Hypocoercivity and functional inequalities

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Abstract method and motivation
The compact case
The non-compact case
Vlasov-Poisson-Fokker-Planck

Outline

- **Abstract method and motivation**
  - ▷ Abstract statement in a Hilbert space
  - ▷ Diffusion limit, toy model

- **The compact case**
  - ▷ Strong confinement
  - ▷ Mode-by-mode decomposition
  - ▷ Application to the torus
  - ▷ Further results

- **The non-compact case**
  - ▷ Without confinement: Nash inequality
  - ▷ With very weak confinement: Caffarelli-Kohn-Nirenberg inequality
  - ▷ With sub-exponential equilibria: weighted Poincaré inequality

- **The Vlasov-Poisson-Fokker-Planck system**
  - ▷ Linearized system and hypocoercivity
  - ▷ Results in the diffusion limit and in the non-linear case

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Abstract method and motivation

▷ Abstract statement

▷ Diffusion limit

▷ A toy model

Collaboration with C. Mouhot and C. Schmeiser

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An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$

$$\mathsf{A} := \left(1 + (\mathsf{T}\Pi)^* \mathsf{T}\Pi\right)^{-1} (\mathsf{T}\Pi)^*$$

* denotes the adjoint with respect to $\langle \cdot, \cdot \rangle$

Π is the orthogonal projection onto the null space of L

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The assumptions

λ_m, λ_M, and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity:*

$$- \langle \mathsf{L}F, F \rangle \geq \lambda_m \, \|(1 - \Pi)F\|^2 \qquad (H1)$$

▷ *macroscopic coercivity:*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \, \|\Pi F\|^2 \qquad (H2)$$

▷ *parabolic macroscopic dynamics:*

$$\Pi \mathsf{T}\Pi \, F = 0 \qquad (H3)$$

▷ *bounded auxiliary operators:*

$$\|\mathsf{A}\mathsf{T}(1 - \Pi)F\| + \|\mathsf{A}\mathsf{L}F\| \leq C_M \, \|(1 - \Pi)F\| \qquad (H4)$$

The estimate

$$\frac{1}{2} \, \frac{d}{dt} \|F\|^2 = \langle \mathsf{L}F, F \rangle \leq - \lambda_m \, \|(1 - \Pi)F\|^2$$

is not enough to conclude that $\|F(t, \cdot)\|^2$ decays exponentially

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Equivalence and entropy decay

For some $\delta > 0$ to be determined later, the L^2 entropy / Lyapunov functional is defined by

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

so that $\langle \mathsf{A}\mathsf{T}\Pi F, F\rangle \sim \|\Pi F\|^2$ and

$$
\begin{aligned}
-\frac{d}{dt}\mathsf{H}[F] =& : \mathsf{D}[F] \\
=& - \langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{A}\mathsf{T}\Pi F, F\rangle \\
& - \delta\,\mathrm{Re}\langle \mathsf{T}\mathsf{A}F, F\rangle + \delta\,\mathrm{Re}\langle \mathsf{A}\mathsf{T}(1-\Pi)F, F\rangle - \delta\,\mathrm{Re}\langle \mathsf{A}\mathsf{L}F, F\rangle
\end{aligned}
$$

▷ *entropy decay rate:* for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$\lambda\,\mathsf{H}[F] \leq \mathsf{D}[F]$$

▷ *norm equivalence* of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2 \leq \mathsf{H}[F] \leq \frac{2+\delta}{4}\,\|F\|^2$$

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Exponential decay of the entropy

$$\lambda = \frac{\lambda_M}{3\,(1+\lambda_M)}\,\min\left\{1, \lambda_m, \frac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\}, \ \delta = \frac{1}{2}\,\min\left\{1, \lambda_m, \frac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\}$$

$$h_1(\delta, \lambda) := (\delta\,C_M)^2 - 4\left(\lambda_m - \delta - \frac{2+\delta}{4}\,\lambda\right)\left(\frac{\delta\,\lambda_M}{1+\lambda_M} - \frac{2+\delta}{4}\,\lambda\right)$$

Theorem

Let L *and* T *be closed linear operators (respectively Hermitian and anti-Hermitian) on* $\mathfrak{H}$. *Under* (H1)–(H4), *for any* $t \geq 0$

$$\mathsf{H}[F(t, \cdot)] \leq \mathsf{H}[F_0]\,e^{-\lambda(\delta)t}$$

where $\delta \in (0, 2)$ *and* $\lambda(\delta)$ *is characterized by*

$$\lambda(\delta) := \sup\left\{\lambda > 0 \ : \ h_1(\delta, \lambda) = 0, \ \lambda_m - \delta - \frac{1}{4}\,(2+\delta)\,\lambda > 0\right\}$$

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Sketch of the proof

- Since $\mathsf{A}\mathsf{T}\Pi = \left(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\right)^{-1}(\mathsf{T}\Pi)^*\mathsf{T}\Pi$, from (H1) and (H2)

$$-\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{A}\mathsf{T}\Pi F, F\rangle \geq \lambda_m\,\|(1-\Pi)F\|^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,\|\Pi F\|^2$$

- By (H4), we know that

$$|\mathrm{Re}\langle \mathsf{A}\mathsf{T}(1-\Pi)F, F\rangle + \mathrm{Re}\langle \mathsf{A}\mathsf{L}F, F\rangle| \leq C_M\,\|\Pi F\|\,\|(1-\Pi)F\|$$

- The equation $G = \mathsf{A}F$ is equivalent to $(\mathsf{T}\Pi)^* F = G + (\mathsf{T}\Pi)^*\,\mathsf{T}\Pi\,G$

$$\langle \mathsf{T}\mathsf{A}F, F\rangle = \langle G, (\mathsf{T}\Pi)^* F\rangle = \|G\|^2 + \|\mathsf{T}\Pi G\|^2 = \|\mathsf{A}F\|^2 + \|\mathsf{T}\mathsf{A}F\|^2$$

$$\langle G, (\mathsf{T}\Pi)^* F\rangle \leq \|\mathsf{T}\mathsf{A}F\|\,\|(1-\Pi)F\| \leq \frac{1}{2\,\mu}\,\|\mathsf{T}\mathsf{A}F\|^2 + \frac{\mu}{2}\,\|(1-\Pi)F\|^2$$

$$\|\mathsf{A}F\| \leq \frac{1}{2}\,\|(1-\Pi)F\|,\ \|\mathsf{T}\mathsf{A}F\| \leq \|(1-\Pi)F\|,\ |\langle \mathsf{T}\mathsf{A}F, F\rangle| \leq \|(1-\Pi)F\|^2$$

- With $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$

$$\mathsf{D}[F] - \lambda\,\mathsf{H}[F] \geq (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{2+\delta}{4}\,\lambda\,(X^2 + Y^2)$$

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Hypocoercivity

Corollary

For any $\delta \in (0, 2)$, if $\lambda(\delta)$ is the largest positive root of $h_1(\delta, \lambda) = 0$ for which $\lambda_m - \delta - \frac{1}{4}(2 + \delta)\lambda > 0$, then for any solution F of the evolution equation

$$\|F(t)\|^2 \leq \frac{2 + \delta}{2 - \delta} e^{-\lambda(\delta)\, t} \|F(0)\|^2 \quad \forall\, t \geq 0$$

From the norm equivalence of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2 - \delta}{4} \|F\|^2 \leq \mathsf{H}[F] \leq \frac{2 + \delta}{4} \|F\|^2$$

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Formal macroscopic (diffusion) limit

Scaled evolution equation

$$\varepsilon \frac{dF}{dt} + \mathsf{T}F = \frac{1}{\varepsilon}\,\mathsf{L}F$$

on the Hilbert space $\mathcal{H}$. $F_\varepsilon = F_0 + \varepsilon\,F_1 + \varepsilon^2\,F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$

$$\varepsilon^{-1}: \qquad \mathsf{L}F_0 = 0\,,$$
$$\varepsilon^0: \qquad \mathsf{T}F_0 = \mathsf{L}F_1\,,$$
$$\varepsilon^1: \qquad \tfrac{dF_0}{dt} + \mathsf{T}F_1 = \mathsf{L}F_2$$

