Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities

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EVOLUTION EQUATIONS AND FUNCTIONAL INEQUALITIES

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Outline

- Introduction
  - Fast diffusion equations: entropy methods and Gagliardo-Nirenberg inequalities
  - Fast diffusion equations: the finite mass regime
  - Fast diffusion equations: the infinite mass regime
- Relative entropy methods and linearization
  - the linearization of the functionals approach: [Blanchet, Bonforte, J.D., Grillo, Vázquez]
  - sharp rates: [Bonforte, J.D., Grillo,Vázquez]
  - An improvement based on the center of mass: [Bonforte, J.D., Grillo, Vázquez]
- An improvement based on the variance: [J.D., Toscani]

Some references

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- Matteo Bonforte, Jean Dolbeault, Gabriele Grillo, and Juan-Luis Vázquez. Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities, submitted to Proc. Nat. Acad. Sciences
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- A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J.-L. Vázquez. Hardy-Poincaré inequalities and applications to nonlinear diffusions. C. R. Math. Acad. Sci. Paris, 344(7): 431-436, 2007

Introduction

Fast diffusion equations: entropy methods and Gagliardo-Nirenberg inequalities

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\; t > 0$$

Porous media / fast diffusion equations

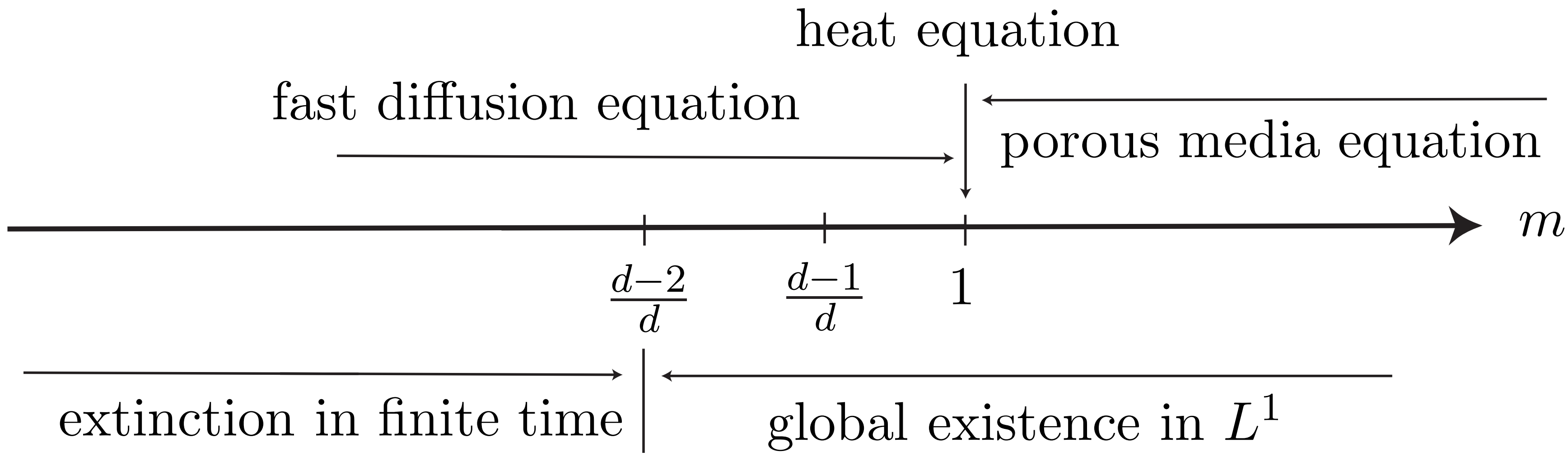
Generalized entropies and nonlinear diffusions (EDP, uncomplete):
[Del Pino, J.D.], [Carrillo, Toscani], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vázquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub],...
[del Pino, Sáez], [Daskalopulos, Sesum]...

1) [J.D., del Pino] relate entropy and Gagliardo-Nirenberg inequalities
2) “entropy – entropy-production method”
3) mass transport techniques
4) hypercontractivity for appropriate semi-groups

Heat equation, porous media & fast diffusion equation

$$u_t = \Delta u^m$$

$$x \in \mathbb{R}^d$$



Existence theory, critical values of the parameter m

Intermediate asymptotics for fast diffusion & porous media

$$\frac{\partial u}{\partial \tau} = \Delta u^m \quad \text{in } \mathbb{R}^d$$

$$u_{|\tau=0} = u_0 \geq 0$$

$$u_0(1+|x|^2) \in L^1\,, \quad u_0^m \in L^1$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0 \, dx = M > 0$

Self-similar (Barenblatt) function: $\mathcal{U}(\tau) = O(\tau^{-d/(2-d(1-m))})$
[Friedmann, Kamin, 1980] As $\tau \to +\infty$

$$\|u(\tau,\cdot) - \mathcal{U}(\tau,\cdot)\|_{L^\infty} = o(\tau^{-d/(2-d(1-m))})$$

$\Longrightarrow$ What about $\|u(\tau,\cdot) - \mathcal{U}(\tau,\cdot)\|_{L^1}$?

