Phase transitions and symmetry in PDEs

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Outline

- *Phase transition* and *symmetry breaking*, a warning
- Preliminaries: two observations

▷ Phase transition and asymptotic behaviour in a *flocking* model with a mean field term

▷ Moving planes and eigenvalues

- *Symmetry and symmetry breaking in interpolation inequalities*

▷ Gagliardo-Nirenberg-Sobolev inequalities on the sphere

▷ [Keller-Lieb-Thirring inequalities on the sphere]

▷ Caffarelli-Kohn-Nirenberg inequalities

- *Ground states with magnetic fields*

▷ *Magnetic rings*, a one-dimensional magnetic interpolation inequality

▷ Interpolation inequalities in dimensions 2 and 3, spectral estimates

- *Aharonov-Bohm magnetic fields in* $\mathbb{R}^2$

▷ Aharonov-Bohm effect

▷ Interpolation [and Keller-Lieb-Thirring] inequalities in $\mathbb{R}^2$

▷ Aharonov-Symmetry and symmetry breaking

Phase transition and symmetry breaking

- The notion of ***phase transition*** in physics

▷ Ehrenfest's classification and more recent definitions

- ***Symmetry breaking***

▷ The principles of Pierre Curie
▷ A mathematical point of view: the symmetry of the ground state

- Bifurcations, interpolation inequalities and evolution equations

▷ Subcritical interpolation inequalities depending on a single parameter
▷ The non-linear problem *versus* the linearized spectral problem
▷ Nonlinear flows as a tool: generalized Bakry-Emery method
▷ Energy and relaxation

Preliminaries

- Phase transition and asymptotic behaviour in a *flocking* model with a mean field term:

  the *homogeneous Cucker-Smale / McKean-Vlasov model*

  PhD thesis of Xingyu Li, https://arxiv.org/abs/1906.07517

- Moving planes and eigenvalues

  (Old) joint work with P. Felmer

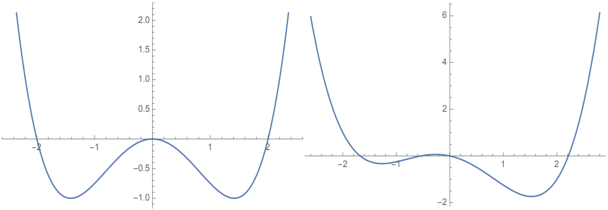
A simple version of the Cucker-Smale model

A model for bird flocking (simplified version)

$$\frac{\partial f}{\partial t} = D\,\Delta_v f + \nabla_v \cdot \left(\nabla_v \phi(v)\, f - \mathbf{u}_f\, f\right)$$

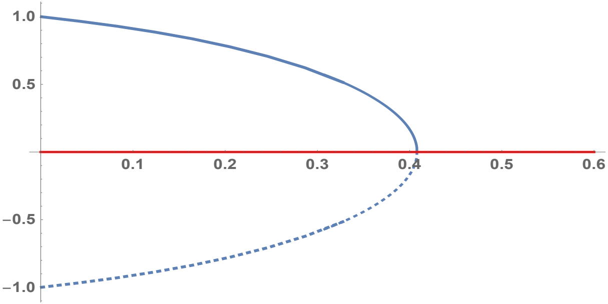
where $\mathbf{u}_f = \int v\, f\, dv$ is the average velocity
f is a probability measure

left: $\phi(v) = \frac{1}{4}\,|v|^4 - \frac{1}{2}\,|v|^2$ right: $\phi(v) - \mathbf{u}_f \cdot v$



[J. Tugaut, 2014]
[A. Barbaro, J. Cañizo, J.A. Carrillo, and P. Degond, 2016]

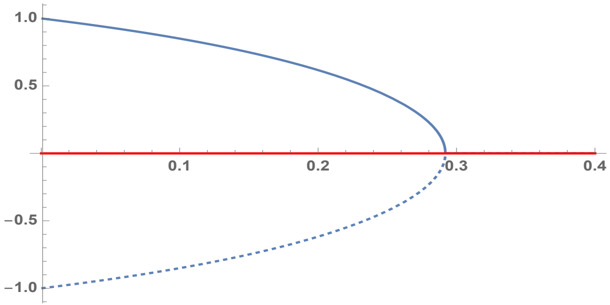
Stationary solutions: phase transition in dimension $d = 1$



- $d = 1$: there exists a bifurcation point $D = D_*$ such that the only stationary solution corresponds to $\mathbf{u}_f = 0$ if $D > D_*$ and there are three solutions corresponding to $\mathbf{u}_f = 0, \pm u(D)$ if $D < D_*$
- $\mathbf{u}_f = 0$ is linearly unstable if $D < D_*$

Notation: $f_\star^{(0)}$, $f_\star^{(+)}$, $f_\star^{(-)}$

Phase transition in dimension $d = 2$



Dynamics and free energy

The *free energy*

$$\mathcal{F}[f] := D \int_{\mathbb{R}^d} f \, \log f \, dv + \int_{\mathbb{R}^d} f \, \phi \, dv - \frac{1}{2} \, |\mathbf{u}_f|^2$$

decays according to

$$\frac{d}{dt} \mathcal{F}[f(t,\cdot)] = - \int_{\mathbb{R}^d} \left| D \, \frac{\nabla_v f}{f} + \nabla_v \phi - \mathbf{u}_f \right|^2 f \, dv$$

- $d = 1$: if $\mathcal{F}[f(t=0,\cdot)] < \mathcal{F}[f_\star^{(0)}]$ and $D < D_*$, then

$$\mathcal{F}[f(t,\cdot)] - \mathcal{F}\left[f_\star^{(\pm)}\right] \le C \, e^{-\lambda \, t}$$

- $d = 1$: λ is the eigenvalue of the linearized problem at $f_\star^{(\pm)}$ in the weighted space $\mathrm{L}^2\left((f_\star^{(\pm)})^{-1}\right)$ with scalar product

$$\langle f, g \rangle_\pm := D \int_{\mathbb{R}} f \, g \left(f_\star^{(\pm)} \right)^{-1} dv - \mathbf{u}_f \, \mathbf{u}_g$$

High and low noise regimes

$$\frac{\partial f}{\partial t} = D\,\Delta f + \nabla \cdot \Big((v - \mathbf{u}_f)\, f + \alpha\, v\, \big(|v|^2 - 1\big)\, f \Big)$$