The first equation reads as $u = F_0 = \Pi F_0$
The second equation is simply solved by $F_1 = -\,(\mathsf{T}\Pi)\,F_0$
After projection, the third equation is
$$\tfrac{d}{dt}\,(\Pi F_0) - \Pi\mathsf{T}\,(\mathsf{T}\Pi)\,F_0 = \Pi\mathsf{L}F_2 = 0$$

$$\partial_t u + (\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)\,u = 0$$

is such that $\frac{d}{dt}\|u\|^2 = -\,2\,\|(\mathsf{T}\Pi)\,u\|^2 \leq -\,2\,\lambda_M\,\|u\|^2$

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⊙ A toy problem

$$\frac{du}{dt} = (\mathsf{L} - \mathsf{T})\,u\,, \quad \mathsf{L} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad \mathsf{T} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}\,, \quad k^2 \geq \Lambda > 0$$
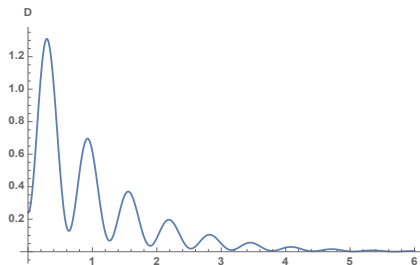
Non-monotone decay, a well known picture:
see for instance (Filbet, Mouhot, Pareschi, 2006)

- H-theorem: $\frac{d}{dt}|u|^2 = \frac{d}{dt}\left(u_1^2 + u_2^2\right) = -2\,u_2^2$
- macroscopic/diffusion limit: $\left(\frac{du_1}{dt} = -k^2\,u_1\right)$
- generalized entropy: $\mathsf{H}(u) = |u|^2 - \frac{\delta\,k}{1+k^2}\,u_1\,u_2$

$$\begin{aligned} \frac{d\mathsf{H}}{dt} &= -\left(2 - \frac{\delta\,k^2}{1+k^2}\right)u_2^2 - \frac{\delta\,k^2}{1+k^2}\,u_1^2 + \frac{\delta\,k}{1+k^2}\,u_1\,u_2 \\[2mm] &\leq -(2-\delta)\,u_2^2 - \frac{\delta\Lambda}{1+\Lambda}\,u_1^2 + \frac{\delta}{2}\,u_1 u_2 \end{aligned}$$

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Plots for the toy problem

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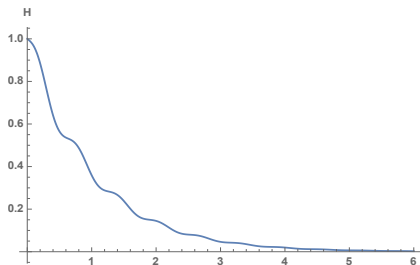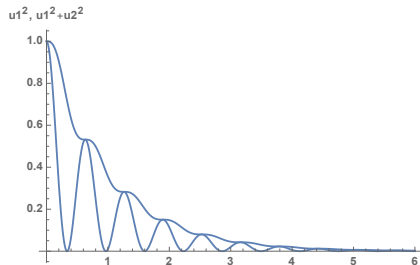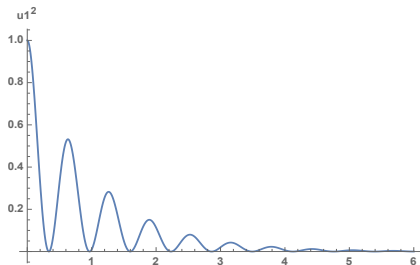
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The compact case

- Fokker-Planck equation and scattering collision operators

▷ A mode-by-mode (Fourier) hypocoercivity result

▷ Enlargement of the space by factorization

▷ Application to the torus and numerical improvements

- Further results: Euclidean space with strong confinement

  Collaboration with E. Bouin, S. Mischler, C. Mouhot, C. Schmeiser

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Fokker-Planck and scattering collision operators

Two basic examples:

- Linear *Fokker-Planck* collision operator

$$\mathsf{L}f = \Delta_v f + \nabla_v \cdot (v\,f)$$

- Linear relaxation operator (linear BGK)

$$\mathsf{L}f = \rho\,(2\pi)^{-d/2}\exp(-|v|^2/2) - f$$

with $\rho = \int f\,dv$

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Fokker-Planck equation with general equilibria

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f, \quad f(0, x, v) = f_0(x, v)$$

for a distribution function $f(t, x, v)$, with *position* variable $x \in \mathbb{R}^d$ or $x \in \mathbb{T}^d$ the flat d-dimensional torus

Fokker-Planck collision operator with a general equilibrium M

$$\mathsf{L} f = \nabla_v \cdot \left[M \nabla_v \left(M^{-1} f \right) \right]$$

Notation and assumptions: an *admissible local equilibrium* M is positive, radially symmetric and

$$\int_{\mathbb{R}^d} M(v)\, dv = 1, \quad d\gamma = \gamma(v)\, dv := \frac{dv}{M(v)}$$

γ is an *exponential weight* if (loose definition)

$$\lim_{|v| \to \infty} \frac{|v|^k}{\gamma(v)} = \lim_{|v| \to \infty} M(v)\, |v|^k = 0 \quad \forall\, k \in (d, \infty)$$

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Definitions

$$\Theta = \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 \, M(v) \, dv = \int_{\mathbb{R}^d} (v \cdot \mathsf{e})^2 \, M(v) \, dv$$

for an arbitrary $\mathsf{e} \in \mathbb{S}^{d-1}$

$$\int_{\mathbb{R}^d} v \otimes v \, M(v) \, dv = \Theta \operatorname{Id}$$

Then

$$\theta = \frac{1}{d} \, \|\nabla_v M\|_{\mathrm{L}^2(d\gamma)}^2 = \frac{4}{d} \int_{\mathbb{R}^d} |\nabla_v \sqrt{M}|^2 \, dv < \infty$$

If $M(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}}$, then $\Theta = 1$ and $\theta = 1$

$$\overline{\sigma} := \frac{1}{2} \sqrt{\theta/\Theta}$$

Microscopic coercivity property (Poincaré inequality): for all $u = M^{-1} \, F \in \mathrm{H}^1(M \, dv)$

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, M \, dv \geq \lambda_m \int_{\mathbb{R}^d} \left(u - \int_{\mathbb{R}^d} u \, M \, dv \right)^2 M \, dv$$

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Scattering collision operators

Scattering collision operator

$$\mathsf{L}f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \left(f(v')\, M(\cdot) - f(\cdot)\, M(v') \right) dv'$$

Main assumption on the *scattering rate* σ: for some positive, finite $\overline{\sigma}$

$$1 \le \sigma(v, v') \le \overline{\sigma} \quad \forall\, v,\, v' \in \mathbb{R}^d$$

Example: linear BGK operator

$$\mathsf{L}f = M\rho_f - f\,, \quad \rho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv$$

Local mass conservation

$$\int_{\mathbb{R}^d} \mathsf{L}f\, dv = 0$$

and we have

$$\int_{\mathbb{R}^d} |\mathsf{L}f|^2\, d\gamma \le 4\, \overline{\sigma}^2 \int_{\mathbb{R}^d} |M\rho_f - f|^2\, d\gamma$$

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(Technicalities !) the symmetry condition

$$\int_{\mathbb{R}^d} \big(\sigma(v, v') - \sigma(v', v)\big) M(v') \, dv' = 0 \quad \forall v \in \mathbb{R}^d$$

implies the *local mass conservation* $\int_{\mathbb{R}^d} \mathsf{L} f \, dv = 0$

Micro-reversibility, i.e., the symmetry of σ, is not required

The null space of L is spanned by the local equilibrium M
L only acts on the velocity variable

Microscopic coercivity property: for some $\lambda_m > 0$

$$\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \sigma(v, v') \, M(v) \, M(v') \big(u(v) - u(v')\big)^2 dv' \, dv$$

$$\geq \lambda_m \int_{\mathbb{R}^d} (u - \rho_{u \, M})^2 \, M \, dv$$

holds according to Proposition 2.2 of (Degond, Goudon, Poupaud, 2000) for all $u = M^{-1} F \in \mathrm{L}^2(M \, dv)$. If $\sigma \equiv 1$, then $\lambda_m = 1$

