{2} I[v]$$

$p = \frac{1}{2m-1}$, $v = w^{2p}$

$$\boxed{\tfrac{1}{2}\left(\tfrac{2m}{2m-1}\right)^2 \int |\nabla w|^2 dx + \left(\tfrac{1}{1-m} - n\right) \int |w|^{1+p} dx + K \geq 0}$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$

$m = \frac{n-1}{n}$: Sobolev, $m \to 1$: logarithmic Sobolev

[Del Pino, J.D.], [Carrillo, Toscani], [Otto]

Optimal constants for Gagliardo-Nirenberg ineq.

[Del Pino, J.D.]

$$1 < p \leq \frac{n}{n-2} \text{ for } n \geq 3$$

$$\|w\|_{2p} \leq A \, \|\nabla w\|_2^\theta \, \|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi n}\right)^{\frac{\theta}{2}} \left(\frac{2y-n}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{n}{2})}\right)^{\frac{\theta}{n}}$$

$$\theta = \frac{n(p-1)}{p(n+2-(n-2)p)}, \; y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$. Uses [Serrin-Pucci], [Serrin-Tang].

$1 < p = \frac{1}{2m-1} \leq \frac{n}{n-2} \iff$ Fast diffusion case: $\frac{n-1}{n} \leq m < 1$

$0 < p < 1 \iff$ Porous medium case: $m > 1$

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau}$+ Csiszár-Kullback inequalities

$\Rightarrow$ Intermediate asymptotics [Del Pino, J.D.]

(i) $\frac{n-1}{n} < m < 1$ if $n \geq 3$

$$\limsup_{t \to +\infty} t^{\frac{1-n(1-m)}{2-n(1-m)}} \|u^m - u_\infty^m\|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t \to +\infty} t^{\frac{1+n(m-1)}{2+n(m-1)}} \|\, [u - u_\infty]\, u_\infty^{m-1}\, \|_{L^1} < +\infty$$

Generalization

Intermediate asymptotics for:

$$u_t = \Delta_p u^m$$

Convergence to a stationary solution for:

$$v_t = \Delta_p v^m + \nabla(x\, v)$$

Let $q = 1 + m - (p-1)^{-1}$. Whether q is bigger or smaller than 1 determines two different regimes like for $p = 1$.

$q < 1 \iff$ Fast diffusion case

$q > 1 \iff$ Porous medium case

For $q > 0$, define the *entropy* by

$$\Sigma[v] = \int \Big[\sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty)\Big]\; dx$$

$\sigma(s) = \frac{s^q - 1}{q-1}$ if $q \neq 1$
$\sigma(s) = s \log s$ if $q = 1$ ($p \neq 2$: see below)

Nonhomogeneous version – Gagliardo-Nirenberg ineq.

$b = \frac{p\,(p-1)}{p^2-p-1}$, $a = b\,q$, $v = w^b$. For $p \neq 2$, let

$$\mathcal{F}[v] = \int v^{-\frac{1}{p-1}} |\nabla v|^p \, dx - \frac{1}{q} \left(\frac{n}{1-\kappa_p} + \frac{p}{p-2} \right) \int v^q \, dx$$

$\kappa_p = \frac{1}{p}\,(p-1)^{\frac{p-1}{p}}$. Based on [Serrin, Tang] (uniqueness result)

Corollary 3 $n \geq 2$, $(2n+1)/(n+1) \leq p < n$. $\forall\, v$ s.t. $\|v\|_{L^1} = \|v_\infty\|_{L^1}$

$$\mathcal{F}[v] \geq \mathcal{F}[v_\infty]$$

$\|w\|_b \leq \mathcal{S}\, \|\nabla w\|_p^\theta\, \|w\|_a^{1-\theta}$	if $a > p$
$\|w\|_a \leq \mathcal{S}\, \|\nabla w\|_p^\theta\, \|w\|_b^{1-\theta}$	if $a < p$

[Del Pino, J.D.] Intermediate asymptotics of $u_t = \Delta_p u^m$

Theorem 2 $n \geq 2$, $1 < p < n$, $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{p}{p-1}$ *and* $q = 1 + m - \frac{1}{p-1}$

$$\text{(i)} \quad \|u(t,\cdot) - u_\infty(t,\cdot)\|_q \leq K\, R^{-(\frac{\alpha}{2}+n(1-\frac{1}{q}))}$$

$$\text{(ii)} \quad \|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}}$$

(i): $\frac{1}{p-1} \leq m \leq \frac{p}{p-1}$ *(ii)*: $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{1}{p-1}$