Here $t \geq 0$ denotes the time variable, $v \in \mathbb{R}^d$ is the velocity variable

$$\mathbf{u}_f(t) = \frac{\int_{\mathbb{R}^d} v\, f(t,v)\, dv}{\int_{\mathbb{R}^d} f(t,v)\, dv} \quad \text{is the } \textit{mean velocity}$$

[J. Tugaut], [A. Barbaro, J. Canizo, J. Carrillo, P. Degond]

Theorem (X. Li)

Let $d \geq 1$ and $\alpha > 0$. There exists a critical $D_* > 0$ such that
(i) $D > D_*$: only one stable stationary distribution with $\mathbf{u}_f = \mathbf{0}$
(ii) $D < D_*$: one instable isotropic stationary distribution with $\mathbf{u}_f = \mathbf{0}$ and a continuum of stable non-negative non-symmetric polarized stationary distributions (unique up to a rotation)

Relative entropy and related quantities

- *Free energy*

$$\mathcal{F}[f] := D \int_{\mathbb{R}^d} f \, \log f \, dv + \int_{\mathbb{R}^d} f \, \phi_\alpha \, dv - \frac{1}{2} \, |\mathbf{u}_f|^2$$

- *Relative entropy* with respect to a stationary solution $f_{\mathbf{u}}$

$$\mathcal{F}[f] - \mathcal{F}[f_{\mathbf{u}}] = D \int_{\mathbb{R}^d} f \, \log \left(\frac{f}{f_{\mathbf{u}}} \right) dv - \frac{1}{2} \, |\mathbf{u}_f - \mathbf{u}|^2$$

- *Relative Fisher information*

$$\mathcal{I}[f] := \int_{\mathbb{R}^d} \left| D \, \frac{\nabla f}{f} + \alpha \, v \, |v|^2 + (1-\alpha) \, v - \mathbf{u}_f \right|^2 f \, dv$$

- The *local Gibbs state*

$$G_f(v) := \frac{e^{-\frac{1}{D} \left(\frac{1}{2} \, |v - \mathbf{u}_f|^2 + \frac{\alpha}{4} \, |v|^4 - \frac{\alpha}{2} \, |v|^2 \right)}}{\int_{\mathbb{R}^d} e^{-\frac{1}{D} \left(\frac{1}{2} \, |v - \mathbf{u}_f|^2 + \frac{\alpha}{4} \, |v|^4 - \frac{\alpha}{2} \, |v|^2 \right)} \, dv}$$

is an equilibrium iff $G_f = f$

Gibbs state *versus* stationary solution

$\mathcal{F}[f]$ is a Lyapunov function in the sense that

$$\frac{d}{dt}\mathcal{F}[f(t,\cdot)] = -\mathcal{I}[f(t,\cdot)]$$

where $\mathcal{F}[f] - \mathcal{F}[f_{\mathbf{u}}] = D \int_{\mathbb{R}^d} f \, \log\left(\frac{f}{f_{\mathbf{u}}}\right) dv - \frac{1}{2}\,|\mathbf{u}_f - \mathbf{u}|^2$ and

$$\mathcal{I}[f] = D^2 \int_{\mathbb{R}^d} \left|\nabla \log\left(\frac{f}{G_f}\right)\right|^2 f\,dv$$

$\frac{d}{dt}\mathcal{F}[f(t,\cdot)] = 0$ if and only if $f = G_f$ is a stationary solution

Stability and coercivity

$$Q_{1,\mathbf{u}}[g] := \lim_{\varepsilon\to 0} \frac{2}{\varepsilon^2}\,\mathcal{F}\big[f_\mathbf{u}(1+\varepsilon\,g)\big] = D\int_{\mathbb{R}^d} g^2\,f_\mathbf{u}\,dv - D^2\,|\mathbf{v}_g|^2$$

where $\mathbf{v}_g := \frac{1}{D}\int_{\mathbb{R}^d} v\,g\,f_\mathbf{u}\,dv$

$$Q_{2,\mathbf{u}}[g] := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\,\mathcal{I}\big[f_\mathbf{u}\,(1+\varepsilon\,g)\big] = D^2\int_{\mathbb{R}^d} |\nabla g - \mathbf{v}_g|^2\,f_\mathbf{u}\,dv$$

Stability: $Q_{1,\mathbf{u}} \ge 0$ *?*

Coercivity: $Q_{2,\mathbf{u}} \ge \lambda\,Q_{1,\mathbf{u}}$ *for some* $\lambda > 0$ *?*

Stability of the isotropic stationary solution

$$Q_{1,\mathbf{0}}[g] = D \int_{\mathbb{R}^d} g^2 \, f_{\mathbf{0}} \, dv - D^2 \, |\mathbf{v}_g|^2$$

We consider the space of the functions $g \in \mathrm{L}^2(f_{\mathbf{0}} \, dv)$ such that

$$\int_{\mathbb{R}^d} g \, f_{\mathbf{0}} \, dv = 0$$

Lemma (X. Li)

$Q_{1,\mathbf{0}}$ is a nonnegative quadratic form if and only if $D \geq D_*$ and

$$Q_{1,\mathbf{0}}[g] \geq \eta(D) \int_{\mathbb{R}^d} g^2 \, f_{\mathbf{0}} \, dv$$

for some explicit $\eta(D) > 0$ if $D > D_*$

Stability of the polarized stationary solution

Corollary (X. Li)

$\mathcal{F}$ has a unique nonnegative minimizer with unit mass, $f_\mathbf{0}$, if $D \geq D_*$. Otherwise, if $D < D_*$, we have

$$\min \mathcal{F}[f] = \mathcal{F}[f_\mathbf{u}] < \mathcal{F}[f_\mathbf{0}]$$

for any $u \in \mathbb{R}^d$ such that $|\mathbf{u}| = u(D)$.