Time-dependent rescaling

Take $u(\tau, y) = R^{-d}(t)\, v\,(t, y/R(\tau))$ where

$$\frac{\partial R}{\partial \tau} = R^{d(1-m)-1}\;, \quad R(0) = 1\;, \quad t = \log R$$

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x\, v)\;, \quad v_{|\tau=0} = u_0$$

[Ralston, Newman, 1984] Lyapunov functional: Entropy or Free energy

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right)\, dx - \Sigma_0$$

$$\frac{d}{d\tau}\Sigma[v] = -I[v]\;, \quad I[v] = \int v \left| \frac{\nabla v^m}{v} + x \right|^2\, dx$$

Entropy and entropy production

Stationary solution: choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left(C + \frac{1-m}{2\,m} \, |x|^2 \right)_+^{-1/(1-m)}$$

Fix Σ_0 so that $\Sigma[v_\infty] = 0$. The entropy can be put in an m-homogeneous form

$$\Sigma[v] = \int \psi\left(\tfrac{v}{v_\infty}\right) \, v_\infty^m \, dx \quad with \; \psi(t) = \tfrac{t^m - 1 - m\,(t-1)}{m-1}$$

Theorem 1. $d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2\,\Sigma[v]$$

An equivalent formulation

$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0 \leq \frac{1}{2} \int v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} I[v]$

$p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int |w|^{1+p} dx + K$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$ and, for some γ, K can be written as

$$K = K_0 \left(\int v \, dx = \int w^{2p} \, dx \right)^\gamma$$

$w = w_\infty = v_\infty^{1/2p}$ is optimal

$m = m_1 := \frac{d-1}{d}$: Sobolev, $m \to 1$: logarithmic Sobolev

Gagliardo-Nirenberg inequalities

Theorem 2. [Del Pino, J.D.] *Assume that* $1 < p \leq \frac{d}{d-2}$ *and* $d \geq 3$

$$\|w\|_{2p} \leq A\,\|\nabla w\|_2^\theta\,\|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}$$

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p)}\,, \quad y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$

Uses [Serrin-Pucci], [Serrin-Tang]

$1 < p = \frac{1}{2m-1} \leq \frac{d}{d-2} \Longleftrightarrow$ Fast diffusion case: $\frac{d-1}{d} \leq m < 1$

$0 < p < 1 \Longleftrightarrow$ Porous medium case: $m > 1$

Intermediate asymptotics

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau}$ + Csiszár-Kullback inequalities

Theorem 3. [Del Pino, J.D.]

(i) $\frac{d-1}{d} < m < 1$ *if* $d \geq 3$

$$\limsup_{t \to +\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}} \| u^m - u_\infty^m \|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t \to +\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}} \| \, [u - u_\infty] \, u_\infty^{m-1} \, \|_{L^1} < +\infty$$

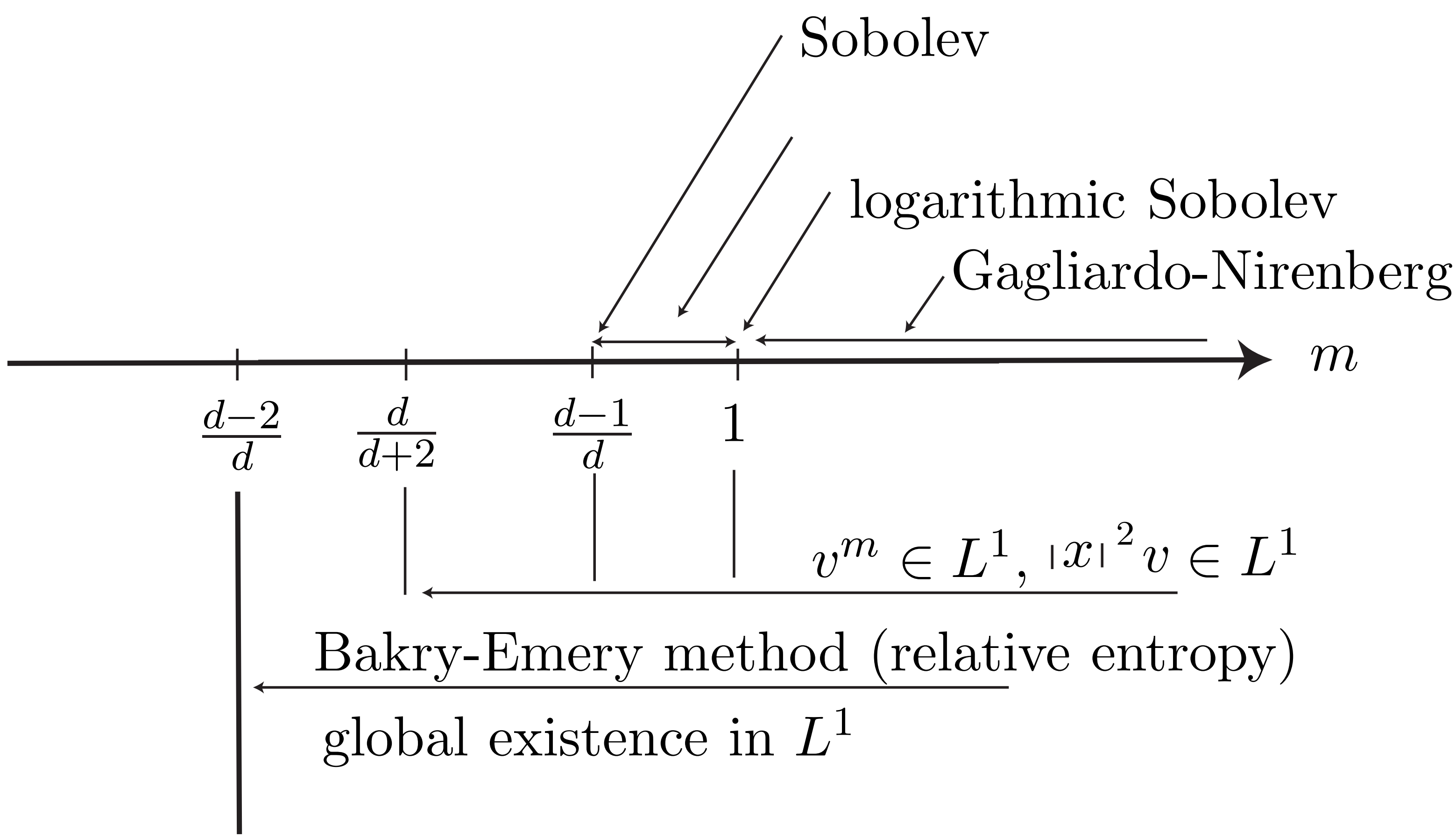
$$u_\infty(t,x) = R^{-d}(t)\, v_\infty\,(x/R(t))$$

Fast diffusion equations: the finite mass regime

- If $m \geq 1$: porous medium regime or $m_1 := \frac{d-1}{d} \leq m < 1$, the decay of the entropy is governed by Gagliardo-Nirenberg inequalities, and to the limiting case $m = 1$ corresponds the logarithmic Sobolev inequality
- If $m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in L^1 and the Barenblatt self-similar solution has finite mass
- The fast diffusion equation can be seen as the gradient flow of the generalized entropy with respect to the Wasserstein distance
- Displacement convexity holds in the same range of exponents, $m \in ((d-1)/d, 1)$, as for the Gagliardo-Nirenberg inequalities

Fast diffusion: finite mass regime

Inequalities...