$\alpha = (1 - \frac{1}{p}(p-1)^{\frac{p-1}{p}})\frac{p}{p-1}$, $R = (1+\gamma\, t)^{1/\gamma}$, $\gamma = (mn+1)(p-1) - (n-1)$

$u_\infty(t,x) = \frac{1}{R^n}\, v_\infty(\log R, \frac{x}{R})$

$v_\infty(x) = (C - \frac{p-1}{mp}(q-1)\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ *if* $m \neq \frac{1}{p-1}$

$v_\infty(x) = C\, e^{-(p-1)^2|x|^{p/(p-1)}/p}$ *if* $m = (p-1)^{-1}$.

Use $v_t = \Delta_p v^m + \nabla\cdot(x\,v)$

$w = v^{(mp+q-(m+1))/p}$, $a = b\,q = p\,\frac{m(p-1)+p-2}{mp(p-1)-1}$.

II-B. The $W^{1,p}$ logarithmic Sobolev inequality and consequences

[Del Pino, JD]

Optimal constants for Gagliardo-Nirenberg ineq.

[Del Pino, J.D.]

Theorem 3 $1<p<n$, $1<a\le\frac{p(n-1)}{n-p}$, $b=p\,\frac{a-1}{p-1}$

$$\|w\|_b \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_a^{1-\theta} \quad \textit{if } a>p$$
$$\|w\|_a \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_b^{1-\theta} \quad \textit{if } a<p$$
$$\textit{Equality if } w(x)=A\,(1+B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$$

$a>p$: $\theta=\frac{(q-p)\,n}{(q-1)\,(n\,p-(n-p)\,q)}$

$a<p$: $\theta=\frac{(p-q)\,n}{q\,(n\,(p-q)+p\,(q-1))}$

The optimal L^p-Euclidean logarithmic Sobolev inequality
(an optimal under scalings form) [Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

Theorem 4 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx \right]$$

$$\mathcal{L}_p = \tfrac{p}{n} \left(\tfrac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\,\frac{p-1}{p}+1)} \right]^{\frac{p}{n}}$$

Equality: $u(x) = \left(\pi^{\frac{n}{2}} (\frac{\sigma}{p})^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)} \right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}$

$p = 2$: Gross' logaritmic Sobolev inequality [Gross, 75], [Weissler, 78]

$p = 1$: [Ledoux 96], [Beckner, 99]

For some purposes, it is sometimes more convenient to use this inequality in a non homogeneous form, which is based upon the fact that

$$\inf_{\mu>0}\left[\frac{n}{p}\log\left(\frac{n}{p\mu}\right)+\mu\,\frac{\|\nabla w\|_p^p}{\|w\|_p^p}\right]=n\,\log\left(\frac{\|\nabla w\|_p}{\|w\|_p}\right)+\frac{n}{p}\,.$$

Corollary 5 *For any* $w\in W^{1,p}(\mathbb{R}^n)$, $w\neq 0$, *for any* $\mu>0$,

$$p\int|w|^p\log\left(\frac{|w|}{\|w\|_p}\right)\,dx+\frac{n}{p}\log\left(\frac{p\,\mu\,e}{n\,\mathcal{L}_p}\right)\int|w|^p\,dx\le\mu\int|\nabla w|^p\,dx\,.$$

II-C. Consequences for $u_t = \Delta_p u^{1/(p-1)}$

[Del Pino, JD, Gentil]

- Existence
- Uniqueness
- Hypercontractivity, Ultracontractivity
- Large deviations

Existence

Consider the Cauchy problem

$$\begin{cases} u_t = \Delta_p(u^{1/(p-1)}) & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(\cdot, t=0) = f \geq 0 \end{cases} \qquad (6)$$

$\Delta_p u^m = \mathrm{div}\left(|\nabla u^m|^{p-2}\nabla u^m\right)$ is 1-homogeneous $\Longleftrightarrow m = 1/(p-1)$.
Notations: $\|u\|_q = (\int_{\mathbb{R}^n} |u|^q\, dx)^{1/q}$, $q \neq 0$. $p^* = p/(p-1)$, $p > 1$.

