Optimal constants
Optimal rates

A variational approach, with applications to nonlinear diffusions

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A BRIEF REVIEW OF THE LITERATURE

1. Comparison techniques
[Friedman-Kamin, 1980]
[Kamin-Vazquez,1988],...

2. Entropy-Entropy production methods
[Bakry], [Emery], [Ledoux], [Coulhon],...
• *linear diffusions*
[Toscani]
[Arnold-Markowich-Toscani-Unterreiter]
• *nonlinear diffusions*
[Carillo-Toscani], [Del Pino, JD], [Otto]
[Juengel-Markowich-Toscani]
[Carillo-Juengel-Markowich-Toscani-Unterreiter]
+ *applications* (...)

3. Variational approach
[Gross]
[Aubin], [Talenti]
[Weissler]
[Carlen], [Carlen-Loss]
[Beckner],...
[Del Pino, JD]

"GROUND STATES"

A 3 steps strategy for the study of *nonlinear (local) scalar fields equations.* The original goal is to identify all the **minimizers** of an energy functional:

1. Get *a priori* **qualitative properties**: positivity, decay at infinity, regularity, Euler-Lagrange equations [1960→] Standard results are easy

2. Characterize the symmetry of the **solutions** [Serrin, Gidas-Ni-Nirenberg, Serrin ~1979→] using moving planes techniques: [JD-Felmer, 1999], [Damascelli-Pacella-Ramaswamy, 2000→] or elaborate symmetrization methods: [Brock, 2001]

3. Prove the **uniqueness** of the solutions of an ODE (radial solutions): [Coffman, 1972], [Peletier-Serrin, 1986], (...) [Erbe-Tang, 1997], [Pucci-Serrin, 1998], [Serrin-Tang, 2000]

"Ground states" is now used in the literature to qualify postive solutions dacaying to zero at ∞.

Entropy and optimal constants in Sobolev type inequalities

Theorem 1 $d \geq 2$, $\frac{2d+1}{d+1} \leq p < d$, $q = 2 - \frac{1}{p-1}$

$$u_t = \Delta_p u \quad \textit{in} \quad \mathbb{R}^d \tag{1}$$

initial data: $u_0 \geq 0$ *in* $L^1 \cap L^\infty$, $|x|^{p/(p-1)} u_0 \in L^1$
$u_0^q \in L^1$ *in case* $p < 2$

$\forall\, s > 1$ $\exists K > 0$ $\forall\, t > 0$

$$\|u(t,\cdot) - u_\infty(t,\cdot)\|_s \leq K\, R^{-(\frac{\alpha}{2}\frac{q}{s} + d(1-\frac{1}{s}))} \quad \textit{if}\ p \geq 2\,,\ s \geq q$$

$$\|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_s \leq K\, R^{-(\frac{\alpha}{2qs} + d(q-\frac{1}{s}))} \quad \textit{if}\ p \leq 2\,,\ s \geq \frac{1}{q}$$

$$\alpha = \left(1 - \frac{1}{p}(p-1)^{\frac{p-1}{p}}\right) \frac{p}{p-1}\,, \qquad q = 2 - \frac{1}{p-1}$$

$R = R(t) = (1 + \gamma\, t)^{1/\gamma}$, $\gamma = (d+1)p - 2d$,

$$u_\infty(t,x) = R^{-d}\, v_\infty(\log R, R^{-1}x)$$

$v_\infty(x) = \left(C - \frac{p-2}{p}\, |x|^{\frac{p}{p-1}}\right)_+^{1/(q-1)}$ *if* $p \neq 2$

$v_\infty(x) = C\, e^{-|x|^2/2}$ *if* $p = 2$.

Comments

$p > 1$: $\Delta_p w = \nabla \cdot (|\nabla w|^{p-2} \nabla w)$
$\|v\|_c = (\int |v|^c \, dx)^{1/c}$ for any $c > 0$.
Existence results, uniform convergence for large time: [diBenedetto, 1993].