... existence of solutions of $u_t = \Delta u^m$

Extensions and related results

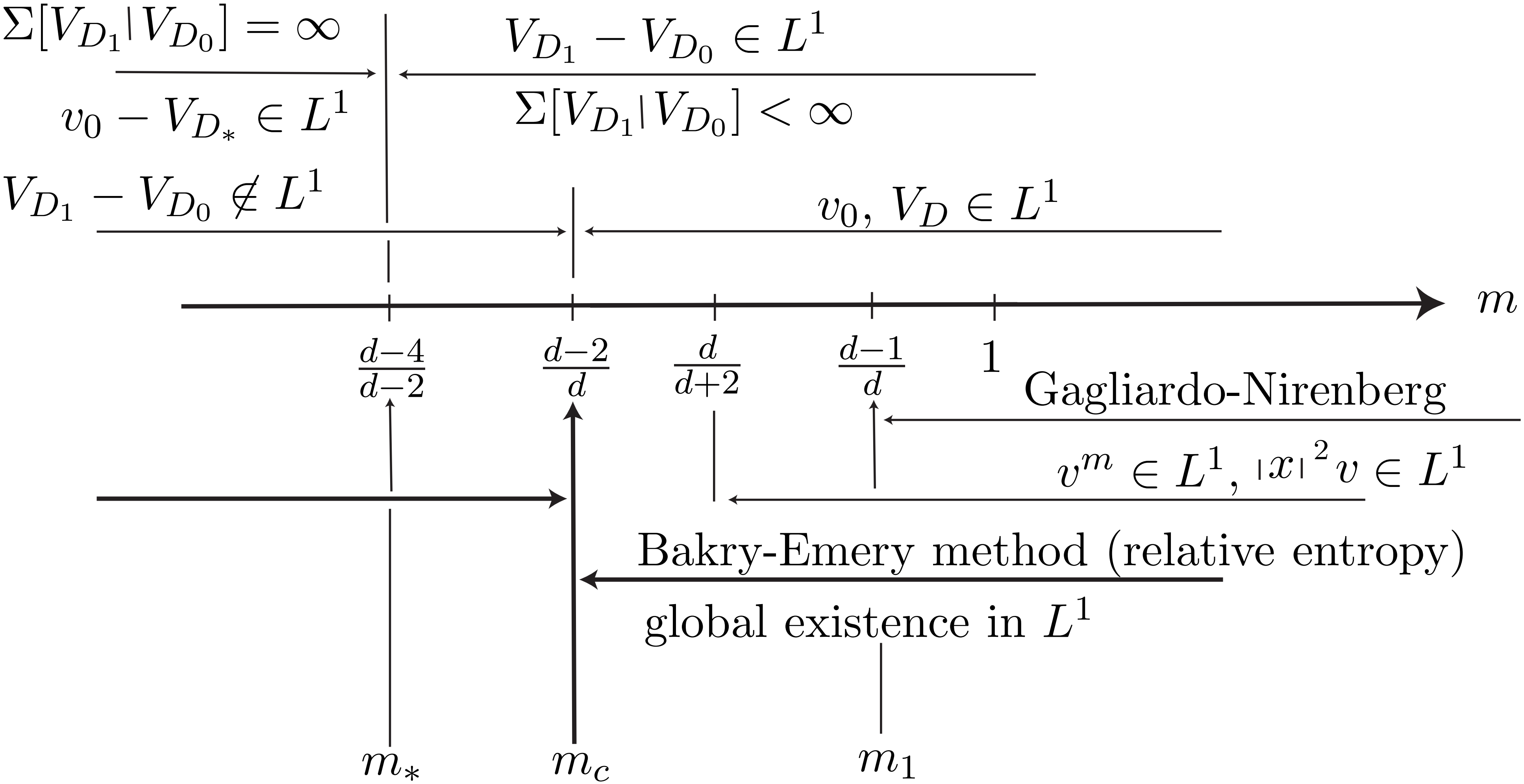
- Mass transport methods: inequalities / rates [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]
- General nonlinearities [Biler, J.D., Esteban], [Carrillo-DiFrancesco], [Carrillo-Juengel-Markowich-Toscani-Unterreiter] and gradient flows [Jordan-Kinderlehrer-Otto], [Ambrosio-Savaré-Gigli], [Otto-Westdickenberg] [J.D.-Nazaret-Savaré], etc
- Non-homogeneous nonlinear diffusion equations [Biler, J.D., Esteban], [Carrillo, DiFrancesco]
- Extension to systems and connection with Lieb-Thirring inequalities [J.D.-Felmer-Loss-Paturel, 2006], [J.D.-Felmer-Mayorga]
- Drift-diffusion problems with mean-field terms. An example: the Keller-Segel model [J.D-Perthame, 2004], [Blanchet-J.D-Perthame, 2006], [Biler-Karch-Laurençot-Nadzieja, 2006], [Blanchet-Carrillo-Masmoudi, 2007], etc
- ... connection with linearized problems [Markowich-Lederman], [Carrillo-Vázquez], [Denzler-McCann], [McCann, Slepčev], [Kim, McCann]

Fast diffusion equations: the infinite mass regime

- If $m > m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in L^1 and the Barenblatt self-similar solution has finite mass.
- For $m \leq m_c$, the Barenblatt self-similar solution has infinite mass

$\Rightarrow$ How to extend to $m \leq m_c$ what has been done for $m > m_c$? Work in relative variables !