Theorem 6 *Let $p > 1$, $f \in L^1(\mathbb{R}^n)$ s.t. $|x|^{p^*} f$, $f \log f \in L^1(\mathbb{R}^n)$. Then there exists a unique weak nonnegative solution $u \in C(\mathbb{R}^+_t, L^1)$ of (6) with initial data f, such that $u^{1/p} \in L^1_{\mathrm{loc}}(\mathbb{R}^+_t, W^{1,p}_{\mathrm{loc}})$.*

[Alt-Luckhaus, 83] [Tsutsumi, 88] [Saa, 91] [Chen, 00] [Agueh, 02]
[Bernis, 88], [Ishige, 96]
Crucial remark: [Benguria, 79], [Benguria, Brezis, Lieb, 81], [Diaz,Saa, 87]
The functional $u \mapsto \int |\nabla u^\alpha|^p\, dx$ is convex for any $p > 1$, $\alpha \in [\frac{1}{p}, 1]$.

Uniqueness

Consider two solutions u_1 and u_2 of (6).

$$\frac{d}{dt}\int u_1 \log\left(\frac{u_1}{u_2}\right) dx$$
$$= \int \left(1 + \log\left(\frac{u_1}{u_2}\right)\right) (u_1)_t \, dx - \int \left(\frac{u_1}{u_2}\right) (u_2)_t \, dx$$
$$= -(p-1)^{-(p-1)} \int u_1 \left[\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right] \cdot \left[\left|\frac{\nabla u_1}{u_1}\right|^{p-2} \frac{\nabla u_1}{u_1} - \left|\frac{\nabla u_2}{u_2}\right|^{p-2} \frac{\nabla u_2}{u_2}\right] dx \, .$$

It is then straightforward to check that two solutions with same initial data f have to be equal since

$$\frac{1}{4\,\|f\|_1} \|u_1(\cdot,t) - u_2(\cdot,t)\|_1^2 \leq \int u_1(\cdot,t) \log\left(\frac{u_1(\cdot,t)}{u_2(\cdot,t)}\right) dx \leq \int f \log\left(\frac{f}{f}\right) dx = 0$$

by the Csiszár-Kullback inequality.

Hyper- and Ultra-contractivity

Understanding the regularizing properties of

$$u_t = \Delta_p u^{1/(p-1)}$$

Theorem 7 *Let* α*,* $\beta \in [1, +\infty]$ *with* $\beta \geq \alpha$*. Under the same assumptions as in the existence Theorem, if moreover* $f \in L^\alpha(\mathbb{R}^n)$*, any solution with initial data* f *satisfies the estimate*

$$\|u(\cdot, t)\|_\beta \leq \|f\|_\alpha \, A(n, p, \alpha, \beta) \, t^{-\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}} \quad \forall\, t > 0$$

with $A(n, p, \alpha, \beta) = (\mathcal{C}_1\,(\beta - \alpha))^{\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}}\,\mathcal{C}_2^{\frac{n}{p}}$, $\mathcal{C}_1 = n\,\mathcal{L}_p\,e^{p-1}\,\frac{(p-1)^{p-1}}{p^{p+1}}$,

$$\mathcal{C}_2 = \frac{(\beta-1)^{\frac{1-\beta}{\beta}}}{(\alpha-1)^{\frac{1-\alpha}{\alpha}}}\,\frac{\beta^{\frac{1-p}{\beta}-\frac{1}{\alpha}+1}}{\alpha^{\frac{1-p}{\alpha}-\frac{1}{\beta}+1}}.$$

Case $p = 2$: $\mathcal{L}_2 = \frac{2}{\pi\,n\,e}$, [Gross 75]; $\beta = \infty$, $p = 2$: [Varopoulos 85]

Large deviations

The three following identities are equivalent:

(i) For any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p \, dx = 1$,

$$\int |w|^p \log |w| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla w|^p \, dx \right]$$

(ii) Let P_t^p be the semigroup associated $u_t = \Delta_p(u^{1/(p-1)})$:

$$\|P_t^p f\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta) \, t^{-\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}}$$