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Euclidean space, confinement, Poincaré inequality

(H1) *Regularity & Normalization:* $V \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} e^{-V} dx = 1$

(H2) *Spectral gap condition* (macroscopic Poincaré inequality): for some $\Lambda > 0$, $\forall u \in H^1(e^{-V} dx)$ such that $\int_{\mathbb{R}^d} u \, e^{-V} dx = 0$
$$\int_{\mathbb{R}^d} |u|^2 \, e^{-V} dx \leq \Lambda \int_{\mathbb{R}^d} |\nabla_x u|^2 \, e^{-V} dx$$

(H3) *Pointwise conditions:*
there exists $c_0 > 0$, $c_1 > 0$ and $\theta \in (0,1)$ s.t.
$$\Delta V \leq \frac{\theta}{2} |\nabla_x V(x)|^2 + c_0, \quad |\nabla_x^2 V(x)| \leq c_1 (1 + |\nabla_x V(x)|) \ \forall x \in \mathbb{R}^d$$

(H4) *Growth condition:* $\int_{\mathbb{R}^d} |\nabla_x V|^2 \, e^{-V} dx < \infty$

Theorem (D., Mouhot, Schmeiser)

Let L be either a Fokker-Planck operator or a linear relaxation operator with a local equilibrium $F(v) = (2\pi)^{-d/2} \exp(-|v|^2/2)$. If f solves
$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \mathsf{L}f$$
then
$$\forall t \geq 0, \quad \|f(t) - F\|^2 \leq (1 + \eta) \|f_0 - F\|^2 \, e^{-\lambda t}$$

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● Mode-by-mode decomposition

▷ Spectral decomposition (Hermite functions): linear Fokker-Planck operator

$$\mathsf{L}f = \Delta_v f + \nabla_v \cdot (v\,f)$$

or the linear relaxation operator (linear BGK)

$$\mathsf{L}f = \rho\,(2\pi)^{-d/2}\,\exp(-|v|^2/2) - f\,,$$

with $\rho = \int f\,dv$: (Arnold, Erb), (Achleitner, Arnold, Stürzer), (Achleitner, Arnold, Carlen), (Arnold, Einav, Wöhrer)

▷ Decomposition in Fourier modes

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Fourier modes

In order to perform a *mode-by-mode hypocoercivity* analysis, we introduce the Fourier representation with respect to x,

$$f(t, x, v) = \int_{\mathbb{R}^d} \hat{f}(t, \xi, v) \, e^{-i \, x \cdot \xi} \, d\mu(\xi)$$

$d\mu(\xi) = (2\pi)^{-d} \, d\xi$ and $d\xi$ is the Lesbesgue measure if $x \in \mathbb{R}^d$
$d\mu(\xi) = (2\pi)^{-d} \sum_{z \in \mathbb{Z}^d} \delta(\xi - z)$ is discrete for $x \in \mathbb{T}^d$

Parseval's identity if $\xi \in \mathbb{Z}^d$ and Plancherel's formula if $x \in \mathbb{R}^d$ read

$$\|f(t, \cdot, v)\|_{L^2(dx)} = \left\| \hat{f}(t, \cdot, v) \right\|_{L^2(d\mu(\xi))}$$

The Cauchy problem is now decoupled in the ξ-direction

$$\partial_t \hat{f} + \mathsf{T} \hat{f} = \mathsf{L} \hat{f}, \quad \hat{f}(0, \xi, v) = \hat{f}_0(\xi, v)$$

$$\mathsf{T} \hat{f} = i \, (v \cdot \xi) \, \hat{f}$$

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For any fixed $\xi \in \mathbb{R}^d$, let us apply the abstract result with

$$\mathcal{H} = L^2(d\gamma), \quad \|F\|^2 = \int_{\mathbb{R}^d} |F|^2 \, d\gamma, \quad \Pi F = M \int_{\mathbb{R}^d} F \, dv = M \, \rho_F$$

and $\mathsf{T} \hat{f} = i(v \cdot \xi) \hat{f}$, $\mathsf{T} \Pi F = i(v \cdot \xi) \rho_F M$,

$$\|\mathsf{T} \Pi F\|^2 = |\rho_F|^2 \int_{\mathbb{R}^d} |v \cdot \xi|^2 M(v) \, dv = \Theta |\xi|^2 |\rho_F|^2 = \Theta |\xi|^2 \|\Pi F\|^2$$

(H2) *Macroscopic coercivity* $\|\mathsf{T} \Pi F\|^2 \geq \lambda_M \|\Pi F\|^2 : \lambda_M = \Theta |\xi|^2$

(H3) $\int_{\mathbb{R}^d} v \, M(v) \, dv = 0$

The operator A is given by

$$\mathsf{A} F = \frac{-i\,\xi \cdot \int_{\mathbb{R}^d} v' \, F(v') \, dv'}{1 + \Theta |\xi|^2} \, M$$

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A mode-by-mode hypocoercivity result

$$\|\mathsf{A}F\| = \|\mathsf{A}(1-\Pi)F\| \leq \frac{1}{1+\Theta\,|\xi|^2} \int_{\mathbb{R}^d} \frac{|(1-\Pi)F|}{\sqrt{M}}\,|v\cdot\xi|\,\sqrt{M}\,dv$$

$$\leq \frac{1}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|\left(\int_{\mathbb{R}^d}(v\cdot\xi)^2\,M\,dv\right)^{1/2}$$

$$= \frac{\sqrt{\Theta}\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

- Scattering operator $\|\mathsf{L}F\|^2 \leq 4\,\overline{\sigma}^2\,\|(1-\Pi)F\|^2$
- Fokker-Planck (FP) operator

$$\|\mathsf{AL}F\| \leq \frac{2}{1+\Theta\,|\xi|^2}\int_{\mathbb{R}^d}\frac{|(1-\Pi)F|}{\sqrt{M}}\,|\xi\cdot\nabla_v\sqrt{M}|\,dv \leq \frac{\sqrt{\theta}\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

In both cases with $\kappa = \sqrt{\theta}$ (FP) or $\kappa = 2\,\overline{\sigma}\,\sqrt{\Theta}$ we obtain

$$\|\mathsf{AL}F\| \leq \frac{\kappa\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

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$$\mathsf{T}\mathsf{A}F(v) = -\frac{(v \cdot \xi)\, M}{1 + \Theta\, |\xi|^2} \int_{\mathbb{R}^d} (v' \cdot \xi)\, (1 - \Pi)F(v')\, dv'$$

is estimated by

$$\|\mathsf{T}\mathsf{A}F\| \leq \frac{\Theta\, |\xi|^2}{1 + \Theta\, |\xi|^2}\, \|(1 - \Pi)F\|$$

(H4) holds with $C_M = \frac{\kappa\, |\xi| + \Theta\, |\xi|^2}{1 + \Theta\, |\xi|^2}$

The two "good" terms

$$-\langle \mathsf{L}F, F \rangle \geq \lambda_m\, \|(1 - \Pi)F\|^2$$

$$\langle \mathsf{A}\mathsf{T}\Pi F, F \rangle = \mathsf{A}F = \frac{\Theta\, |\xi|^2}{1 + \Theta\, |\xi|^2}\, \|\Pi F\|^2$$

Two elementary estimates

$$\frac{\Theta\, |\xi|^2}{1 + \Theta\, |\xi|^2} \geq \frac{\Theta}{\max\{1, \Theta\}}\, \frac{|\xi|^2}{1 + |\xi|^2}\,, \qquad \frac{\lambda_M}{(1 + \lambda_M)\, C_M^2} = \frac{\Theta\, (1 + \Theta\, |\xi|^2)}{(\kappa + \Theta\, |\xi|)^2} \geq \frac{\Theta}{\kappa^2 + \Theta}$$

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Mode-by-mode hypocoercivity with exponential weights