Fast diffusion: infinite mass regime



Entropy methods and linearization...

... intermediate asymptotics, vanishing

A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez

- use the properties of the flow
- write everything as relative quantities (to the Barenblatt profile)
- compare the functionals (entropy, Fisher information) to their linearized counterparts

$\Longrightarrow$ Extend the domain of validity of the method to the price of a restriction of the set of admissible solutions

Setting of the problem

We consider the solutions $u(\tau, y)$ of

$$\begin{cases} \partial_\tau u = \Delta u^m \\ u(0,\cdot) = u_0 \end{cases}$$

where $m \in (0,1)$ (fast diffusion) and $(\tau, y) \in Q_T = (0,T) \times \mathbb{R}^d$
Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where

$$m_c := \frac{d-2}{d}$$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$
- $0 < m < m_c$, $T < +\infty$: vanishing in finite time

$$\lim_{\tau \nearrow T} u(\tau, y) = 0$$

Relative entropy methods and linearization

Fast diffusion equation and Barenblatt solutions

Consider the fast diffusion equation

$$\frac{\partial u}{\partial \tau} = -\nabla \cdot (u \, \nabla u^{m-1}) = \frac{1-m}{m} \, \Delta u^m \tag{1}$$

with $m < 1$. We look for positive solutions $u(\tau, y)$ for $\tau \geq 0$ and $y \in \mathbb{R}^d$, $d \geq 1$, corresponding to nonnegative initial-value data

$$u(\tau = 0, \cdot) = u_0 \in L^1_{\mathrm{loc}}(dx)$$

In the limit case $m = 0$, u^m/m has to be replaced by $\log u$
Barenblatt solutions play for $m < 1$ a role similar to Gaussian kernel for $m = 1$

$$U_{D,T}(\tau, y) := \frac{1}{R(\tau)^d} \left(D + \tfrac{1-m}{2\,d\,|m-m_c|} \left| \tfrac{y}{R(\tau)} \right|^2 \right)_+^{-\frac{1}{1-m}}$$

whenever $m > m_c := \frac{d-2}{d}$ and $m \neq 1$, with $R(\tau) := (T + \tau)^{\frac{1}{d\,(m-m_c)}}$

Extension of the family of the Barenblatt solutions

- If $m > m_c := (d-2)/d$, the Barenblatt solution describes the large time asymptotics of the solutions of equation (1) as $\tau \to \infty$ provided $M = \int_{\mathbb{R}^d} u_0 \, dy$ is finite, a condition that uniquely determines $D = D(M)$.
Notice that in the range $m \geq m_c$, solutions of (1) with $u_0 \in L^1_+(dx)$ exist globally in time and mass is conserved: $\int_{\mathbb{R}^d} u(\tau, y) \, dy = M$ for any $\tau \geq 0$
- If $m < m_c$ the family of Barenblatt functions can be extended by considering

$$R(\tau) := (T - \tau)^{-\frac{1}{d\,(m_c - m)}}$$

The parameter T now denotes the *extinction time*
- If $m = m_c$ take $R(\tau) = e^\tau$, $U_{D,T}(\tau, y) = e^{-d\,\tau} \left(D + e^{-2\tau} \, |y|^2/2 \right)^{-d/2}$

Two crucial values of m:

$$m_* := \frac{d-4}{d-2} \quad \text{and} \quad m_c := \frac{d-2}{d}$$

Rescaling

A time-dependent change of variables

$$t := \tfrac{1-m}{2} \log\left(\tfrac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\tfrac{1}{2\,d\,|m-m_c|}}\,\frac{y}{R(\tau)}$$

If $m = m_c$, we take $t = \tau/d$ and $x = e^{-\tau}\,y/\sqrt{2}$

The generalized Barenblatt functions $U_{D,T}(\tau,y)$ are transformed into stationary *generalized Barenblatt profiles* $V_D(x)$

$$V_D(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}} \quad x \in \mathbb{R}^d$$

If u is a solution to (1), the function $v(t,x) := R(\tau)^d\,u(\tau,y)$ solves

$$\frac{\partial v}{\partial t} = -\nabla \cdot \left[v\,\nabla\left(v^{m-1} - V_D^{m-1}\right)\right] \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{2}$$

with initial condition $v(t=0,x) = v_0(x) := R(0)^{-d}\,u_0(y)$

Goal

We are concerned with the *sharp rate* of convergence of a solution v of the rescaled equation to the *generalized Barenblatt profile* V_D in the whole range $m < 1$. Convergence is measured in terms of the relative entropy

$$\mathcal{E}[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - V_D^m - m\, V_D^{m-1}(v - V_D) \right] \, dx$$

for all $m \neq 0$, $m < 1$

Assumptions on the initial datum v_0

(H1) $V_{D_0} \leq v_0 \leq V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \geq 3$ *and* $m \leq m_*$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

The case $m = m_* = \frac{d-4}{d-2}$ will be discussed later
If $m > m_*$, we define D as the unique value in $[D_1, D_0]$ such that $\int_{\mathbb{R}^d} (v_0 - V_D)\, dx = 0$

Our goal is to find the best possible rate of decay of $\mathcal{E}[v]$ if v solves (2)

Sharp rates of convergence

Theorem 4. [Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \neq m_*$*, the entropy decays according to*

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,(1-m)\,\Lambda\,t} \quad \forall\, t \geq 0$$

The sharp decay rate Λ *is equal to the best constant* $\Lambda_{\alpha,d} > 0$ *in the Hardy–Poincaré inequality of Theorem* 17 *with* $\alpha := 1/(m-1) < 0$
The constant $C > 0$ *depends only on* m, d, D_0, D_1, D *and* $\mathcal{E}[v_0]$