The minimum is taken on $\mathrm{L}^1_+\big(\mathbb{R}^d, (1+|v|^4)\,dv\big)$ such that $\int_{\mathbb{R}^d} f\,dv = 1$

Corollary (X. Li)

Let $D < D_*$, $|\mathbf{u}| = u(D) \neq 0$. Then

$$Q_{1,\mathbf{u}}[g] \geq 0$$

Hint: $f_\mathbf{u}$ minimizes the free energy

A coercivity result

Poincaré inequality: if $\int_{\mathbb{R}^d} h\, f_{\mathbf{u}}\, dv = 0$

$$\int_{\mathbb{R}^d} |\nabla h|^2\, f_{\mathbf{u}}\, dv \geq \Lambda_D \int_{\mathbb{R}^d} |h|^2\, f_{\mathbf{u}}\, dv$$

Let $f \in \mathrm{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} f\, dv = 1$, $g = (f - f_{\mathbf{u}})/f_{\mathbf{u}}$ and let $\mathbf{u}[f] = \frac{u(D)}{|\mathbf{u}_f|}\, \mathbf{u}_f$ if $D < D_*$ and $\mathbf{u}_f \neq \mathbf{0}$. Otherwise take $\mathbf{u}[f] = \mathbf{0}$

Proposition (X. Li)

Let $d \geq 1$, $\alpha > 0$, $D > 0$. If $\mathbf{u} = \mathbf{0}$, then

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D\, Q_{1,\mathbf{u}}[g]$$

Otherwise, if $|\mathbf{u}| = u(D) \neq 0$ for some $D \in (0, D_*)$, then

$$Q_{2,\mathbf{u}}[g] \geq \mathcal{C}_D\, \big(1 - \kappa(D)\big)\, \frac{(\mathbf{v}_g \cdot \mathbf{u})^2}{|\mathbf{v}_g|^2\, |\mathbf{u}|^2}\, Q_{1,\mathbf{u}}[g]$$

High noise: convergence to the isotropic solution

Theorem (X. Li)

For any $d \geq 1$ and any $\alpha > 0$, if $D > D_*$, then for any solution f with nonnegative initial datum $f_{\rm in}$ of mass 1 such that $\mathcal{F}[f_{\rm in}] < \infty$, there is a positive constant C such that, for any time $t > 0$,

$$0 \leq \mathcal{F}[f(t,\cdot)] - \mathcal{F}[f_{\mathbf{0}}] \leq C\, e^{-\mathcal{C}_D\, t}$$

Non-local scalar product and linearized evolution operator

In terms of $f = f_0\,(1+g)$ the evolution equation is

$$f_0\,\frac{\partial g}{\partial t} = D\,\nabla\cdot\Big((\nabla g - \mathbf{v}_g)\,f_0 - \mathbf{v}_g\,g\,f_0\Big)$$

with $\mathbf{v}_g = \frac{1}{D}\int_{\mathbb{R}^d} v\,g\,f_0\,dv$, $Q_{1,\mathbf{0}}[g] = \langle g, g\rangle = D\int_{\mathbb{R}^d} g\,g\,f_0\,dv - D^2\,\mathbf{v}_g\cdot\mathbf{v}_g$
$\langle\cdot,\cdot\rangle$ is a *scalar product* on $\mathcal{X} := \big\{g\in \mathrm{L}^2(f_0\,dv)\ :\ \int_{\mathbb{R}^d} g\,f_0\,dv = 0\big\}$

$$\frac{\partial g}{\partial t} = \mathcal{L}\,g - \mathbf{v}_g\cdot\Big(D\,\nabla g - (v+\nabla\phi_\alpha)\,g\Big)\,,\quad \mathcal{L}\,g := D\,\Delta g - (v+\nabla\phi_\alpha)\cdot(\nabla g - \mathbf{v}_g)$$

Lemma (X. Li)

Assume that $D > D_*$ and $\alpha > 0$. The norm $g\mapsto\sqrt{\langle g, g\rangle}$ is equivalent to the standard norm on $\mathrm{L}^2(f_0\,dv)$
The linearized operator $\mathcal{L}$ is self-adjoint on $\mathcal{X}$ and

$$-\,\langle g, \mathcal{L}\,g\rangle = Q_{2,\mathbf{0}}[g]$$

The scalar product $\langle \cdot, \cdot \rangle$ is well adapted to the linearized evolution operator in the sense that a solution of the *linearized equation*

$$\frac{\partial g}{\partial t} = \mathcal{L}\, g$$

with initial datum $g_0 \in \mathcal{X}$ is such that

$$\frac{1}{2}\,\frac{d}{dt} Q_{1,\mathbf{0}}[g] = \frac{1}{2}\,\frac{d}{dt} \langle g, g \rangle = \langle g, \mathcal{L}\, g \rangle = -\, Q_{2,\mathbf{0}}[g]$$

and has exponential decay. We know that

$$\langle g(t,\cdot), g(t,\cdot) \rangle = \langle g_0, g_0 \rangle \, e^{-2\,\mathcal{C}_D\, t} \quad \forall\, t \geq 0$$

Low noise and partially symmetric solutions

Proposition (X. Li)

Let $\alpha > 0$, $D > 0$ and consider a solution $f \in C^0\left(\mathbb{R}^+, \mathrm{L}^1(\mathbb{R}^d)\right)$ with initial datum $f_{\rm in} \in \mathrm{L}^1_+(\mathbb{R}^d)$ such that $\mathcal{F}[f_{\rm in}] < \mathcal{F}[f_{\mathbf{0}}]$ and $\mathbf{u}_{f_{\rm in}} = (u, 0 \dots 0)$ for some $u \neq 0$. We further assume that $f_{\rm in}(v_1, v_2, \dots v_{i-1}, v_i, \dots) = f_{\rm in}(v_1, v_2, \dots v_{i-1}, -v_i, \dots)$ for any $i = 2$, $3, \dots d$. Then

$$\limsup_{t \to +\infty} e^{+\lambda t} \|f(t,\cdot) - f_{\mathbf{u}}\|_{\mathrm{L}^1(\mathbb{R}^d)} < \infty$$

$$0 \le \mathcal{F}[f(t,\cdot)] - \mathcal{F}[f_{\mathbf{u}}] \le C\, e^{-\lambda t} \quad \forall\, t \ge 0$$

holds with $\lambda = \mathcal{C}_D \left(1 - \kappa(D)\right) > 0$

Without symmetry assumption, the question of the rate of convergence to a solution / to the set of polarized solutions is still open

Moving planes and eigenvalues

(Old) joint work with P. Felmer

The theorem of Gidas, Ni and Nirenberg

Theorem

[Gidas, Ni and Nirenberg, 1979 and 1980] Let $u \in C^2(B)$, $B = B(0,1) \subset \mathbb{R}^d$, *be a solution of*

$$\Delta u + f(u) = 0 \text{ in } B\,, \quad u = 0 \text{ on } \partial B$$

and assume that f *is Lipschitz. If* u *is positive, then it is radially symmetric and decreasing along any radius:* $u'(r) < 0$ *for any* $r \in (0,1]$

Extension: $\Delta u + f(r,u) = 0$, $r = |x|$ if $\frac{\partial f}{\partial r} \leq 0$... a "cooperative" case