Theorem

Let us consider an admissible M and a collision operator L satisfying the assumptions, and take $\xi \in \mathbb{R}^d$. If $\hat{f}$ is a solution such that $\hat{f}_0(\xi, \cdot) \in \mathrm{L}^2(d\gamma)$, then for any $t \geq 0$, we have

$$\left\| \hat{f}(t, \xi, \cdot) \right\|_{\mathrm{L}^2(d\gamma)}^2 \leq 3\, e^{-\mu_\xi\, t} \left\| \hat{f}_0(\xi, \cdot) \right\|_{\mathrm{L}^2(d\gamma)}^2$$

where

$$\mu_\xi := \frac{\Lambda\, |\xi|^2}{1 + |\xi|^2} \quad and \quad \Lambda = \frac{\Theta}{3\, \max\{1, \Theta\}}\, \min\left\{ 1, \frac{\lambda_m\, \Theta}{\kappa^2 + \Theta} \right\}$$

with $\kappa = 2\, \overline{\sigma}\, \sqrt{\Theta}$ for scattering operators and $\kappa = \sqrt{\theta}$ for (FP) operators

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Exponential convergence to equilibrium in $\mathbb{T}^d$

The unique global equilibrium in the case $x \in \mathbb{T}^d$ is given by

$$f_\infty(x,v) = \rho_\infty \, M(v) \quad \text{with} \quad \rho_\infty = \frac{1}{|\mathbb{T}^d|} \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dx \, dv$$

Theorem

Assume that γ has an exponential growth. We consider an admissible M, a collision operator L satisfying the assumptions. There exists a positive constant C such that the solution f of the Cauchy problem on $\mathbb{T}^d \times \mathbb{R}^d$ with initial datum $f_0 \in \mathrm{L}^2(dx\,d\gamma)$ satisfies

$$\|f(t,\cdot,\cdot) - f_\infty\|_{\mathrm{L}^2(dx\,d\gamma)} \leq C \, \|f_0 - f_\infty\|_{\mathrm{L}^2(dx\,d\gamma)} \, e^{-\frac{1}{4}\Lambda t} \quad \forall \, t \geq 0$$

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Theorem

Assume that M is admissible, $k \in (d, \infty]$ and

$$d\gamma_k := \gamma_k(v) \, dv \quad \text{where} \quad \gamma_k(v) = \left(1 + |v|^2\right)^{k/2} \quad \text{and} \quad k > d$$

There exists a positive constant C_k such that the solution $f(t, \cdot, \cdot)$ on $\mathbb{T}^d \times \mathbb{R}^d$ with initial datum $f_0 \in \mathrm{L}^2(dx \, d\gamma_k)$ satisfies

$$\|f(t, \cdot, \cdot) - f_\infty\|_{\mathrm{L}^2(dx \, d\gamma_k)} \le C_k \, \|f_0 - f_\infty\|_{\mathrm{L}^2(dx \, d\gamma_k)} \, e^{-\frac{1}{4} \Lambda t} \quad \forall t \ge 0$$

Tool: Enlargement of the space by the *factorization method* of (Gualdani, Mischler, Mouhot)

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Further references

- Exponential rates in kinetic equations: (Talay 2001), (Wu 2001)

- In presence of a strongly confining potential (Hérau 2006 & 2007), (Mouhot, Neumann, 2006)

- Hypo-elliptic methods (Hérau, Nier 2004), (Eckmann, Hairer, 2003), (Hörmander, 1967), (Kolmogorov, 1934), (Il'in, Has'min'ski, 1964) with applications to the Vlasov-Poisson-Fokker-Planck equation: (Victory, O'Dwyer, 1990), (Bouchut, 1993)

- Related topics:
  ▷ H^1-hypocoercivity (Gallay), (Villani...)...(Monmarché), (Evans)
  ▷ Diffusion limits (Degond, Poupaud, Schmeiser, Goudon,...)
  ▷ Poincaré inequalities and Lyapunov functions
  ▷ Harris type methods, use of coupling

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The global picture:
▷ by what can we replace the *macroscopic* Poincaré inequality ?

🌐 Nash's inequality and a decay rate when $V = 0$

🌐 Very weak confinement: Caffarelli-Kohn-Nirenberg inequalities and moments

🌐 With sub-exponential equilibria: weighted Poincaré inequality or Hardy-Poincaré inequality

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A result based on Nash's inequality

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f, \quad (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$$

$$\mathsf{D}[f] = -\frac{d}{dt} \mathsf{H}[f] \geq \mathsf{a} \left(\|(1 - \Pi)f\|^2 + 2 \langle \mathsf{ATΠ}f, f \rangle \right)$$

We observe that

$$\mathsf{A}^* f = \mathsf{TΠ} \left(1 + (\mathsf{TΠ})^* \mathsf{TΠ} \right)^{-1} f$$

$$= \mathsf{T} \left(1 + (\mathsf{TΠ})^* \mathsf{TΠ} \right)^{-1} \Pi f = M \, \mathsf{T} u_f = v \, M \cdot \nabla_x u_f$$

if u_f is the solution in $\mathrm{H}^1(dx)$ of $u_f - \Theta \, \Delta_x u_f = \rho_f$, and

$$\|u_f(t, \cdot)\|_{\mathrm{L}^1(dx)} = \|\rho_f(t, \cdot)\|_{\mathrm{L}^1(dx)} = \|f_0\|_{\mathrm{L}^1(dx \, dv)}$$

$$\|\nabla_x u_f\|^2_{\mathrm{L}^2(dx)} \leq \frac{1}{\Theta} \langle \mathsf{ATΠ}f, f \rangle$$

$$\|\Pi f\|^2 \leq \|u_f\|^2_{\mathrm{L}^2(dx)} + 2 \langle \mathsf{ATΠ}f, f \rangle$$

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Nash's inequality

$$\|u\|_{\mathrm{L}^2(dx)}^2 \leq \mathcal{C}_{Nash} \|u\|_{\mathrm{L}^1(dx)}^{\frac{4}{d+2}} \|\nabla u\|_{\mathrm{L}^2(dx)}^{\frac{2d}{d+2}} \quad \forall\, u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)$$

Use $\|\Pi f\|^2 \leq \Phi^{-1}\big(2\,\langle \mathsf{AT}\Pi f, f\rangle\big)$ with $\Phi^{-1}(y) := y + \big(\frac{y}{\mathsf{c}}\big)^{\frac{d}{d+2}}$ to get

$$\|(1-\Pi)f\|^2 + 2\,\langle \mathsf{AT}\Pi f, f\rangle \geq \Phi(\|f\|^2) \geq \Phi\big(\tfrac{2}{1+\delta}\,\mathsf{H}[f]\big)$$

$$\mathsf{D}[f] = -\frac{d}{dt}\mathsf{H}[f] \geq \mathsf{a}\,\Phi\big(\tfrac{2}{1+\delta}\,\mathsf{H}[f]\big)$$

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Algebraic decay rates in $\mathbb{R}^d$

On the whole Euclidean space, we can define the entropy

$$\mathsf{H}[f] := \tfrac{1}{2} \|f\|^2_{\mathrm{L}^2(dx\, d\gamma_k)} + \delta \langle \mathsf{A}f, f\rangle_{dx\, d\gamma_k}$$

Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|^2_{\mathrm{L}^2(dx)} \leq \mathfrak{C}_{\mathrm{Nash}} \|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(dx)} \|\nabla u\|^{\frac{2d}{d+2}}_{\mathrm{L}^2(dx)}$$

proves that

$$\mathsf{H}[f] \leq C \left(\mathsf{H}[f_0] + \|f_0\|^2_{\mathrm{L}^1(dx\, dv)} \right) (1+t)^{-\frac{d}{2}}$$

Theorem

Assume that γ_k has an exponential growth ($k = \infty$) or a polynomial growth of order $k > d$

There exists a constant $C > 0$ such that, for any $t \geq 0$

$$\|f(t, \cdot, \cdot)\|^2_{\mathrm{L}^2(dx\, d\gamma_k)} \leq C \left(\|f_0\|^2_{\mathrm{L}^2(dx\, d\gamma_k)} + \|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\, \mathrm{L}^1(dx))} \right) (1+t)^{-\frac{d}{2}}$$