- Notion of *sharp rate* has to be discussed
- Rates of convergence in more standard norms: $L^q(dx)$ for $q \geq \max\{1, d\,(1-m)/\,[2\,(2-m)+d\,(1-m)]\}$, or C^k by interpolation
- By undoing the time-dependent change of variables, we deduce results on the *intermediate asymptotics* of (1), i.e. rates of decay of $u(\tau,y) - U_{D,T}(\tau,y)$ as $\tau \to +\infty$ if $m \in [m_c, 1)$, or as $\tau \to T$ if $m \in (-\infty, m_c)$

Basin of attraction of the Barenblatt solutions

Barenblatt solutions $U_{D,T}$ have two parameters:
D corresponds to the *mass*
T has the meaning of the *extinction time* of the solution for $m < m_c$ and of a time-delay parameter otherwise

The basin of attraction of $U_{D,T}$ contains all solutions corresponding to data which are trapped between two Barenblatt profiles $U_{D_0,T}(0,\cdot), U_{D_1,T}(0,\cdot)$ for the same value of T and satisfy a relative mass condition

- if $m > m_c$, $U_{D,T}$ attracts all solutions with corresponding mass
- if $m_* < m \leq m_c$, $U_{D,T}$ attracts all solutions such that $\int_{\mathbb{R}^d} [u_0 - U_{D,T}(0,\cdot)]\,dy = 0$ for some $D \in [D_1, D_0]$
- If $m < m_*$, $U_{D,T}$ attracts all solutions corresponding to data which are integrable perturbations of $U_{D,T}(0,\cdot)$

Strategy of proof

Assume that $D = 1$ and consider $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$, with $\alpha = 1/(m-1) < 0$, and

$$\mathcal{L}_{\alpha,d} := -h_{1-\alpha}\, \mathrm{div}\, [\, h_\alpha\, \nabla \cdot\,]$$

on the weighted space $L^2(d\mu_\alpha)$: $\int_{\mathbb{R}^d} f\, (\mathcal{L}_{\alpha,d}\, f)\, d\mu_{\alpha-1} = \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha$

A first order expansion of $v(t,x) = h_\alpha(x) \left[1 + \varepsilon\, f(t,x)\, h_\alpha^{1-m}(x)\right]$ solves

$$\frac{\partial f}{\partial t} + \mathcal{L}_{\alpha,d}\, f = 0$$

Theorem 4 follows from a spectral gap for $\mathcal{L}_{\alpha,d}$

- For $d \geq 3$, let $\alpha_* := -(d-2)/2$ corresponding to $m = m_*$, $m_1 := (d-1)/d$ with corresponding $\alpha_1 = -d$, and $m_2 := d/(d+2)$ with corresponding $\alpha_2 = -(d+2)/2$. We have $m_* < m_c < m_2 < m_1 < 1$
- $d = 2$ gives $m_* = -\infty$ so that $\alpha_* = 0$, as well as $m_c = 0$, and $m_1 = m_2 = 1/2$

A table of correspondence

$$\alpha = 1/(m-1) \quad \Longleftrightarrow m = 1 + \frac{1}{\alpha}$$

$m \in (-\infty, 1)$ means $\alpha \in (-\infty, 0)$

$m =$	$-\infty$	m_*	m_c	m_2	m_1	1
$m =$	$-\infty$	$\frac{d-4}{d-2}$	$\frac{d-2}{d}$	$\frac{d}{d+2}$	$\frac{d-1}{d}$	1
$\alpha =$	0	$-\frac{d-2}{2}$	$-\frac{d}{2}$	$-\frac{d+2}{2}$	$-d$	$-\infty$

Sharp Hardy-Poincaré inequalities

Theorem 5. *Let $d \geq 3$. For any $\alpha \in (-\infty, 0) \setminus \{\alpha_*\}$, there is a positive constant $\Lambda_{\alpha,d}$ such that*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 \, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_\alpha \quad \forall \, f \in H^1(d\mu_\alpha) \tag{3}$$

under the additional condition $\int_{\mathbb{R}^d} f \, d\mu_{\alpha-1} = 0$ if $\alpha < \alpha_*$

$$\Lambda_{\alpha,d} = \begin{cases} \frac{1}{4}\,(d-2+2\,\alpha)^2 & \text{if } \alpha \in \left[-\frac{d+2}{2}, \alpha_*\right) \cup (\alpha_*, 0) \\ -\,4\,\alpha - 2\,d & \text{if } \alpha \in \left[-d, -\frac{d+2}{2}\right) \\ -\,2\,\alpha & \text{if } \alpha \in (-\infty, -d) \end{cases}$$

For $d = 2$, inequality (3) holds for all $\alpha < 0$, with $\Lambda_{\alpha,2} = \alpha^2$ for $\alpha \in [-2, 0)$ and $\Lambda_{\alpha,2} = -2\alpha$ for $\alpha \in (-\infty, -2)$
For $d = 1$, (3) holds, with $\Lambda_{\alpha,1} = -2\,\alpha$ if $\alpha < -1/2$ and $\Lambda_{\alpha,1} = (\alpha - 1/2)^2$ if $\alpha \in [-1/2, 0)$

Comments

The Hardy-Poincaré inequalities (3) share many properties with Hardy's inequalities, because of homogeneity reasons.By scaling

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 \,(D+|x|^2)^{\alpha-1}\,dx \leq \int_{\mathbb{R}^d} |\nabla f|^2 \,(D+|x|^2)^{\alpha}\,dx$$

holds for any $f \in H^1((D+|x|^2)^\alpha dx)$ and any $D \geq 0$, under the additional conditions $\int_{\mathbb{R}^d} f\,(D+|x|^2)^{\alpha-1}\,dx = 0$ and $D > 0$ if $\alpha < \alpha_*$

- $\Lambda_{\alpha,d}$ is independent of D
- $D \to 0$: [Hardy, Caffarelli-Kohn-Nirenberg] (standard Hardy ineq.: $\alpha = 0$)
- $m \to 1$: Poincaré inequality (gaussian measure): rescale first to get $(1+(1-m)\,|x|^2)^{-1/(1-m)}$