An extension of the theorem of Gidas, Ni and Nirenberg

Theorem (JD, P. Felmer, 1999)

$$\Delta u + \lambda\, f(r,u) = 0 \text{ in } B\,, \quad u = 0 \text{ on } \partial B$$

and assume that $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ *(no assumption on the sign of* $\frac{\partial f}{\partial r}$*)*

There exists λ_1, λ_2 *with* $0 < \lambda_1 \le \lambda_2$ *such that*

(i) Monotonicity: if $\lambda \in (0, \lambda_1)$, *then* $\frac{d}{dr}(u - \lambda\, u_0) < 0$ *where* u_0 *is the solution of* $\Delta u_0 + \lambda f(r,0) = 0$

(ii) Symmetry: if $\lambda \in (0, \lambda_2)$, *then* u *is radially symmetric*

Symmetry and symmetry breaking in interpolation inequalities *without magnetic field*

- Gagliardo-Nirenberg-Sobolev inequalities on the sphere
- [Keller-Lieb-Thirring inequalities on the sphere]
- Caffarelli-Kohn-Nirenberg inequalities

Joint work with M.J. Esteban, M. Loss, M. Kowalczyk,...

A result of uniqueness on a classical example

On the sphere $\mathbb{S}^d$, let us consider the positive solutions of

$$-\Delta u + \lambda\, u = u^{p-1}$$

$p \in [1,2) \cup (2,2^*]$ if $d \geq 3$, $2^* = \frac{2\,d}{d-2}$

$p \in [1,2) \cup (2,+\infty)$ if $d = 1,\, 2$

Theorem

If $\lambda \leq d$, $u \equiv \lambda^{1/(p-2)}$ is the unique solution

[Gidas & Spruck, 1981], [Bidaut-Véron & Véron, 1991]

Bifurcation point of view and symmetry breaking

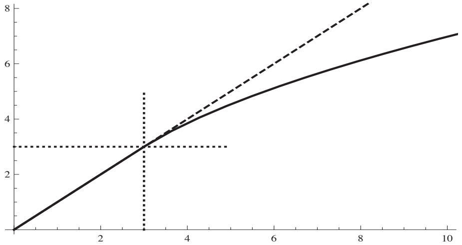


Figure: $(p-2)\,\lambda \mapsto (p-2)\,\mu(\lambda)$ *with* $d=3$

$$\boxed{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > \tfrac{d}{p-2}$$

▷ The inequality holds with $\mu(\lambda) = \lambda = \frac{d}{p-2}$ [Bakry & Emery, 1985] [Beckner, 1993], [Bidaut-Véron & Véron, 1991, Corollary 6.1]

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2} \left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}} \, d\mu - \left(\int_{\mathbb{S}^d} \rho \, d\mu \right)^{\frac{2}{p}} \right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log \left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}} \right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\mu$$

[Bakry & Emery, 1985] *carré du champ* method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and observe that $\frac{d}{dt}\mathcal{E}_p[\rho] = -\mathcal{I}_p[\rho]$

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

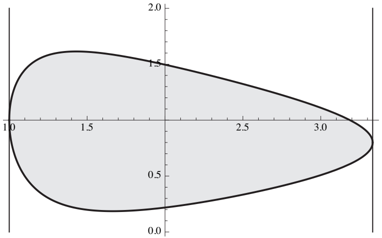
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^{\#}$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0$$



(p, m) *admissible region*, $d = 5$

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- JD, M.J. Esteban, and M. Loss. Interpolation inequalities on the sphere: linear *versus* nonlinear flows. Annales de la faculté des sciences de Toulouse Sér. 6, 26 (2): 351-379, 2017
- JD, M.J. Esteban. Improved interpolation inequalities and stability. Preprint, 2019 arXiv:1908.08235

Optimal inequalities

With $\mu(\lambda) = \lambda = \frac{d}{p-2}$: [Bakry & Emery, 1985] [Beckner, 1993], [Bidaut-Véron & Véron, 1991, Corollary 6.1]

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

- $d \geq 3$, $p \in [1,2)$ or $p \in (2, \frac{2\,d}{d-2})$
- $d = 1$ or $d = 2$, $p \in [1,2)$ or $p \in (2,\infty)$
- $p = -2 = 2\,d/(d-2) = -2$ with $d = 1$ [Exner, Harrell, Loss, 1998]

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{4}\left(\int_{\mathbb{S}^1} \frac{1}{u^2}\,d\mu\right)^{-1} \geq \frac{1}{4}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \quad \forall\, u \in \mathrm{H}^1_+(\mathbb{S}^1)$$

Caffarelli-Kohn-Nirenberg, symmetry and symmetry breaking results, and weighted nonlinear flows

Joint work with M.J. Esteban and M. Loss

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

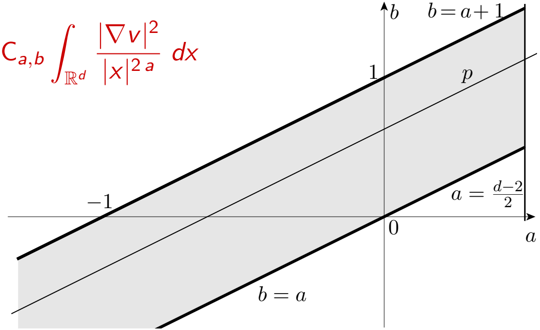
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d = 3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

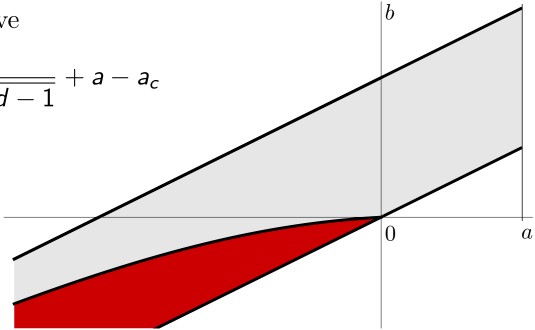
$a < a_c := (d-2)/2$
$a \le b \le a+1$ if $d \ge 3$,
$a + 1/2 < b \le a+1$ if $d = 1$
and $a < b \le a+1 < 1$,
$p = 2/(b-a)$ if $d = 2$

[Il'in (1961)]
[Glaser, Martin, Grosse, Thirring (1976)]
[Caffarelli, Kohn, Nirenberg (1984)]
[F. Catrina, Z.-Q. Wang (2001)]