Time-dependent rescaling
$u(t,x) = R^{-d} \, v(\log R, \frac{x}{R})$

$$v_t = \Delta_p v + \nabla \cdot (x \, v) \tag{2}$$

with initial data u_0

$$v_\infty(x) = (C - \frac{p-2}{p} \, |x|^{\frac{p}{p-1}})_+^{1/(q-1)} \quad \text{if} \quad p \neq 2$$

is a stationary, nonnegative radial and nonincreasing solution of (2).
C is uniquely determined by the condition $M = \|v_\infty\|_1$ and can be explicitely computed

For $q > 0$, define the *entropy* by

$$\Sigma[v] = \int \Big[\sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty) \Big] \; dx$$

$\sigma(s) = \frac{s^q - 1}{q-1}$ if $q \neq 1$
$\sigma(s) = s \log s$ if $q = 1$ ($p = 2$)
$\frac{d\Sigma}{dt} = -q\,(I_1 + I_2 + I_3 + I_4)$ where

$$I_1 = \int v^{-\frac{1}{p-1}} \, |\nabla v|^p \, dx \quad I_2 = \int |x|^{\frac{p}{p-1}} v \, dx$$
$$I_3 = -\frac{d}{q} \int v^q \, dx \quad I_4 = \int |\nabla v|^{p-2} \nabla v \cdot |x|^{\frac{1}{p-1} - 1} x \, dx$$

Heat equation ($p = 2$): [Toscani, 1997].
From now on: $p \neq 2$

Lemma 1 *Let* $\kappa_p = \frac{1}{p}\,(p-1)^{\frac{p-1}{p}}$

$$\frac{1}{q}\frac{d\Sigma}{dt} \leq -(1 - \kappa_p)(I_1 + I_2) - I_3$$

Theorem 2 *Let* $1 < p < d$, $1 < a \le \frac{p(d-1)}{d-p}$, *and* $b = p\,\frac{a-1}{p-1}$. *There exists a positive constant* $\mathcal{S}$ *such that for any function* $w \in W^{1,p}_{\mathrm{loc}}$ *with* $\|w\|_a + \|w\|_b < +\infty$, *if* $a > p$ *(I) or* $a < p$ *(II),*

$$\begin{cases} \|w\|_b \le \mathcal{S}\, \|\nabla w\|_p^\theta\, \|w\|_a^{1-\theta}, & \theta = \frac{(a-p)d}{(a-1)(dp-(d-p)a)} \quad (I) \\ \|w\|_a \le \mathcal{S}\, \|\nabla w\|_p^\theta\, \|w\|_b^{1-\theta}, & \theta = \frac{(p-a)d}{a(d(p-a)+p(a-1))} \quad (II) \end{cases}$$

and equality holds for any function taking, up to a translation, the form $\overline{w}(x) = A\,(1 + B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$ *for some* $(A, B) \in \mathbb{R}^2$, *where* B *has the sign of* $a - p$.

$b = \frac{p\,(p-1)}{p^2-p-1}$, $a = b\,q$, $v = w^b$. For $p \neq 2$, let $\mathcal{F}[v] = \int v^{-\frac{1}{p-1}} |\nabla v|^p\, dx - \frac{1}{q}\left(\frac{d}{1-\kappa_p} + \frac{p}{p-2}\right) \int v^q\, dx$.

Corollary 3 $d \ge 2$, $(2d+1)/(d+1) \le p < d$.

$$\mathcal{F}[v] \ge \mathcal{F}[v_\infty]$$

$\forall\, v$ *such that* $\|v\|_{L^1} = \|v_\infty\|_{L^1}$.

Proof. Use a scaling leaving $\|v\|_1$ invariant in the first case, $\|v\|_q$ in the second case. □

$$\Sigma \leq \frac{q}{1-\kappa_p}\,\frac{p-1}{p}\,[(1-\kappa_p)(I_1+I_2)+I_3]$$

The inequality is autonomous (the moment in $|x|^{\frac{p}{p-1}}$ is the same on both sides of the inequality).

Corollary 4 *If v is a solution of (2) corresponding to an initial data satisfying the conditions of Theorem 1, then, with $\alpha = (1-\kappa_p)\,\frac{p}{p-1}$, for any $t > 0$, $\Sigma(t) \leq e^{-\alpha\, t}\,\Sigma(0)$.*

A variant of the Csiszár-Kullback inequality [Caceres-Carrillo-JD]

Lemma 2 *Let f and g be two nonnegative functions in $L^q(\Omega)$ for a given domain Ω in $\mathbb{R}^d$. Assume that $q \in (1,2]$. Then*

$$\int_\Omega [\sigma(\frac{f}{g}) - \sigma'(1)(\frac{f}{g} - 1)]\, g^q\, dx$$

$$\geq \frac{q}{2} \max(\|f\|^{q-2}_{L^q(\Omega)}, \|g\|^{q-2}_{L^q(\Omega)})\, \|f-g\|^2_{L^q(\Omega)}\ .$$