Relative entropy and relative Fisher information

The *relative entropy* of J. Ralston and W.I. Newmann is

$$\mathcal{F}[w] := \frac{m}{1-m} \int_{\mathbb{R}^d} \left[w - 1 - \frac{1}{m} \big(w^m - 1 \big) \right] V_D^m \, dx$$

for any $m < 1$, $m \neq 0$. In the limit case $m = 1$, we recover
$\mathcal{F}[w] := \int_{\mathbb{R}^d} [w \, \log w - (w-1)] \; d\mu$

The *generalized relative Fisher information* is

$$\mathcal{I}[w] := \int_{\mathbb{R}^d} \left| \frac{1}{m-1} \, \nabla \left[\left(w^{m-1} - 1 \right) V_D^{m-1} \right] \right|^2 v \, dx$$

where $w = \frac{v}{V_D}$. If v is a solution of (2), then

$$\frac{d}{dt} \mathcal{F}[w(t,\cdot)] = - \mathcal{I}[w(t,\cdot)] \quad \forall \, t > 0$$

Linearization and interpolation

Method of [Blanchet, Bonforte, J.D., Grillo, Vázquez]: let
$f := (w-1)\,V_D^{m-1}$, $h_1(t) := \inf_{\mathbb{R}^d} w(t,\cdot)$, $h_2(t) := \sup_{\mathbb{R}^d} w(t,\cdot)$ and
$h := \max\{h_2, 1/h_1\}$. We notice that $h(t) \to 1$ as $t \to +\infty$

$$h^{m-2} \int_{\mathbb{R}^d} |f|^2\, V_D^{2-m}\, dx \le \frac{2}{m}\,\mathcal{F}[w] \le h^{2-m} \int_{\mathbb{R}^d} |f|^2\, V_D^{2-m}\, dx$$

$$\int_{\mathbb{R}^d} |\nabla f|^2\, V_D\, dx \le [1 + X(h)]\,\mathcal{I}[w] + Y(h) \int_{\mathbb{R}^d} |f|^2\, V_D^{2-m}\, dx$$

where X and Y are functions such that $\lim_{h\to 1} X(h) = \lim_{h\to 1} Y(h) = 0$

$$h_2^{2(2-m)}/h_1 \le h^{5-2m} =: 1 + X(h)$$

$$\left[\,(h_2/h_1)^{2(2-m)} - 1\right] \le d\,(1-m)\,\left[h^{4(2-m)} - 1\right] =: Y(h)$$

A new interpolation inequality: for $h > 0$ small enough

$$\mathcal{F}[w] \le \frac{h^{2-m}\,[1 + X(h)]}{2\left[\Lambda_{\alpha,d} - m\,Y(h)\right]}\, m\,\mathcal{I}[w]$$

Gronwall estimates

(...) one more interpolation allows to close the system of estimates: for some $\mathsf{C} = \mathsf{C}(d, m, D, D_0, D_1)$,

$$0 \leq h - 1 \leq \mathsf{C}\,\mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$$

Hence we have a nonlinear differential inequality

$$\frac{d}{dt}\mathcal{F}[w(t,\cdot)] \leq -2\,\frac{\Lambda_{\alpha,d} - m\,Y(h)}{\big[1 + X(h)\big]\,h^{2-m}}\,\mathcal{F}[w(t,\cdot)]$$

A Gronwall lemma (take $h = 1 + \mathsf{C}\,\mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$) then shows that

$$\limsup_{t\to\infty}\, \mathrm{e}^{\,2\,\Lambda_{\alpha,d}\,t}\mathcal{F}[w(t,\cdot)] < +\infty$$

Explicit dependence on the initial data

Corollary 6. [Bonforte, J.D., Grillo, Vázquez] *Let* $h(0) < h_*$, h_* *small enough. Under the assumptions of Theorem* 4

$$\mathcal{F}[w(t,\cdot)] \leq G\big(t, h(0), \mathcal{F}[w(0,\cdot)]\big) \quad \forall\, t \geq 0$$

where G *is the unique solution of the nonlinear ODE*

$$\frac{dG}{dt} = -2\,\frac{\Lambda_{\alpha,d} - m\,Y(h)}{[1 + X(h)]\,h^{2-m}}\,G \quad \textit{with} \quad h = 1 + \mathsf{C}\,G^{\frac{1-m}{d+2-(d+1)m}}$$

and initial condition $G(0) = \mathcal{F}[w(0,\cdot)]$

Operator equivalence

- [Bonforte, J.D., Grillo,Vázquez - CRAS]: The operator

$$\mathcal{L}_{\alpha,d} = -h_{1-\alpha}\,\mathrm{div}\,[\,h_\alpha\,\nabla\cdot\,]$$

defined on $L^2(d\mu_{\alpha-1})$ has a spectral gap for any $\alpha \neq \alpha_* = (2-d)/2$

- J. Denzler and R.J. McCann formally linearized the fast diffusion flow in the framework of mass transportation and gave for all $m \in (m_c, 1)$ the spectrum of the operator closure of $\mathcal{L}_{\alpha,d}$, initially defined on $\mathcal{D}(\mathbb{R}^d)$, in the Hilbert space $H^{1,*}(d\mu_\alpha) := \big\{f \in L^2(d\mu_{\alpha-1}) : \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu_\alpha < \infty$ *and* $\int_{\mathbb{R}^d} f\,d\mu_{\alpha-1} = 0$ *if* $\alpha < \alpha_*\big\}$, so that

$$\mathcal{H}_{\alpha,d}\,f := h_{1-\alpha}\,\nabla\cdot\big[h_\alpha\,\nabla\big(h_{1-\alpha}\,\nabla\cdot(h_\alpha\,\nabla f)\big)\big]$$

Proposition 7. *$\mathcal{L}_{\alpha,d}$ on $L^2(d\mu_{\alpha-1})$ is unitarily equivalent to $\mathcal{H}_{\alpha,d}$ on $H^{1,*}(d\mu_\alpha)$. As a consequence, they have the same spectrum*

The unitary operator $U = \sqrt{\mathcal{L}_{\alpha,d}} : H^{1,*}(d\mu_\alpha) \to L^2(d\mu_{\alpha-1})$ is such that $U\,\mathcal{H}_{\alpha,d}\,U^{-1} = \mathcal{L}_{\alpha,d}$

How to compute the spectrum of $\mathcal{L}_{\alpha,d}$?