(iii) Let Q_t^p be the semigroup associated to $v_t + \frac{1}{p} |\nabla v|^p = 0$:

$$\|e^{Q_t^p g}\|_\beta \leq \|e^g\|_\alpha \, B(n,p,\alpha,\beta) \, t^{-\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}}$$

III — L^1 *intermediate asymptotics*

for scalar conservation laws

Joint work with Miguel ESCOBEDO

Let $q > 1$ and consider a nonnegative entropy solution of

$$\begin{cases} U_\tau + (U^q)_\xi = 0 \,, \quad \xi \in \mathbb{R} \,, \quad \tau > 0 \\ U(\tau = 0, \cdot) = U_0 \end{cases} \tag{7}$$

Question: what is the asymptotic behavior as $t \to +\infty$?

P. Lax (1957): $\|U(\tau,\cdot) - W_\infty(\tau,\cdot)\|_1 = O(\tau^{-1/2})$ as $\tau \to \infty$ if

$$U_\tau + f(U)_\xi = 0$$

with $f \in C^2$ near the origin + additional conditions.

T.-P. Liu & M. Pierre: $\lim_{\tau\to\infty} \tau^{\frac{1}{q}(1-\frac{1}{p})} \|U(\tau) - U_\infty(\tau)\|_p = 0$
where U_∞ is the self-similar solution

Y.-J. Kim (2001): $q \in (1,2)$, intermediate asymptotics in L^1

M. Escobedo, J.D. (2003) $q \in (1,2)$, intermediate asymptotics in L^1 + additional estimates

Theorem 8 *Let U be a global, piecewise C^1 entropy solution of (7) corresponding to a nonnegative initial data U_0 in $L^1 \cap L^\infty(\mathbb{R})$ which is compactly supported in $(\xi_0, +\infty)$ for some $\xi_0 \in \mathbb{R}$ and such that*

$$\liminf_{\xi \to (\xi_0)_+} \quad \frac{U_0(\xi)}{|\xi - \xi_0|^{1/(q-1)}} > 0$$

Then, for any $\alpha \in (0, \frac{q}{q-1})$ and $\epsilon > 0$,

$$\limsup_{\tau \to +\infty} \tau^{\alpha - \epsilon} \int_{\mathbb{R}} |U(\tau, \xi) - U_\infty(\tau, \xi - \xi_0)| \, \frac{d\xi}{|\xi - \xi_0|^\alpha} = 0$$

Self-similar solution: $U_\infty(\tau, \xi) = (|\xi|/q\,\tau)^{1/(q-1)} \chi_{\xi \leq c(\tau)}$

Corollary 9 *For any $\beta < 1$, there exists a constant C_β such that*

$$\|U(\tau, \cdot) - U_\infty(\tau, \xi - \xi_0)\|_1 \leq C_\beta \, \tau^{-\beta}$$

Uniform estimates: *Graph convergence*

Theorem 10 *Under the same assumptions as above,*

$$\lim_{\tau \to +\infty} \sup_{\xi \in \mathrm{supp}(U(\tau,\cdot))} \tau^{1/q} \, |U(\tau,\xi) - U_\infty(\tau, \cdot - \xi_0)| = 0$$

$$\lim_{\tau \to +\infty} (1 + q\,\tau)^{-1/q} \max\,[\mathrm{supp}(U(\tau,\cdot))] = Const(q, U_0)$$

Notions of solution, time-dependent rescaling, shocks

Let U be a nonnegative piecewise C^1 entropy solution of (7), whose points of discontinuity are given by the curves $\xi_1(\tau) < \xi_2(\tau) < \cdots < \xi_n(\tau)$. Then the rescaled function

$$u(t,x) = e^t\, U\left((e^{qt} - 1)/q, e^t x\right)$$

is a piecewise C^1 function, whose points of discontinuity are given by the curves $s_i(t) \equiv e^{-t}\xi_i((e^{qt}-1)/q)$

Rankine-Hugoniot condition

$$s_i'(t) = \frac{(u_i^+)^q - s_i(t)\,u_i^+ - (u_i^-)^q + s_i(t)\,u_i^-}{u_i^+ - u_i^-}$$

Out of the curves $x = s_i(t)$ the function u is a classical solution of

$$u_t = (x\,u - u^q)_x \tag{8}$$

and across these curves it satisfies

$$u_i^- := \lim_{\substack{x \to s_i(t) \\ x < s_i(t)}} u(t,x) > \lim_{\substack{x \to s_i(t) \\ x > s_i(t)}} u(t,x) := u_i^+$$

Moreover u and U have the same initial data $U_0 := U(0,\cdot) = u(0,\cdot)$. Finally, if $U_0 \in L^1(\mathbb{R})$, then $||u(t)||_1 = ||U_0||_1$, for all $t > 0$.