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\rm FS}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c-a)^2 + d - 1}} + a - a_c$$



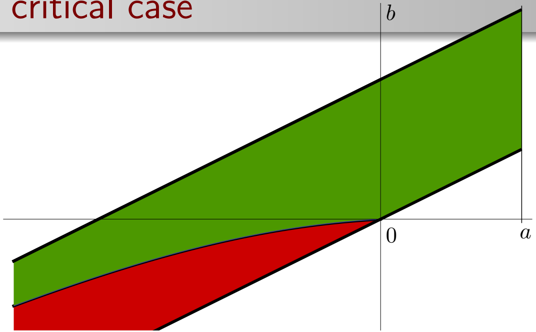
[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]

$$v \mapsto \mathsf{C}^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v = v_\star$

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (2016)]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The symmetry proof in one slide

- A change of variables: $v(|x|^{\alpha-1}\,x)=w(x)$, $\mathsf{D}_\alpha v=\left(\alpha\,\frac{\partial v}{\partial s},\frac1s\,\nabla_\omega v\right)$

$$\|v\|_{\mathrm L^{2p,d-n}(\mathbb R^d)}\le\mathsf K_{\alpha,n,p}\,\|\mathsf D_\alpha v\|_{\mathrm L^{2,d-n}(\mathbb R^d)}^\vartheta\,\|v\|_{\mathrm L^{p+1,d-n}(\mathbb R^d)}^{1-\vartheta}\quad\forall\,v\in\mathrm H^p_{d-n,d-n}(\mathbb R^d)$$

- Concavity of the Rényi entropy power: with $\mathcal L_\alpha=-\,\mathsf D_\alpha^*\,\mathsf D_\alpha=\alpha^2\left(u''+\frac{n-1}s\,u'\right)+\frac1{s^2}\,\Delta_\omega\,u$ and $\frac{\partial u}{\partial t}=\mathcal L_\alpha\,u^m$

$$\begin{aligned}
&-\tfrac{d}{dt}\,\mathcal G[u(t,\cdot)]\left(\textstyle\int_{\mathbb R^d}u^m\,d\mu\right)^{1-\sigma}\\
&\ge(1-m)\,(\sigma-1)\int_{\mathbb R^d}u^m\left|\mathcal L_\alpha\mathsf P-\frac{\int_{\mathbb R^d}u\,|\mathsf D_\alpha\mathsf P|^2\,d\mu}{\int_{\mathbb R^d}u^m\,d\mu}\right|^2d\mu\\
&+2\int_{\mathbb R^d}\left(\alpha^4\left(1-\frac1n\right)\left|\mathsf P''-\frac{\mathsf P'}s-\frac{\Delta_\omega\,\mathsf P}{\alpha^2\,(n-1)\,s^2}\right|^2+\frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf P'-\frac{\nabla_\omega\mathsf P}s\right|^2\right)u^m\,d\mu\\
&+2\int_{\mathbb R^d}\left((n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)|\nabla_\omega\mathsf P|^2+c(n,m,d)\,\frac{|\nabla_\omega\mathsf P|^4}{\mathsf P^2}\right)u^m\,d\mu
\end{aligned}$$

- Elliptic regularity and the Emden-Fowler transformation: justifying the integrations by parts

The variational problem on the cylinder

▷ *With the Emden-Fowler transformation*

$$v(r,\omega)=r^{a-a_c}\,\varphi(s,\omega)\quad\text{with}\quad r=|x|\,,\quad s=-\log r\quad\text{and}\quad \omega=\frac{x}{r}$$

the variational problem becomes

$$\Lambda\mapsto\mu(\Lambda):=\min_{\varphi\in\mathrm{H}^1(\mathcal{C})}\frac{\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}}$$

is a concave increasing function

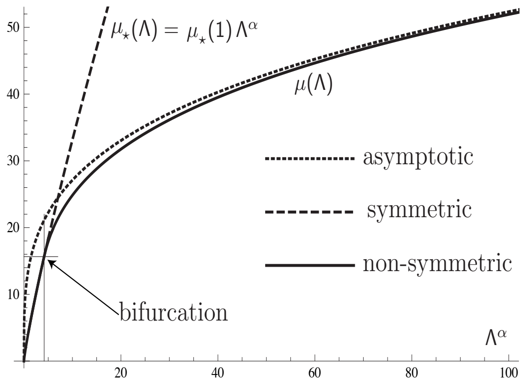
Restricted to symmetric functions, the variational problem becomes

$$\mu_\star(\Lambda):=\min_{\varphi\in\mathrm{H}^1(\mathbb{R})}\frac{\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\varphi\|^2_{\mathrm{L}^p(\mathbb{R}^d)}}=\mu_\star(1)\,\Lambda^\alpha$$

Symmetry means $\mu(\Lambda)=\mu_\star(\Lambda)$

Symmetry breaking means $\mu(\Lambda)<\mu_\star(\Lambda)$

Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point* Λ_1 *computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as shown by F. Catrina and Z.-Q. Wang*

Three references

- Lecture notes on *Symmetry and nonlinear diffusion flows*... a course on entropy methods (see webpage)
- [JD, Maria J. Esteban, and Michael Loss] *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs* ... the elliptic point of view: Proc. Int. Cong. of Math., Rio de Janeiro, 3: 2279-2304, 2018.
- [JD, Maria J. Esteban, and Michael Loss] *Interpolation inequalities, nonlinear flows, boundary terms, optimality and linearization*... the parabolic point of view
Journal of elliptic and parabolic equations, 2: 267-295, 2016.