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The rate of decay of the heat equation

If ρ is a solution of the heat equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

with initial datum ρ_0, then

$$\|\rho(t, \cdot)\|_{\mathrm{L}^1(dx)} = \|\rho_0\|_{\mathrm{L}^1(dx)}$$

$$\frac{d}{dt} \|\rho(t, \cdot)\|_{\mathrm{L}^2(dx)}^2 = -\|\nabla\rho(t, \cdot)\|_{\mathrm{L}^2(dx)}^2 \leq -\mathcal{C}\,\|\rho(t, \cdot)\|_{\mathrm{L}^2(dx)}^{2+\frac{4}{d}}$$

by Nash's inequality, and so

$$\|\rho(t, \cdot)\|_{\mathrm{L}^2(dx)}^2 = C\,\|\rho_0\|_{\mathrm{L}^2(dx)}^2\,(1 + t)^{-\frac{d}{2}}$$

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● The global picture

● Depending on the local equilibria and on the external potential (H1) and (H2) (which are Poincaré type inequalities) can be replaced by other functional inequalities:

▷ *microscopic coercivity* (H1)

$$- \langle \mathsf{L}F, F \rangle \geq \lambda_m \, \|(1 - \Pi)F\|^2$$

$$\implies \textbf{\textit{weak Poincaré inequalities}} \text{ or}$$
$$\textbf{\textit{Hardy-Poincaré inequalities}}$$

▷ *macroscopic coercivity* (H2)

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \, \|\Pi F\|^2$$

$$\implies \textbf{\textit{Nash inequality}}, \textbf{\textit{weighted Nash}} \text{ or}$$
$$\textbf{\textit{Caffarelli-Kohn-Nirenberg inequalities}}$$

● This can be done at the level of the *diffusion equation* (homogeneous case) or at the level of the *kinetic equation* (non-homogeneous case)

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Diffusion (Fokker-Planck) equations

Potential	$V = 0$	$V(x) = \gamma \log|x|$ $\gamma < d$	$V(x) = |x|^\alpha$ $\alpha \in (0,1)$	$V(x) = |x|^\alpha$ $\alpha \geq 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 1: $\partial_t u = \Delta u + \nabla \cdot (n \nabla V)$

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Kinetic Fokker-Planck equations

B = Bouin, L = Lafleche, M = Mouhot, MM = Mischler, Mouhot
S = Schmeiser

Potential	$V = 0$	$V(x) = \gamma \log|x|$ $\gamma < d$	$V(x) = |x|^\alpha$ $\alpha \in (0,1)$	$V(x) = |x|^\alpha$ $\alpha \geq 1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v) = e^{-\langle v \rangle^\beta}$, $\beta \geq 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b < 1$, $\beta = 2$ convergence	DMS, Mischler-Mouhot $e^{-\lambda t}$ convergence
$F(v) = e^{-\langle v \rangle^\beta}$, $\beta \in (0,1)$	BDLS: $t^{-\zeta}$, $\zeta = \min\left\{\frac{d}{2}, \frac{k}{\beta}\right\}$ decay			
$F(v) = \langle v \rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 1: $\partial_t f + v \cdot \nabla_x f = F \nabla_v(F^{-1} \nabla_v f)$. Notation: $\langle v \rangle = \sqrt{1 + |v|^2}$

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Further references

● *Weak Poincaré inequality*: (Röckner & Wang, 2001), (Kavian, Mischler), (Cao, PhD thesis), (Hu, Wang, 2019) + (Ben-Artzi, Einav) for recent spectral considerations

● *Weighted Nash inequalities*: (Bakry, Bolley, Gentil, Maheux, 2012), (Wang, 2000, 2002, 2010)

● Related topics:
▷ fractional diffusion (Cattiaux, Puel, Fournier, Tardif,...) + (Bouin, D., Lafleche, Schmeiser): work in progress

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The macroscopic Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \Delta_x u + \nabla_x \cdot (\nabla_x V \, u) = \nabla_x \left(e^{-V} \, \nabla_x \left(e^V \, u \right) \right)$$

Here $x \in \mathbb{R}^d$, $d \geq 3$, and V is a potential such that $e^{-V} \notin \mathrm{L}^1(\mathbb{R}^d)$ corresponding to a *very weak confinement*

Two examples

$$V_1(x) = \gamma \, \log |x| \quad \text{and} \quad V_2(x) = \gamma \, \log \langle x \rangle$$

with $\gamma < d$ and $\langle x \rangle := \sqrt{1 + |x|^2}$ for any $x \in \mathbb{R}^d$

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Potential	$V = 0$	$V(x) = \gamma \log|x|$ $\gamma < d$	$V(x) = |x|^\alpha$ $\alpha \in (0,1)$	$V(x) = |x|^\alpha$ $\alpha \geq 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 2: $\partial_t u = \Delta u + \nabla \cdot (n \nabla V)$

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A first decay result

> **Theorem**
>
> *Assume that $d \geq 3$, $\gamma < (d-2)/2$ and $V = V_1$ or $V = V_2$*
> *F any solution u with initial datum $u_0 \in L^1_+ \cap L^2(\mathbb{R}^d)$,*
>
> $$\|u(t,\cdot)\|_2^2 \leq \frac{\|u_0\|_2^2}{(1 + c\,t)^{\frac{d}{2}}} \quad with \quad c := \frac{4}{d} \min\left\{1, 1 - \frac{2\,\gamma}{d-2}\right\} \mathcal{C}_{\text{Nash}}^{-1} \frac{\|u_0\|_2^{4/d}}{\|u_0\|_1^{4/d}}$$

Here $\mathcal{C}_{\text{Nash}}$ denotes the optimal constant in Nash's inequality

$$\|u\|_2^{2+\frac{4}{d}} \leq \mathcal{C}_{\text{Nash}} \|u\|_1^{\frac{4}{d}} \|\nabla u\|_2^2 \quad \forall u \in L^1 \cap H^1(\mathbb{R}^d)$$

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An extended range of exponents: with moments

Theorem

Let $d \geq 1$, $0 < \gamma < d$, $V = V_1$ or $V = V_2$, and $u_0 \in L^1_+ \cap L^2(e^V)$
with $\qquad\qquad \left\| |x|^k u_0 \right\|_1 < \infty$ for some $k \geq \max\{2, \gamma/2\}$

$$\forall\, t \geq 0\,, \quad \|u(t, \cdot)\|^2_{L^2(e^V\, dx)} \leq \|u_0\|^2_{L^2(e^V\, dx)}\, (1 + c\, t)^{-\frac{d-\gamma}{2}}$$

for some c depending on d, γ, k, $\|u_0\|_{L^2(e^V\, dx)}$, $\|u_0\|_1$, and $\left\| |x|^k u_0 \right\|_1$

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An extended range of exponents: in self-similar variables

$$u_\star(t, x) = \frac{c_\star}{(1 + 2\,t)^{\frac{d-\gamma}{2}}} \, |x|^{-\gamma} \exp\left(-\frac{|x|^2}{2\,(1 + 2\,t)}\right)$$

Here the initial data need have a sufficient decay...
$c_\star$ is chosen such that $\|u_\star\|_1 = \|u_0\|_1$

Theorem

Let $d \geq 1$, $\gamma \in (0, d)$, $V = V_1$ assume that

$$\forall\, x \in \mathbb{R}^d, \quad 0 \leq u_0(x) \leq K\, u_\star(0, x)$$

for some constant $K > 1$

$$\forall\, t \geq 0, \quad \|u(t, \cdot) - u_\star(t, \cdot)\|_p \leq K\, c_\star^{1 - \frac{1}{p}} \|u_0\|_1^{\frac{1}{p}} \left(\frac{e}{2\,|\gamma|}\right)^{\frac{\gamma}{2}\left(1 - \frac{1}{p}\right)} (1 + 2\,t)^{-\zeta_p}$$

for any $p \in [1, +\infty)$, where $\zeta_p := \frac{d}{2}\left(1 - \frac{1}{p}\right) + \frac{1}{2\,p} \min\left\{2, \frac{d}{d-\gamma}\right\}$