Entropy

For every $c > 0$, let u_∞^c be the *stationary solution* of (8) :

$$u_\infty^c(x) = \begin{cases} x^{1/(q-1)} & 0 \le x \le c \\ 0 & \text{if } x < 0 \text{ or } x > c \end{cases}$$

Relative entropy Σ of the solution u with respect to u_∞^c: For any positive constant c, let

$$\Sigma(t) = \int_0^c \mu(x) \left| u(t,x) - u_\infty^c(x) \right| dx$$

Define $f(v) = v - v^q$ for $v > 0$

$$\frac{d\Sigma}{dt} \le \int_0^c \mu'(u_\infty^c)^q \left| f\left(\frac{u}{u_\infty^c}\right) \right| dx \le 0$$

Assume for simplicity that $u(t,.)$ has exactly one shock. Let $v^{\pm} = u^{\pm}/u_{\infty}^{c}$ at $x = s(t) \in (0,c)$: $v^{-} > v^{+}$ and

$$s'(t) = -(u_{\infty}^{c'})^{q-1} \frac{f(v^{+}) - f(v^{-})}{v^{+} - v^{-}}$$

$$\frac{d\Sigma}{dt} = \int_0^c \mu\, u_t \left[\mathbb{1}_{u>u_{\infty}^{c}} - \mathbb{1}_{u<u_{\infty}^{c}} \right] dx$$

$$+ \quad [\mu(s)|u - u_{\infty}^{c}(s)| \cdot s'(t)\,]_{u=u^{+}}^{u=u^{-}}$$

$$\frac{d\Sigma}{dt} \le \int_0^c \mu'\,(u_{\infty}^{c})^q \left| f\left(\frac{u}{u_{\infty}^{c}}\right)\right| dx + \mu(s)\,(u_{\infty}^{c}(s))^q\; \Psi(v^{-}, v^{+})$$

$+$ boundary terms

$$\Psi(v^-,v^+) := \Big[f(v^+) - f(v^-)\Big] \cdot \frac{|v^+-1|-|v^--1|}{v^+-v^-} + |f(v^+)| - |f(v^-)|$$

- $1 \le v^+ \le v^-$: $f(v^-) \le f(v^+) \le 0$ and $\Psi(v^-,v^+) = 0$

- $v^+ < 1 \le v^-$: $f(v^-) \le 0 < f(v^+)$

$$\begin{aligned} \tfrac{1}{2}\Psi(v^-,v^+) = & \ \frac{v^--1}{v^--v^+} f(v^+) + \frac{1-v^+}{v^--v^+} f(v^-) \\ \le & \ \ f\left(\frac{v^--1}{v^--v^+} v^+ + \frac{1-v^+}{v^--v^+} v^-\right) = f(1) = 0 \end{aligned}$$

- $v^+ < v^- \le 1$: $f(v^-) \ge 0$ and $f(v^+) \ge 0$ and $\Psi(v^-,v^+) = 0$

$$\frac{d\Sigma_\alpha}{dt} \leq -\alpha \int_0^c x^{-\alpha-1+\frac{q}{q-1}} \left| f\left(\frac{u}{u_\infty^c}\right)\right| dx + \text{boundary terms}$$

Taylor expansion:

$$f\left(\frac{u}{u_\infty^c}\right) = (1-q)\left(\frac{u}{u_\infty^c}-1\right) + q(1-q)\left(\frac{u}{u_\infty^c}-1\right)^2 \int_0^1 (1-\theta)\left(\theta\frac{u}{u_\infty^c}+1-\theta\right)^{q-2} d\theta$$

$$\int_0^c x^{-\alpha+\frac{1}{q-1}} \left(\frac{u}{u_\infty^c}-1\right)^2 dx \leq \underbrace{\left\|\frac{u}{u_\infty^c}-1\right\|_{L^\infty(0,c(t))}}_{\to 0} \int_0^c \mu\, |u-u_\infty^c|\, dx$$

is neglectible compared to $\Sigma_\alpha(t)$ as $t \to +\infty$.