With magnetic fields (1/3) in dimensions 2 and 3

- Interpolation inequalities and spectral estimates
- Estimates, numerics; an open question on constant magnetic fields

Magnetic interpolation inequalities in the Euclidean space

▷ Three interpolation inequalities and their dual forms

▷ Estimates in dimension $d = 2$ for constant magnetic fields

- Lower estimates
- Upper estimates and numerical results
- A linear stability result (numerical) and an open question

Assumptions are not detailed: $\mathbf{A} \in \mathrm{L}^{d+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^d)$, $\varepsilon > 0$ + integral conditions as in [Esteban, Lions, 1989]

Estimates are given (almost) only in the case $p > 2$ but similar estimates hold in the other cases

Joint work with M.J. Esteban, A. Laptev and M. Loss

Magnetic Laplacian and spectral gap

In dimensions $d = 2$ and $d = 3$: the *magnetic Laplacian* is

$$-\Delta_{\mathbf{A}}\,\psi = -\Delta\,\psi - 2\,i\,\mathbf{A}\cdot\nabla\psi + |\mathbf{A}|^2\psi - i\,(\mathrm{div}\,\mathbf{A})\,\psi$$

where the magnetic potential (resp. field) is $\mathbf{A}$ (resp. $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$) and

$$\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) := \left\{\psi \in \mathrm{L}^2(\mathbb{R}^d)\,:\,\nabla_{\mathbf{A}}\psi \in \mathrm{L}^2(\mathbb{R}^d)\right\}\,,\quad \nabla_{\mathbf{A}} := \nabla + i\,\mathbf{A}$$

Spectral gap inequality

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \Lambda[\mathbf{B}]\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$$

- Λ depends only on $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$
- Assumption: *equality holds for some* $\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$
- If $\mathbf{B}$ is a constant magnetic field, $\Lambda[\mathbf{B}] = |\mathbf{B}|$
- If $d = 2$, $\mathrm{spec}(-\Delta_{\mathbf{A}}) = \{(2j+1)\,|\mathbf{B}|\,:\,j \in \mathbb{N}\}$ is generated by the *Landau levels*. The *Lowest Landau Level* corresponds to $j = 0$

Magnetic interpolation inequalities

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mu_{\mathbf{B}}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$$

for any $\alpha \in (-\Lambda[\mathbf{B}], +\infty)$ and any $p \in (2, 2^*)$,

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \beta\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \ge \nu_{\mathbf{B}}(\beta)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$$

for any $\beta \in (0, +\infty)$ and any $p \in (1, 2)$

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \gamma \int_{\mathbb{R}^d} |\psi|^2\,\log\left(\frac{|\psi|^2}{\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right) dx + \xi_{\mathbf{B}}(\gamma)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

(limit case corresponding to $p = 2$) for any $\gamma \in (0, +\infty)$

$$\mathsf{C}_p := \begin{cases} \min_{u \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \dfrac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}} & \text{if} \quad p \in (2, 2^*) \\ \min_{u \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \dfrac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} & \text{if} \quad p \in (1, 2) \end{cases}$$

$\mu_{\mathbf{0}}(1) = \mathsf{C}_p$ if $p \in (2, 2^*)$, $\nu_{\mathbf{0}}(1) = \mathsf{C}_p$ if $p \in (1, 2)$
$\xi_{\mathbf{0}}(\gamma) = \gamma\,\log\left(\pi\,e^2/\gamma\right)$ if $p = 2$

A statement

Theorem

$p \in (2, 2^*)$: $\mu_{\mathbf{B}}$ is monotone increasing on $(-\Lambda[\mathbf{B}], +\infty)$, concave and

$$\lim_{\alpha \to (-\Lambda[\mathbf{B}])_+} \mu_{\mathbf{B}}(\alpha) = 0 \quad \textit{and} \quad \lim_{\alpha \to +\infty} \mu_{\mathbf{B}}(\alpha)\, \alpha^{\frac{d-2}{2} - \frac{d}{p}} = \mathsf{C}_p$$

$p \in (1, 2)$: $\nu_{\mathbf{B}}$ is monotone increasing on $(0, +\infty)$, concave and

$$\lim_{\beta \to 0_+} \nu_{\mathbf{B}}(\beta) = \Lambda[\mathbf{B}] \quad \textit{and} \quad \lim_{\beta \to +\infty} \nu_{\mathbf{B}}(\beta)\, \beta^{-\frac{2\,p}{2\,p + d\,(2-p)}} = \mathsf{C}_p$$

$\xi_{\mathbf{B}}$ is continuous on $(0, +\infty)$, concave, $\xi_{\mathbf{B}}(0) = \Lambda[\mathbf{B}]$ and

$$\xi_{\mathbf{B}}(\gamma) = \frac{d}{2}\, \gamma \log\left(\frac{\pi\, e^2}{\gamma}\right)(1 + o(1)) \quad \textit{as} \quad \gamma \to +\infty$$

Constant magnetic fields: equality is achieved
Nonconstant magnetic fields: only partial answers are known

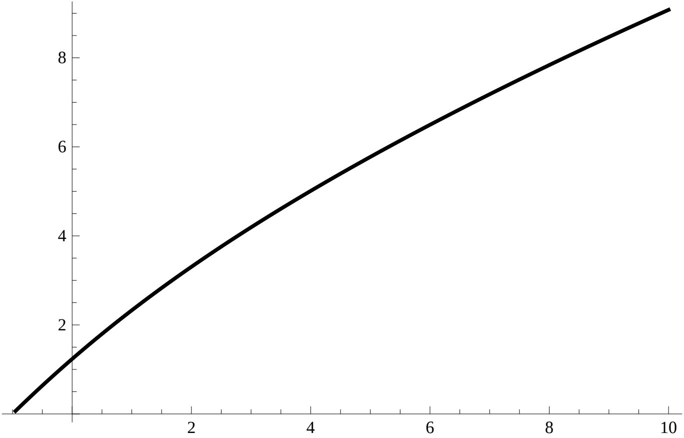


Figure: Case $d = 2$, $p = 3$, $B = 1$: plot of $\alpha \mapsto (2\,\pi)^{\frac{2}{p}-1}\,\mu_{\mathbf{B}}(\alpha)$

Numerical results and the symmetry issue

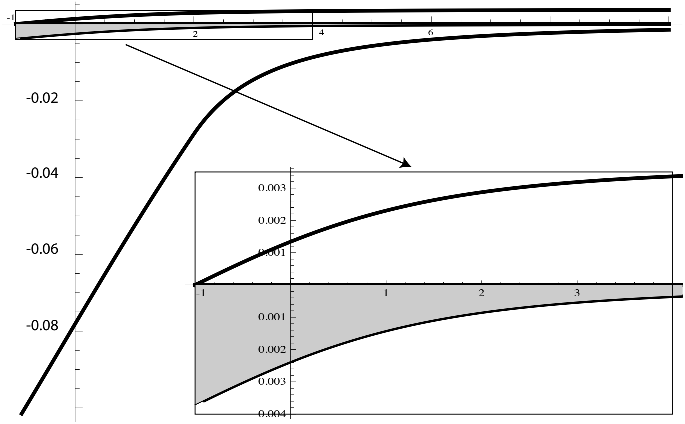


Figure: Case $d = 2$, $p = 3$, $B = 1$
Upper estimates: $\alpha \mapsto \mu_{\text{Gauss}}(\alpha)$, $\mu_{\text{EL}}(\alpha)$
Lower estimates: $\alpha \mapsto \mu_{\text{interp}}(\alpha)$, $\mu_{\text{LT}}(\alpha)$
The exact value associated with $\mu_{\mathbf{B}}$ lies in the grey area.
Plots represent the curves $\log_{10}(\mu/\mu_{\text{EL}})$