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Proofs: basic case

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^2 \, dx = -2 \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} \Delta V \, |u|^2 \, dx$$

with either $V = V_1$ or $V = V_2$ and

$$\Delta V_1(x) = \gamma \, \frac{d-2}{|x|^2} \quad \text{and} \quad \Delta V_2(x) = \gamma \, \frac{d-2}{1+|x|^2} + \frac{2\,\gamma}{(1+|x|^2)^2}$$

For $\gamma \leq 0$: apply Nash's inequality

$$\frac{d}{dt} \|u\|_2^2 \leq -2 \|\nabla u\|_2^2 \leq -\frac{2}{\mathcal{C}_{\text{Nash}}} \|u_0\|_1^{-4/d} \|u\|_2^{2+4/d}$$

For $0 < \gamma < (d-2)/2$: Hardy-Nash inequalities

Lemma

Let $d \geq 3$ and $\delta < (d-2)^2/4$

$$\|u\|_2^{2+\frac{4}{d}} \leq \mathcal{C}_\delta \left(\|\nabla u\|_2^2 - \delta \int_{\mathbb{R}^d} \frac{u^2}{|x|^2} \, dx \right) \|u\|_1^{\frac{4}{d}} \quad \forall \, u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)$$

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Proofs: moments

Growth of the moment

$$M_k(t) := \int_{\mathbb{R}^d} |x|^k u \, dx$$

From the equation

$$M_k' = k \left(d + k - 2 - \gamma \right) \int_{\mathbb{R}^d} u \, |x|^{k-2} \, dx \leq k \left(d + k - 2 - \gamma \right) M_0^{\frac{2}{k}} M_k^{1 - \frac{2}{k}}$$

then use the Caffarelli-Kohn-Nirenberg inequality

$$\int_{\mathbb{R}^d} |x|^\gamma \, u^2 \, dx \leq \mathcal{C} \left(\int_{\mathbb{R}^d} |x|^{-\gamma} \, \left| \nabla \left(|x|^\gamma u \right) \right|^2 \, dx \right)^a \left(\int_{\mathbb{R}^d} |x|^k \, |u| \, dx \right)^{2(1-a)}$$

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Proofs: self-similar solutions

The proof relies on *uniform decay estimates* + Poincaré inequality in self-similar variables

Proposition

Let $\gamma \in (0, d)$ and assume that

$$0 \le u(0, x) \le c_\star \left(\sigma + |x|^2\right)^{-\gamma/2} \exp\left(-\frac{|x|^2}{2}\right) \quad \forall\, x \in \mathbb{R}^d$$

with $\sigma = 0$ if $V = V_1$ and $\sigma = 1$ if $V = V_2$. Then

$$0 \le u(t, x) \le \frac{c_\star}{(1 + 2\, t)^{\frac{d-\gamma}{2}}} \left(\sigma + |x|^2\right)^{-\gamma/2} \exp\left(-\frac{|x|^2}{2\,(1 + 2\, t)}\right)$$

for any $x \in \mathbb{R}^d$ and $t \ge 0$

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The kinetic Fokker-Planck equation

Let us consider the kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \mathsf{L} f$$

where $\mathsf{L} f$ is one of the two following collision operators

(a) a Fokker-Planck operator

$$\mathsf{L} f = \nabla_v \cdot \left(M \, \nabla_v \left(M^{-1} \, f \right) \right)$$

(b) a scattering collision operator

$$\mathsf{L} f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \left(f(v') \, M(\cdot) - f(\cdot) \, M(v') \right) dv'$$

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Potential	$V = 0$	$V(x) = \gamma \log|x|$ $\gamma < d$	$V(x) = |x|^\alpha$ $\alpha \in (0,1)$	$V(x) = |x|^\alpha$ $\alpha \geq 1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v) = e^{-\langle v \rangle^\beta}, \, \beta \geq 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b < 1, \, \beta = 2$ convergence	DMS, Mischler-Mouhot $e^{-\lambda t}$ convergence
$F(v) = e^{-\langle v \rangle^\beta}$, $\beta \in (0,1)$	BDLS: $t^{-\zeta}$, $\zeta = \min\{\frac{d}{2}, \frac{k}{\beta}\}$ decay			
$F(v) = \langle v \rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 2: $\partial_t f + v \cdot \nabla_x f = F \nabla_v (F^{-1} \nabla_v f)$. Notation: $\langle v \rangle = \sqrt{1 + |v|^2}$

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Decay rates

$$\forall\, (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \mathcal{M}(x,v) = M(v)\, e^{-V(x)}, \quad M(v) = (2\pi)^{-\frac{d}{2}}\, e^{-\frac{1}{2}|v|^2}$$

$$(\mathbf{H1}) \quad 1 \leq \sigma(v,v') \leq \overline{\sigma}, \quad \forall\, v,\, v' \in \mathbb{R}^d, \quad \text{for some} \quad \overline{\sigma} \geq 1$$

$$(\mathbf{H2}) \quad \int_{\mathbb{R}^d} \big(\sigma(v,v') - \sigma(v',v)\big) M(v')\, dv' = 0 \quad \forall\, v \in \mathbb{R}^d$$

Theorem

Let $d \geq 1$, $V = V_2$ with $\gamma \in [0,d)$, $k > \max\{2, \gamma/2\}$ and
$f_0 \in \mathrm{L}^2(\mathcal{M}^{-1} dx\, dv)$ such that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \langle x \rangle^k f_0\, dx\, dv + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^k f_0\, dx\, dv < +\infty$$

If (H1)–(H2) *hold, then there exists $C > 0$ such that*

$$\forall\, t \geq 0, \quad \|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathcal{M}^{-1} dx\, dv)} \leq C\, (1+t)^{-\frac{d-\gamma}{2}}$$

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● With sub-exponential equilibria

▷ The homogeneous Fokker-Planck equation with sub-exponential equilibria $F(v) = C_\alpha \, e^{-\langle v \rangle^\alpha}$, $\alpha \in (0, 1)$
– decay rates based on the weak Poincaré inequality (Kavian, Mischler)
– decay rates based on a weighted Poincaré / Hardy-Poincaré inequality

▷ The kinetic Fokker-Planck equation with sub-exponential local equilibria and ***no confinement***, the equation with linear scattering

In collaboration with Emeric Bouin, Laurent Lafleche and Christian Schmeiser

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Fokker-Planck with sub-exponential equilibria

We consider the ***homogeneous Fokker-Planck equation***

$$\partial_t g = \nabla_v \cdot \left(F \, \nabla_v \left(F^{-1} \, g \right) \right)$$

associated with ***sub-exponential equilibria***

$$F(v) = C_\alpha \, e^{-\langle v \rangle^\alpha} \, , \quad \alpha \in (0, 1)$$

The corresponding Ornstein-Uhlenbeck equation for $h = g/F$ is

$$\partial_t h = F^{-1} \, \nabla_v \cdot \left(F \, \nabla_v h \right)$$

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Potential	$V = 0$	$V(x) = \gamma \log|x|$ $\gamma < d$	$V(x) = |x|^\alpha$ $\alpha \in (0,1)$	$V(x) = |x|^\alpha$ $\alpha \geq 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 3: $\partial_t u = \Delta u + \nabla \cdot (n \nabla V)$