$$\boxed{\frac{d\Sigma_\alpha}{dt} + (q-1)\,\alpha\,\Sigma_\alpha(t) = o(\Sigma_\alpha(t))}$$

IV — Fourth order operators

$$
\begin{cases}
u_t + (u(\log u)_{xx})_{xx} = 0 \\
u(\cdot, 0) = u_0 \quad \text{in } S^1
\end{cases}
\tag{9}
$$

Joint work with Ansgar JÜNGEL and Ivan GENTIL,
in progress

[Jüngel et al.]
[Cáceres, Carrillo, Toscani]

$$u_t + (u(\log u)_{xx})_{xx} = 0, \quad u(\cdot, 0) = u_0 \quad \text{in } S^1$$

There are several Lyapunov functionals:

$$\frac{d}{dt}\int_{S^1} u(\log u - 1)dx + \int_{S^1} u(\log u)_{xx}^2 dx = 0$$

$$\frac{d}{dt}\int_{S^1} (u - \log u)dx + \int_{S^1} (\log u)_{xx}^2 dx = 0$$

Existence of periodic solutions

Theorem 4 *Let $u_0 : S^1 \to \mathbb{R}$ be a measurable function such that $\int (u_0 - \log u_0)dx < \infty$. Then there exists a global weak solution u of (9) satisfying*

$$u \in L^q_{\text{loc}}(0,\infty; W^{1,p}(S^1)) \cap W^{1,1}_{\text{loc}}(0,\infty; H^{-2}(S^1)),$$
$$u \geq 0 \quad \text{in } S^1 \times (0,\infty), \quad \log u \in L^2_{\text{loc}}(0,\infty; H^2(S^1)),$$

where $p \in (1, 4/3)$, $q = 5p/(4p-2) \in (2, 5/2)$, and for all $T > 0$ and all smooth test functions ϕ

$$\int_0^T \langle u_t, \phi \rangle_{(H^2)^*, H^2} dt + \int_0^T \int_{S^1} u(\log u)_{xx} \phi_{xx} dx dt = 0.$$

Optimal logarithmic Sobolev inequality on S^1

Theorem 5 *Let* $H = \{u \in H^1(S^1) : u_x \not\equiv 0 \text{ a.e.}\}$ *and* $\|u\|_2^2 = \int_{S^1} u^2 dx/L$. *Then*

$$\inf_{u \in H} \frac{\int_{S^1} u_x^2 dx}{\int_{S^1} u^2 \log(u^2/\|u\|_2^2) dx} = \frac{\pi^2}{2L^2}.$$

Lower bound: Expand the quotient for $u = 1 + \varepsilon v$ with $\int_{S^1} v dx = 0$ in powers of ε and use the Poincaré inequality.
Upper bound: entropy - entropy-production method:

$$v_t = v_{xx} \quad \text{in } S^1 \times (0, \infty), \qquad v(\cdot, 0) = u^2 \quad \text{in } S^1$$

Then

$$\frac{d}{dt} \int_{S^1} (\sqrt{v}_x)^2 dx - \frac{\pi^2}{2L^2} \int_{S^1} v \log v dx \le -\frac{2}{3} \int_{S^1} \frac{(\sqrt{v}_x)^4}{v} dx \le 0$$

Corollary 6 *Let* $\mathcal{H} = \{u \in H^2(S^1) : u_x \not\equiv 0 \text{ a.e.}\}$. *Then*

$$\inf_{u \in \mathcal{H}} \frac{\int_{S^1} u_{xx}^2 dx}{\int_{S^1} u^2 \log(u^2/\|u\|_2^2) dx} = \frac{\pi^2}{2L^4}.$$

- Asymptotic behavior

$$\frac{d}{dt}\int_{S^1} u \log\left(\frac{u}{\bar{u}}\right) dx \leq -\frac{2L^4}{\pi^2}\int_{S^1} (\sqrt{u})_{xx} dx$$

- Hyper-contractivity: in progress