An open question of symmetry

- [Bonheure, Nys, Van Schaftingen, 2016] for a fixed $\alpha > 0$ and for $\mathbf{B}$ small enough, the optimal functions are radially symmetric functions, *i.e.*, belong to $\mathcal{C}_0$
This regime is equivalent to the regime as $\alpha \to +\infty$ for a given $\mathbf{B}$, at least if the magnetic field is constant
- Numerically our upper and lower bounds are (in dimension $d = 2$, for a constant magnetic field) numerically extremely close
- The optimal function in $\mathcal{C}_0$ is linearly stable with respect to perturbations in $\mathcal{C}_1$
- A reference: JD, M.J. Esteban, A. Laptev, M. Loss. Interpolation inequalities and spectral estimates for magnetic operators. Annales Henri Poincaré, 19 (5): 1439-1463, May 2018

▷ *Prove that the optimality case is achieved among radial function if* $d = 2$ *and* $\mathbf{B}$ *is a constant magnetic field*

With magnetic fields (2/3)

Magnetic rings: the case of $\mathbb{S}^1$

▷ A magnetic interpolation inequality on $\mathbb{S}^1$: with $p > 2$

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \geq \mu_{a,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

▷ Consequences

- [A Keller-Lieb-Thirring inequality]
- A new Hardy inequality for Aharonov-Bohm magnetic fields in $\mathbb{R}^2$

Joint work with M.J. Esteban, A. Laptev and M. Loss

Magnetic flux, a reduction

Assume that $a : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function such that its restriction to $(-\pi, \pi] \approx \mathbb{S}^1$ is in $\mathrm{L}^1(\mathbb{S}^1)$ and define the space

$$X_a := \left\{ \psi \in C_{\mathrm{per}}(\mathbb{R}) \,:\, \psi' + i\, a\, \psi \in \mathrm{L}^2(\mathbb{S}^1) \right\}$$

- A standard change of gauge (see *e.g.* [Ilyin, Laptev, Loss, Zelik, 2016])

$$\psi(s) \mapsto e^{i \int_{-\pi}^{s} (a(s) - \bar{a})\, \mathrm{d}\sigma}\, \psi(s)$$

where $\bar{a} := \int_{-\pi}^{\pi} a(s)\, \mathrm{d}\sigma$ is the *magnetic flux*, reduces the problem to

$$a \textit{ is a constant function}$$

- For any $k \in \mathbb{Z}$, ψ by $s \mapsto e^{iks}\, \psi(s)$ shows that $\mu_{a,p}(\alpha) = \mu_{k+a,p}(\alpha)$

$$a \in [0, 1]$$

- $\mu_{a,p}(\alpha) = \mu_{1-a,p}(\alpha)$ because $|\psi' + i\, a\, \psi|^2 = |\chi' + i\,(1-a)\,\chi|^2 = \left|\overline{\psi}' - i\, a\, \overline{\psi}\right|^2$ if $\chi(s) = e^{-is}\, \overline{\psi(s)}$

$$a \in [0, 1/2]$$

Optimal interpolation

We want to characterize the *optimal constant* in the inequality

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \mu_{a,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

written for any $p > 2$, $a \in (0, 1/2]$, $\alpha \in (-a^2, +\infty)$, $\psi \in X_a$

$$\mu_{a,p}(\alpha) := \inf_{\psi \in X_a \setminus \{0\}} \frac{\int_{-\pi}^{\pi} \left(|\psi' + i\,a\,\psi|^2 + \alpha\,|\psi|^2\right) d\sigma}{\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

$p = -2 = 2\,d/(d-2)$ with $d = 1$ [Exner, Harrell, Loss, 1998]
$p = +\infty$ [Galunov, Olienik, 1995] [Ilyin, Laptev, Loss, Zelik, 2016]
$\lim_{\alpha \to -a^2} \mu_{a,p}(\alpha) = 0$ [JD, Esteban, Laptev, Loss, 2016]

Using a Fourier series $\psi(s) = \sum_{k\in\mathbb{Z}} \psi_k\,e^{iks}$, we obtain that

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \sum_{k\in\mathbb{Z}} (a+k)^2\,|\psi_k|^2 \ge a^2\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

$\psi \mapsto \|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$ is coercive for any $\alpha > -a^2$

An interpolation result for the magnetic ring

Theorem

For any $p > 2$, $a \in \mathbb{R}$, and $\alpha > -a^2$, $\mu_{a,p}(\alpha)$ is achieved and
(i) if $a \in [0, 1/2]$ and $a^2\,(p+2) + \alpha\,(p-2) \le 1$, then $\mu_{a,p}(\alpha) = a^2 + \alpha$ and equality is achieved only by the constant functions
(ii) if $a \in [0, 1/2]$ and $a^2\,(p+2) + \alpha\,(p-2) > 1$, then $\mu_{a,p}(\alpha) < a^2 + \alpha$ and equality is not achieved among the constant functions
If $\alpha > -a^2$, $a \mapsto \mu_{a,p}(\alpha)$ is monotone increasing on $(0, 1/2)$

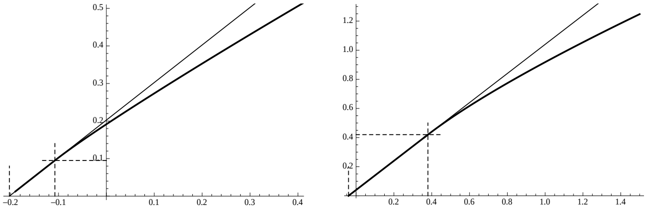


Figure: $\alpha \mapsto \mu_{a,p}(\alpha)$ with $p = 4$ and (left) $a = 0.45$ or (right) $a = 0.2$

Elimination of the phase

Let us define

$$\mathcal{Q}_{a,p,\alpha}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

Lemma

For any $a \in (0,1/2)$, $p > 2$, $\alpha > -\,a^2$,

$$\mu_{a,p}(\alpha) = \min_{u \in \mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\alpha}[u]$$

is achieved by a function $u > 0$

A new Hardy inequality

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2\,dx \ge \tau \int_{\mathbb{R}^2} \frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\,|\Psi|^2\,\mathrm{d}\mathbf{x} \quad \forall\,\varphi \in \mathrm{L}^q(\mathbb{S}^1)\,, \quad q \in (1,+\infty)$$