For $q > 1$ ($p > 2$)

$$\sigma(s) = \frac{s^q - 1}{q-1}$$

Note that for $q < 1$ ($p < 2$)
$\frac{s^q-1}{q-1} - \frac{q}{q-1}(s-1)$
$= (\frac{1}{q} - 1)^{-1}[(s^q)^{\frac{1}{q}} - 1 - \frac{1}{q}((s^q) - 1)]$

Extension to other nonlinear diffusions involving the p-Laplacian

$$u_t = \Delta_p u^m \tag{3}$$

Formal in the sense that apparently a complete existence theory for such an equation is not yet available, except in the special cases $p = 2$ or $m = 1$. Let

$$q = 1 + m - (p-1)^{-1}$$

Whether q is bigger or smaller than 1 determines two different regimes like for $m = 1$ (depending if p is bigger or smaller than 2). The case $q = 1$ ($m = (p-1)^{-1}$) is a limiting case, which involves a generalized logarithmic Sobolev inequality. We shall assume that:

(H) the solution corresponding to a given non-negative initial data u_0 in $L^1 \cap L^\infty$, such that $u_0^q \in L^1$ (in case $m < \frac{1}{p-1}$) and $\int |x|^{\frac{p}{p-1}} u_0 \, dx < +\infty$, is well defined for any $t > 0$, belongs to $C^0(\mathbb{R}^+; L^1(\mathbb{R}^d, (1 + |x|^{\frac{p}{p-1}})\, dx) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$, such that u^q and $t \mapsto \int u^{-\frac{1}{p-1}} |\nabla u|^p \, dx$ belong to $C^0(\mathbb{R}^+; L^1(\mathbb{R}^d)$ and $L^1_{\mathrm{loc}}(\mathbb{R}^+)$ respectively.

Theorem 5 *Assume that* $d \geq 2$, $1 < p < d$, $\frac{d-(p-1)}{d(p-1)} \leq m \leq \frac{p}{p-1}$ *and* $q = 1 + m - \frac{1}{p-1}$. *Let* u *be a solution of (3) satisfying (H). Then there exists a constant* $K > 0$ *such that for any* $t > 0$,

$$\|u(t,\cdot)-u_\infty(t,\cdot)\|_q \leq K\, R^{-(\frac{\alpha}{2}+d(1-\frac{1}{q}))} \quad \text{if } \frac{1}{p-1} \leq m \leq \frac{p}{p-1}$$

$$\|u^q(t,\cdot)-u_\infty^q(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}} \quad \text{if } \frac{d-(p-1)}{d(p-1)} \leq m \leq \frac{1}{p-1}$$

$\alpha = (1 - \frac{1}{p}\,(p-1)^{\frac{p-1}{p}})\,\frac{p}{p-1}$
$R = R(t) = (1 + \gamma\, t)^{1/\gamma}$

$$\gamma = (md+1)(p-1) - (d-1)$$

$u_\infty(t,x) = R^{-d}\, v_\infty(\log R, R^{-1}x)$

$$v_\infty(x) = \Big(C - \frac{p-1}{mp}\,(q-1)\,|x|^{\frac{p}{p-1}}\Big)_+^{1/(q-1)}$$

$v_\infty(x) = C\, e^{-(p-1)^2|x|^{p/(p-1)}/p} \quad$ *if* $m = (p-1)^{-1}$

Use $v_t = \Delta_p v^m + \nabla \cdot (x\, v)$
$w = v^{(mp+q-(m+1))/p}$, $a = b\, q = p\,\frac{m(p-1)+p-2}{mp(p-1)-1}$.

The case $q = 1$ (which corresponds to $m = (p-1)^{-1}$) is a limiting case, for which we can use the generalization to $W^{1,p}$ of the logarithmic Sobolev inequality.

Theorem 6 *Let $1 < p < d$. Then for any $w \in W^{1,p}$, $w \neq 0$,*

$$\int |w|^p \log\left(\frac{|w|}{\|w\|_p}\right) dx + \frac{d}{p^2}\|w\|_p^p \cdot \left(1 - \log \mathcal{L}_p - \log(\frac{d}{p^2\lambda})\right) \leq \lambda \|\nabla w\|_p^p$$

with

$$\mathcal{L}_p = \frac{p}{d}\left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d\,\frac{p-1}{p}+1)}\right]^{\frac{p}{d}}$$

For $\lambda = (p-1)\,p^{p-1}$, Inequality (6) is morevover optimal with equality if and only if $w = v^{1/p}$ is equal, up to a translation and a multiplication by a constant, to $v_\infty^{1/p}$.