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Weak Poincaré inequality

$$\int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \mathrm{d}\xi \leq \mathcal{C}_{\alpha,\tau} \left(\int_{\mathbb{R}^d} |\nabla h|^2 \, \mathrm{d}\xi \right)^{\frac{\tau}{1+\tau}} \left\| h - \widetilde{h} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}^{\frac{2}{1+\tau}}$$

for some explicit positive constant $\mathcal{C}_{\alpha,\tau}$, $\widetilde{h} := \int_{\mathbb{R}^d} h \, \mathrm{d}\xi$. Using

$$\frac{d}{dt} \int_{\mathbb{R}^d} \left| h(t,\cdot) - \widetilde{h} \right|^2 \mathrm{d}\xi = -2 \int_{\mathbb{R}^d} |\nabla_v h|^2 \, \mathrm{d}\xi$$

where $h = g/F$ and $\mathrm{d}\xi = F \, \mathrm{d}v$ + Hölder's inequality

$$\int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \mathrm{d}\xi \leq \left(\int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \langle v \rangle^{-\beta} \, \mathrm{d}\xi \right)^{\frac{\tau}{\tau+1}} \left(\int_{\mathbb{R}^d} \left\| h - \widetilde{h} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}^2 \langle v \rangle^{\beta\,\tau} \, \mathrm{d}\xi \right)^{\frac{1}{1+\tau}}$$

with $(\tau + 1)/\tau = \beta/\eta$, then for with $\mathcal{M} = \sup_{s \in (0,t)} \left\| h(s,\cdot) - \widetilde{h} \right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}^{2/\tau}$

$$\int_{\mathbb{R}^d} \left| h(t,\cdot) - \widetilde{h} \right|^2 \mathrm{d}\xi \leq \left(\left(\int_{\mathbb{R}^d} \left| h(0,\cdot) - \widetilde{h} \right|^2 \mathrm{d}\xi \right)^{-\frac{1}{\tau}} + \frac{2\,\tau^{-1}}{\mathcal{C}_{\alpha,\tau}^{1+1/\tau}\,\mathcal{M}}\,t \right)^{-\tau}$$

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Weighted Poincaré inequality

There exists a constant $\mathcal{C} > 0$ such that

$$\int_{\mathbb{R}^d} |\nabla h|^2 \, F \, dv \geq \mathcal{C} \int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \langle v \rangle^{-\beta} \, F \, dv$$

with $\beta = 2 \, (1 - \alpha)$, $\widetilde{h} := \int_{\mathbb{R}^d} h \, F \, dv$ and $F(v) = C_\alpha \, e^{-\langle v \rangle^\alpha}$ and $\alpha \in (0, 1)$

Written in terms of $g = h \, F$, the inequality becomes

$$\int_{\mathbb{R}^d} \left| \nabla_v \left(F^{-1} g \right) \right|^2 \, F^2 \, d\mu \geq \mathcal{C} \int_{\mathbb{R}^d} |g - \overline{g}|^2 \, \langle v \rangle^{-2 \, (1-\alpha)} \, d\mu$$

where $d\mu = F \, dv$

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$$\frac{d}{dt} \int_{\mathbb{R}^d} |h(t,v)|^2 \langle v \rangle^k \, F \, \mathrm{d}v + 2 \int_{\mathbb{R}^d} |\nabla_v h|^2 \langle v \rangle^k \, F \, \mathrm{d}v$$

$$= - \int_{\mathbb{R}^d} \nabla_v(h^2) \cdot \left(\nabla_v \langle v \rangle^k \right) F \, \mathrm{d}v$$

With $\ell = 2 - \alpha$, $a \in \mathbb{R}$, $b \in (0, +\infty)$

$$\nabla_v \cdot \left(F \, \nabla_v \langle v \rangle^k \right) = \frac{k}{\langle v \rangle^4} \left(d + (k+d-2) \, |v|^2 - \alpha \, \langle v \rangle^\alpha \, |v|^2 \right) \leq a - b \, \langle v \rangle^{-\ell}$$

Proposition (Weighted L^2 norm)

There exists a constant $\mathcal{K}_k > 0$ such that, if h solves the Ornstein-Uhlenbeck equation, then

$$\forall \, t \geq 0 \quad \|h(t,\cdot)\|_{\mathrm{L}^2(\langle v \rangle^k \, \mathrm{d}\xi)} \leq \mathcal{K}_k \, \|h^{\mathrm{in}}\|_{\mathrm{L}^2(\langle v \rangle^k \, \mathrm{d}\xi)}$$

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$$\frac{d}{dt} \int_{\mathbb{R}^d} \left| h(t,\cdot) - \widetilde{h} \right|^2 \mathrm{d}\xi = -2 \int_{\mathbb{R}^d} |\nabla_v h|^2 \, \mathrm{d}\xi \leq -2\,\mathcal{C} \int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \langle v \rangle^{-\beta} \, \mathrm{d}\xi$$

+ Hölder

Theorem

Assume that $\alpha \in (0,1)$. Let $g^{\mathrm{in}} \in \mathrm{L}^1_+(\mathrm{d}\mu) \cap \mathrm{L}^2(\langle v \rangle^k \mathrm{d}\mu)$ for some $k > 0$ and consider the solution g to the homogeneous Fokker-Planck equation with initial datum g^{in}. If $\overline{g} = \left(\int_{\mathbb{R}^d} g \, \mathrm{d}v \right) F$, then

$$\int_{\mathbb{R}^d} |g(t,\cdot) - \overline{g}|^2 \, \mathrm{d}\mu \leq \left(\left(\int_{\mathbb{R}^d} |g^{\mathrm{in}} - \overline{g}|^2 \, \mathrm{d}\mu \right)^{-\beta/k} + \frac{2\,\beta\,\mathcal{C}}{k\,\mathcal{K}^{\beta/k}}\, t \right)^{-k/\beta}$$

with $\beta = 2\,(1-\alpha)$ and $\mathcal{K} := \mathcal{K}_k^2 \, \left\| g^{\mathrm{in}} \right\|^2_{\mathrm{L}^2(\langle v \rangle^k \, \mathrm{d}\mu)} + \Theta_k \left(\int_{\mathbb{R}^d} g^{\mathrm{in}} \, \mathrm{d}v \right)^2$

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The kinetic equation with sub-exponential local equilibria

- the *Fokker-Planck* operator

$$\mathsf{L}_1 f = \nabla_v \cdot \left(F \, \nabla_v \left(F^{-1} f \right) \right)$$

- the *scattering* collision operator

$$\mathsf{L}_2 f = \int_{\mathbb{R}^d} \mathrm{b}(\cdot, v') \left(f(v') \, F(\cdot) - f(\cdot) \, F(v') \right) \mathrm{d}v'$$

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Micro Poincaré $F(v) = e^{-\langle v \rangle^\beta}, \ \beta \geq 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b < 1$, $\beta = 2$ convergence	DMS, Mischler-Mouhot $e^{-\lambda t}$ convergence
$F(v) = e^{-\langle v \rangle^\beta},$ $\beta \in (0,1)$	BDLS: $t^{-\zeta}$, $\zeta = \min\left\{\frac{d}{2}, \frac{k}{\beta}\right\}$ decay			
$F(v) = \langle v \rangle^{-d-\beta}$	BDLS, fractional in progress			

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The decay rate with sub-exponential local equilibria

Theorem

Let $\alpha \in (0,1)$, $\beta > 0$, $k > 0$ and let $F(v) = C_\alpha\, e^{-\langle v \rangle^\alpha}$. Assume that either $\mathsf{L} = \mathsf{L}_1$ and $\beta = 2\,(1-\alpha)$, or $\mathsf{L} = \mathsf{L}_2$ + Assumptions. There exists a numerical constant $\mathcal{C} > 0$ such that any solution f of

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad f(0,\cdot,\cdot) = f^{\mathrm{in}} \in \mathrm{L}^2(\langle v \rangle^k \mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^1_+(\mathrm{d}x\,\mathrm{d}v)$$

satisfies

$$\forall\, t \geq 0\,, \quad \|f(t,\cdot,\cdot)\|^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left| f(t,x,v) \right|^2 \mathrm{d}x\,\mathrm{d}\mu \leq \mathcal{C}\, \frac{\left\| f^{\mathrm{in}} \right\|^2}{(1 + \kappa\, t)^\zeta}$$

with rate $\zeta = \min\{d/2, k/\beta\}$, for some positive κ which is an explicit function of the two quotients, $\left\| f^{\mathrm{in}} \right\| / \left\| f^{\mathrm{in}} \right\|_k$ and $\left\| f^{\mathrm{in}} \right\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)} / \left\| f^{\mathrm{in}} \right\|$

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Preliminaries

$$D[f] := - \langle Lf, f \rangle + \delta \langle AT\Pi f, \Pi f \rangle$$
$$+ \delta \langle AT(\mathrm{Id} - \Pi)f, \Pi f \rangle - \delta \langle TA(\mathrm{Id} - \Pi)f, (\mathrm{Id} - \Pi)f \rangle$$
$$- \delta \langle AL(\mathrm{Id} - \Pi)f, \Pi f \rangle$$