The spectrum of the Laplace-Beltrami operator on S^{d-1} is described by

$$-\Delta_{S^{d-1}} Y_{\ell\mu} = \ell\,(\ell + d - 2)\,Y_{\ell\mu}$$

with $\ell = 0, 1, 2, \ldots$ and $\mu = 1, 2, \ldots M_\ell := \frac{(d+\ell-3)!\,(d+2\ell-2)}{\ell!\,(d-2)!}$ with $M_0 = 1$, and $M_1 = 1$ if $d = 1$. Using spherical coordinates and separation of variables, the discrete spectrum of $\mathcal{L}_{\alpha,d}$ is given by λ such that

$$v'' + \left(\frac{d-1}{r} + \frac{2\,\alpha\,r}{1+r^2}\right) v' + \left(\frac{\lambda}{1+r^2} - \frac{\ell\,(\ell+d-2)}{r^2}\right) v = 0$$

has a solution on $\mathbb{R}^+ \ni r$, in the domain of $\mathcal{L}_{\alpha,d}$. The change of variables $v(r) = r^\ell\,w(-r^2)$ allows to express w in terms of the hypergeometric function ${}_2F_1(a, b, c; z)$ with $c = \ell + d/2$, $a + b + 1 = \ell + \alpha + d/2$ and $a\,b = (2\,\ell\,\alpha + \lambda)/4$, as the solution for $s = -r^2$ of

$$s\,(1 - s)\,y'' + [\,c - (a + b + 1)\,s\,]\,y' - a\,b\,y = 0$$

The spectrum of $\mathcal{L}_{\alpha,d}$

Proposition 8. *The bottom of the continuous spectrum of the operator $\mathcal{L}_{\alpha,d}$ on $L^2(d\mu_{\alpha-1})$ is*

$$\lambda_{\alpha,d}^{\mathrm{cont}} := \frac{1}{4}(d+2\,\alpha-2)^2$$

$\mathcal{L}_{\alpha,d}$ has some discrete spectrum only for $m > m_2 = d/(d+2)$

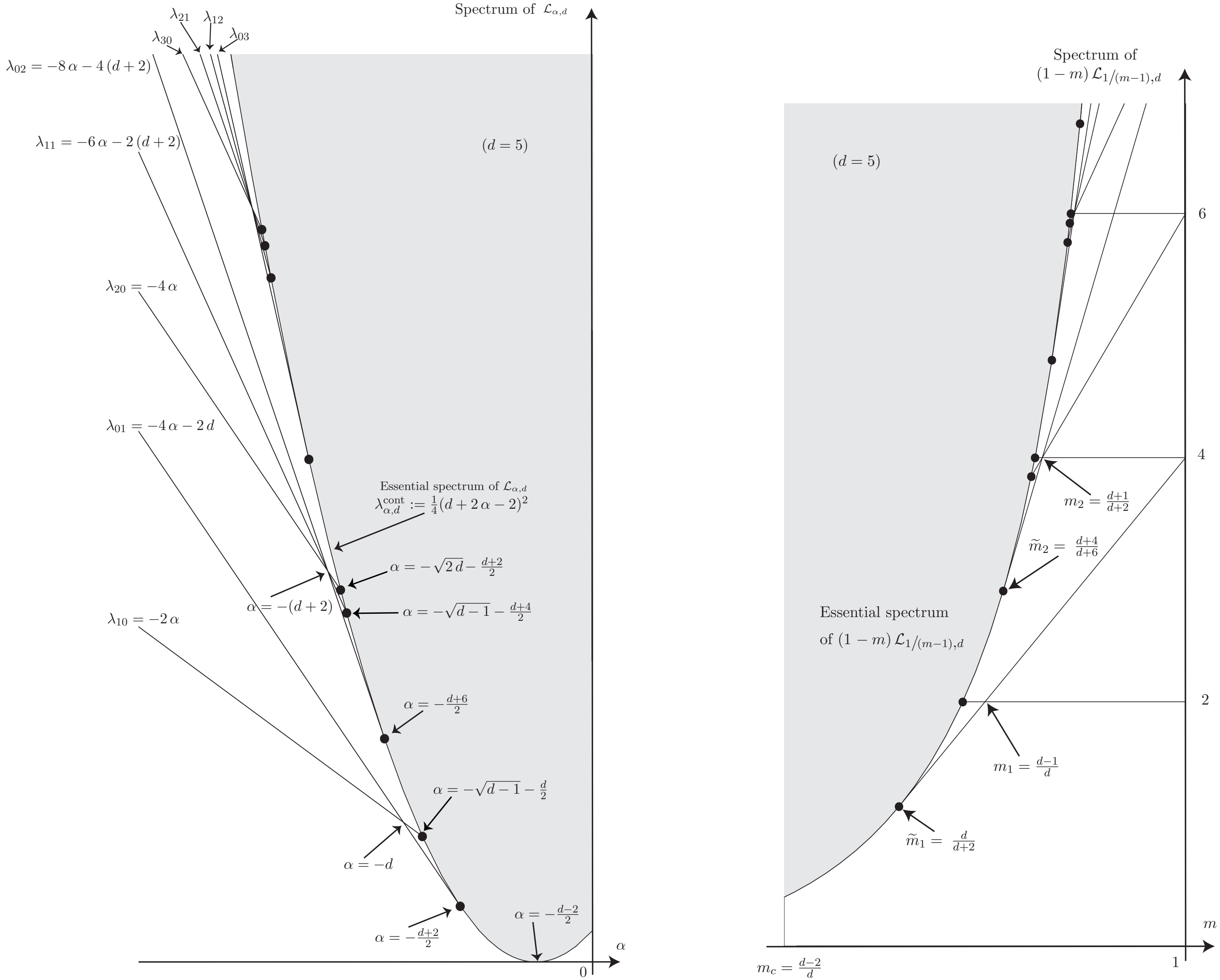
- *For $d \geq 2$, the discrete spectrum is made of the eigenvalues*

$$\lambda_{\ell k} = -2\,\alpha\,(\ell+2\,k) - 4\,k\left(k+\ell+\frac{d}{2}-1\right) \tag{4}$$

with $\ell,\,k = 0,\,1,\ldots$ provided $(\ell,k) \neq (0,0)$ and $\ell+2k-1 < -(d+2\,\alpha)/2$