An optimal under scalings form:

Theorem 7 *Let us assume* $1 < p < d$. *Then for any* $u \in W^{1,p}(\mathbb{R}^d)$ *with* $\int |u|^p \, dx = 1$

$$\int |u|^p \log |u| \, dx \leq \frac{d}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx \right] \qquad (4)$$

$$\mathcal{L}_p = \frac{p}{d} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d\,\frac{p-1}{p}+1)} \right]^{\frac{p}{d}} \qquad (5)$$

Inequality (4) is optimal and equality holds if and only if for some $\sigma > 0$ *and* $\bar{x} \in \mathbb{R}^d$

$$u(x) = \pi^{-\frac{d}{2}} \sigma^{-d\,\frac{p-1}{p}} \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d\,\frac{p-1}{p}+1)} \, e^{-\frac{1}{\sigma}|x-\bar{x}|^{\frac{p}{p-1}}} \qquad (6)$$

- An inequality stronger than the corresponding Gagliardo-Nirenberg inequalities
- $p = 2$: [Gross, 1975], [Weissler, 1979]
- $p = 1$: [Beckner, 1999]

$$\int |u| \log |u| \, dx \leq d \log \left[\mathcal{L}_1 \int |\nabla u| \, dx \right]$$

Optimal constants for Gagliardo-Nirenberg ineq.
[Del Pino, J.D.]

Theorem 3 $1<p<d$, $1<a\le\frac{p(d-1)}{d-p}$, $b=p\,\frac{a-1}{p-1}$

$$\|w\|_b \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_a^{1-\theta} \qquad \textit{if}\; a>p$$
$$\|w\|_a \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_b^{1-\theta} \qquad \textit{if}\; a<p$$
$$\textit{Equality if}\; w(x)=A\left(1+B\,|x|^{\frac{p}{p-1}}\right)_+^{-\frac{p-1}{a-p}}$$

$a>p$: $\theta=\frac{(q-p)\,d}{(q-1)\,(d\,p-(d-p)\,q)}$
$a<p$: $\theta=\frac{(p-q)\,d}{q\,(d\,(p-q)+p\,(q-1))}$

Proof based on [Serrin, Tang]

A new logarithmic Sobolev inequality, with optimal constant [Del Pino, J.D.]

Theorem 4 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \leq \frac{d}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx \right]$$
$$\mathcal{L}_p = \frac{p}{d} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{d}{2}+1)}{\Gamma(d\,\frac{p-1}{p}+1)} \right]^{\frac{p}{d}}$$

Equality: $u = \pi^{-\frac{d}{2}} \sigma^{-d\,\frac{p-1}{p}} \dfrac{\Gamma(\frac{d}{2}+1)}{\Gamma(d\,\frac{p-1}{p}+1)} \, e^{-\frac{1}{\sigma}|x-\bar{x}|^{\frac{p}{p-1}}}$

$p = 2$: Gross' logaritmic Sobolev inequality
$p = 1$: [Beckner]

[Del Pino, J.D.] Intermediate asymptotics of

$$u_t = \Delta_p u^m$$

Theorem 5 $d \geq 2$, $1 < p < d$
$\frac{d-(p-1)}{d(p-1)} \leq m \leq \frac{p}{p-1}$ *and* $q = 1 + m - \frac{1}{p-1}$

(i) $\|u(t,\cdot) - u_\infty(t,\cdot)\|_q \leq K\, R^{-(\frac{\alpha}{2}+d(1-\frac{1}{q}))}$
(ii) $\|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}}$

(i): $\frac{1}{p-1} \leq m \leq \frac{p}{p-1}$, *(ii)*: $\frac{d-(p-1)}{d(p-1)} \leq m \leq \frac{1}{p-1}$
$\alpha = (1 - \frac{1}{p}\,(p-1)^{\frac{p-1}{p}})\,\frac{p}{p-1}$, $R = (1 + \gamma\, t)^{1/\gamma}$
$\gamma = (md + 1)(p-1) - (d-1)$

$$u_\infty(t,x) = \frac{1}{R^d}\, v_\infty(\log R, \frac{x}{R})$$

$v_\infty(x) = (C - \frac{p-1}{mp}\,(q-1)\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ $m \neq \frac{1}{p-1}$
$v_\infty(x) = C\, e^{-(p-1)^2|x|^{p/(p-1)}/p}$ *if* $m = (p-1)^{-1}$.