Corollary

Let $p > 2$, $a \in [0, 1/2]$, $q = p/(p-2)$ and assume that φ is a non-negative function in $\mathrm{L}^q(\mathbb{S}^1)$. Then the inequality holds with $\tau > 0$ given by

$$\alpha_{a,p}\left(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\right) = 0$$

Moreover, $\tau = a^2/\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$ if $4\,a^2 + \|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2) \le 1$

For any $a \in (0, 1/2)$, by taking φ constant, small enough in order that $4\,a^2 + \|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2) \le 1$, we recover the inequality

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2\,\mathrm{d}\mathbf{x} \ge a^2 \int_{\mathbb{R}^2} \frac{|\Psi|^2}{|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}$$

[Laptev, Weidl, 1999] constant magnetic fields; [Hoffmann-Ostenhof, Laptev, 2015] in $\mathbb{R}^d$, $d \ge 3$

With magnetic fields (3/3)

Aharonov-Bohm magnetic fields in $\mathbb{R}^2$

- Aharonov-Bohm effect
- [Interpolation and Keller-Lieb-Thirring inequalities in $\mathbb{R}^2$]
- Aharonov-Symmetry and symmetry breaking

Joint work with D. Bonheure, M.J. Esteban, A. Laptev, & M. Loss

Aharonov-Bohm effect

A major difference between classical mechanics and quantum mechanics is that particles are described by a non-local object, the wave function. In 1959 Y. Aharonov and D. Bohm proposed a series of experiments intended to put in evidence such phenomena which are nowadays called *Aharonov-Bohm effects*

One of the proposed experiments relies on a long, thin solenoid which produces a magnetic field such that the region in which the magnetic field is non-zero can be approximated by a line in dimension $d = 3$ and by a point in dimension $d = 2$

▷ [Physics today, 2009] *"The notion, introduced 50 years ago, that electrons could be affected by electromagnetic potentials without coming in contact with actual force fields was received with a skepticism that has spawned a flourishing of experimental tests and expansions of the original idea."* Problem solved by considering appropriate weak solutions !

▷ Is the wave function a physical object or is its modulus ? Decisive experiments have been done only 20 years ago

The interpolation inequality

Let us consider an Aharonov-Bohm vector potential

$$\mathbf{A}(x)=\frac{a}{|x|^2}\,(x_2,-x_1)\;,\quad x=(x_1,x_2)\in\mathbb{R}^2\setminus\{0\}\,,\quad a\in\mathbb{R}$$

Magnetic Hardy inequality [Laptev, Weidl, 1999]

$$\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\,\psi|^2\,dx\ge\min_{k\in\mathbb{Z}}\,(a-k)^2\int_{\mathbb{R}^2}\frac{|\psi|^2}{|x|^2}\,dx$$

where $\nabla_{\mathbf{A}}\,\psi:=\nabla\psi+\,i\,\mathbf{A}\,\psi$, so that, with $\psi=|\psi|\,e^{iS}$

$$\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\,\psi|^2\,dx=\int_{\mathbb{R}^2}\left[(\partial_r\,|\psi|)^2+(\partial_r S)^2\,|\psi|^2+\frac{1}{r^2}\,(\partial_\theta S+A)^2\,|\psi|^2\right]dx$$

Magnetic interpolation inequality

$$\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\,\psi|^2\,dx+\lambda\int_{\mathbb{R}^2}\frac{|\psi|^2}{|x|^2}\,dx\ge\mu(\lambda)\left(\int_{\mathbb{R}^2}\frac{|\psi|^p}{|x|^2}\,dx\right)^{2/p}$$

▷ Symmetrization: [Erdös, 1996], [Boulenger, Lenzmann], [Lenzmann, Sok]

A magnetic Hardy-Sobolev inequality

Theorem

Let $a \in [0, 1/2]$ *and* $p > 2$. *For any* $\lambda > -a^2$, *there is an optimal, monotone increasing, concave function* $\lambda \mapsto \mu(\lambda)$ *which is such that*

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 \, dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2} \, dx \geq \mu(\lambda) \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2} \, dx \right)^{2/p}$$

If $\lambda \leq \lambda_\star = 4 \, \frac{1-4\,a^2}{p^2-4} - a^2$ *equality is achieved by*

$$\psi(x) = \left(|x|^\alpha + |x|^{-\alpha} \right)^{-\frac{2}{p-2}} \quad \forall \, x \in \mathbb{R}^2 \,, \quad \textit{with} \quad \alpha = \frac{p-2}{2} \sqrt{\lambda + a^2}$$

If $\lambda > \lambda_\bullet$ *with*

$$\lambda_\bullet := \frac{8 \left(\sqrt{p^4 - a^2 \, (p-2)^2 \, (p+2) \, (3\,p-2)} + 2 \right) - 4\,p\,(p+4)}{(p-2)^3 \, (p+2)} - a^2$$

there is symmetry breaking: optimal functions are not radially symmetric

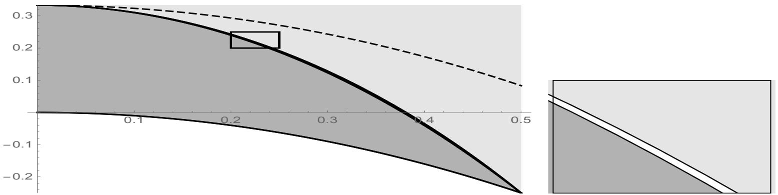


Figure: Case $p = 4$
Symmetry breaking region: $\lambda > \lambda_\bullet(a)$
Symmetry breaking region: $\lambda < \lambda_\star$

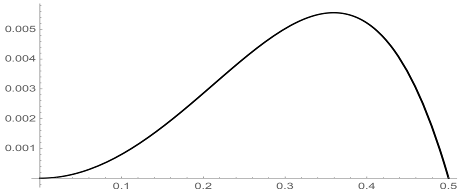


Figure: The curve $a \mapsto \lambda_\bullet(a) - \lambda_\star(a)$

References

- D. Bonheure, J. Dolbeault, M.J. Esteban, A. Laptev, M. Loss. *Inequalities involving Aharonov-Bohm magnetic potentials in dimensions 2 and 3.* Preprint arXiv:1902.06454
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