● ***microscopic coercivity***. If $L = L_1$, we rely on the weighted Poincaré inequality

$$\langle Lf, f \rangle \leq - \mathcal{C} \, \|(\mathrm{Id} - \Pi)f\|_{-\beta}^2$$

If $L = L_2$, we assume that there exists a constant $\mathcal{C} > 0$ such that

$$\int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \langle v \rangle^{-\beta} F \, dv \leq \mathcal{C} \iint_{\mathbb{R}^d \times \mathbb{R}^d} b(v, v') \left| h' - h \right|^2 F \, F' \, dv \, dv'$$

● ***Weighted*** L^2 ***norms*** Let $k > 0$, $f^{\mathrm{in}} \in L^2(\langle v \rangle^k \, dx \, d\mu)$ a solution. $\exists \mathcal{K}_k > 1$ such that

$$\forall t \geq 0, \quad \|f(t, \cdot, \cdot)\|_{L^2(\langle v \rangle^k \, dx \, d\mu)} \leq \mathcal{K}_k \, \|f^{\mathrm{in}}\|_{L^2(\langle v \rangle^k \, dx \, d\mu)}$$

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Proof

$$\mathsf{H}_\delta[f] := \frac{1}{2}\,\|f\|^2 + \delta\,\langle \mathsf{A}f, f\rangle\,, \quad \frac{d}{dt}\mathsf{H}_\delta[f] = -\,\mathsf{D}[f]$$

- There exists $\kappa > 0$ such that $\forall\, f \in \mathrm{L}^2\left(\langle v\rangle^{-\beta}\,\mathrm{d}x\,\mathrm{d}\mu\right) \cap \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$,

$$\mathsf{D}[f] \geq \kappa\left(\|(\mathrm{Id} - \Pi)f\|_{-\beta}^2 + \langle \mathsf{ATI}f, \Pi f\rangle\right)$$

- For any $f \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^2(\mathrm{d}x\,\mathrm{d}v)$,

$$\langle \mathsf{ATI}f, \Pi f\rangle \geq \Phi\left(\|\Pi f\|^2\right)$$

$$\Phi^{-1}(y) := 2\,y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}}, \quad \mathsf{c} = \Theta\,\mathfrak{C}_{\mathrm{Nash}}^{-\frac{d+2}{d}}\,\|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}^{-\frac{4}{d}}$$

- For any $f \in \mathrm{L}^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$,

$$\|(\mathrm{Id} - \Pi)f\|_{-\beta}^2 \geq \Psi\left(\|(\mathrm{Id} - \Pi)f\|^2\right)$$

$$\Psi(y) := C_0\,y^{1+\beta/k}, \quad C_0 := \left(\mathcal{K}_k\left(1 + \Theta_k\right)\|f^{\mathrm{in}}\|_k\right)^{-\frac{2\,\beta}{k}}$$

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The Vlasov-Poisson-Fokker-Planck system: linearization and hypocoercivity

▷ Linearized Vlasov-Poisson-Fokker-Planck system

▷ A result in the non-linear case, $d = 1$

In collaboration with Lanoir Addala, Xingyu Li and Lazhar M. Tayeb

🟢 L^2-Hypocoercivity and large time asymptotics of the linearized Vlasov-Poisson-Fokker-Planck system. Preprint hal-02299535 and arxiv: 1909.12762

(Hérau, Thomann, 2016), (Herda, Rodrigues, 2018)

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● Linearized Vlasov-Poisson-Fokker-Planck system

The *Vlasov-Poisson-Fokker-Planck system* in presence of an external potential V is

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v\, f)$$

$$-\Delta_x \phi = \rho_f = \int_{\mathbb{R}^d} f \, dv \tag{VPFP}$$

Linearized problem around $f_\star$: $f = f_\star (1 + \eta\, h)$, $\iint_{\mathbb{R}^d \times \mathbb{R}^d} h\, f_\star \, dx\, dv = 0$

$$\partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h = \eta\, \nabla_x \psi_h \cdot \nabla_v h$$

$$-\Delta_x \psi_h = \int_{\mathbb{R}^d} h\, f_\star \, dv$$

Drop the $\mathcal{O}(\eta)$ term : *linearized Vlasov-Poisson-Fokker-Planck* system

$$\partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h = 0$$

$$-\Delta_x \psi_h = \int_{\mathbb{R}^d} h\, f_\star \, dv\,, \qquad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h\, f_\star \, dx\, dv = 0$$

$$\tag{VPFPlin}$$

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Hypocoercivity

Let us define the norm

$$\|h\|^2 := \iint_{\mathbb{R}^d \times \mathbb{R}^d} h^2 \, f_\star \, dx \, dv + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2 \, dx$$

Theorem

Let us assume that $d \geq 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. Then there exist two positive constants $\mathcal{C}$ and λ such that any solution h of (VPFPlin) *with an initial datum h_0 of zero average with $\|h_0\|^2 < \infty$ is such that*

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \, \|h_0\|^2 \, e^{-\lambda t} \quad \forall \, t \geq 0$$

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Diffusion limit

Linearized problem in the parabolic scaling

$$\varepsilon\,\partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \frac{1}{\varepsilon}\left(\Delta_v h - v \cdot \nabla_v h\right) = 0$$

$$-\Delta_x \psi_h = \int_{\mathbb{R}^d} h\, f_\star\, dv\,, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h\, f_\star\, dx\, dv = 0$$

(VPFPscal)

Expand $h_\varepsilon = h_0 + \varepsilon\, h_1 + \varepsilon^2\, h_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$. With $W_\star = V + \phi_\star$

$\varepsilon^{-1}:$ $\qquad \Delta_v h_0 - v \cdot \nabla_v h_0 = 0$

$\varepsilon^0:$ $\qquad v \cdot \nabla_x h_0 - \nabla_x W_\star \cdot \nabla_v h_0 + v \cdot \nabla_x \psi_{h_0} = \Delta_v h_1 - v \cdot \nabla_v h_1$

$\varepsilon^1:$ $\qquad \partial_t h_0 + v \cdot \nabla_x h_1 - \nabla_x W_\star \cdot \nabla_v h_1 = \Delta_v h_2 - v \cdot \nabla_v h_2$

With $u = \Pi h_0$, $-\Delta\psi = u\,\rho_\star$, $w = u + \psi$, equations simply mean

$$u = h_0\,, \quad v \cdot \nabla_x w = \Delta_v h_1 - v \cdot \nabla_v h_1$$

from which we deduce that $h_1 = -v \cdot \nabla_x w$ and

$$\partial_t u - \Delta w + \nabla_x W_\star \cdot \nabla u = 0$$

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Theorem

Let us assume that $d \geq 1$, $V(x) = |x|^{\alpha}$ for some $\alpha > 1$ and $M > 0$. For any $\varepsilon > 0$ small enough, there exist two positive constants $\mathcal{C}$ and λ, which do not depend on ε, such that any solution h of (VPFPscal) with an initial datum h_0 of zero average and such that $\|h_0\|^2 < \infty$ satisfies

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0$$

Corollary

Assume that $d = 1$, $V(x) = |x|^{\alpha}$ for some $\alpha > 1$ and $M > 0$. If f solves (VPFP) with initial datum $f_0 = (1 + h_0) f_\star$ such that h_0 has zero average, $\|h_0\|^2 < \infty$ and $(1 + h_0) \geq 0$, then

$$\|h(t, \cdot, \cdot)\|^2 \leq \mathcal{C} \|h_0\|^2 e^{-\lambda t} \quad \forall t \geq 0$$

holds with $h = f/f_\star - 1$ for some positive constants $\mathcal{C}$ and λ

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These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/
▷ Lectures

The papers can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/
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Thank you for your attention !

Abstract method and motivation
The compact case
The non-compact case
Vlasov-Poisson-Fokker-Planck

Linearized system and hypocoercivity
Results in the diffusion limit / non-linear case