- *If $d=1$, the discrete spectrum is made of the eigenvalues $\lambda_k = k\,(1-2\,\alpha-k)$ with $k \in \mathbb{N} \cap [1, 1/2-\alpha]$*

Plots $(d = 5)$



Spectrum of $\mathcal{L}_{\alpha,d}$
λ_{21} λ_{12}
λ_{30} λ_{03}
$\lambda_{02} = -8\,\alpha - 4\,(d+2)$
$\lambda_{11} = -6\,\alpha - 2\,(d+2)$
$(d = 5)$
$\lambda_{20} = -4\,\alpha$
$\lambda_{01} = -4\,\alpha - 2\,d$
Essential spectrum of $\mathcal{L}_{\alpha,d}$
$\lambda^{\rm cont}_{\alpha,d} := \frac{1}{4}(d + 2\,\alpha - 2)^2$
$\alpha = -\sqrt{2\,d} - \frac{d+2}{2}$
$\alpha = -(d+2)$
$\alpha = -\sqrt{d-1} - \frac{d+4}{2}$
$\lambda_{10} = -2\,\alpha$
$\alpha = -\frac{d+6}{2}$
$\alpha = -\sqrt{d-1} - \frac{d}{2}$
$\alpha = -d$
$\alpha = -\frac{d-2}{2}$
$\alpha = -\frac{d+2}{2}$
α
0
Spectrum of
$(1-m)\,\mathcal{L}_{1/(m-1),d}$
$(d = 5)$
6
4
$m_2 = \frac{d+1}{d+2}$
$\widetilde{m}_2 = \frac{d+4}{d+6}$
Essential spectrum
of $(1-m)\,\mathcal{L}_{1/(m-1),d}$
2
$m_1 = \frac{d-1}{d}$
$\widetilde{m}_1 = \frac{d}{d+2}$
m
1
$m_c = \frac{d-2}{d}$

- Continuous spectrum: Persson's characterization

$$\lambda^{\rm cont}_{\alpha,d}\int_{\mathbb{R}^d}|f|^2\,|x|^{2(\alpha-1)}\;dx\le\int_{\mathbb{R}^d}|\nabla f|^2\,|x|^{2\,\alpha}dx$$

for any $f\in\mathcal{D}(\mathbb{R}^d\setminus\{0\})$

- The condition that the solution of the eigenvalue equation is in the domain of $\mathcal{L}_{\alpha,d}$ determines the eigenvalues
- $\alpha=1/(m-1)$
- For $d\ge 2$

$$\alpha=-d\quad\Longleftrightarrow\quad m=m_1=(d-1)/d$$

(corresponds to $-2\,\alpha=\lambda_{10}=\lambda_{01}=-4\,\alpha-2\,d$)

$$\alpha=-(d+2)/2\quad\Longleftrightarrow\quad m=m_2=d/(d+2)$$

(corresponds to $\lambda_{01}=\lambda^{\rm cont}_0:=\frac{1}{4}\,(d+2\,\alpha-2)^2$)

- $d=2$ and $d=1$ are slightly different

The critical case $(m = m_*$, $d \geq 3)$: slow asymptotics

Theorem 9. [Bonforte, Grillo, Vázquez] *Assume that* $d \geq 3$, $m = m_*$, *and suppose that* (H1)-(H2) *hold. If* $|v_0 - V_D|$ *is bounded a.e. by a radial* $L^1(dx)$ *function, then there exists a positive constant* C^* *such that*

$$\mathcal{E}[v(t,\cdot)] \leq C^* \, t^{-1/2} \quad \forall \, t \geq 0 \tag{5}$$

where C^* *depends only on* m, d, D_0, D_1, D *and* $\mathcal{E}[v_0]$

There exists a positive continuous and monotone function $\mathcal{N}$ on $\mathbb{R}^+$ such that for any nonnegative smooth function f with $M = \int_{\mathbb{R}^d} f \, d\mu_{-d/2}$

$$\frac{1}{M^2} \int_{\mathbb{R}^d} |f|^2 \, d\mu_{-d/2} \leq \mathcal{N}\left(\frac{1}{M^2} \int_{\mathbb{R}^d} |\nabla f|^2 \; d\mu_{(2-d)/2} \right)$$

$\lim_{s \to 0^+} s^{-1/3} \mathcal{N}(s) = c_1 > 0$ and $\lim_{s \to \infty} s^{-d/(d+2)} \mathcal{N}(s) = c_2 > 0$
Up to technicalities, for some $K > 0$, $t \geq t_0$ large enough

$$(\mathcal{F}[w(t,\cdot)])^3 \leq K \, \mathcal{I}[w(t,\cdot)]$$

Faster convergence ? Improved Hardy-Poincaré inequality

Can we improve the rates of convergence by imposing restrictions on the initial data ?

- [Carrillo, Lederman, Markowich, Toscani (2002)] Poincaré inequalities for linearizations of very fast diffusion equations (radially symmetric solutions)
- Formal or partial results: [Denzler, McCann (2005)], [McCann, Slepčev (2006)]

Lemma 10. *Let* $\widetilde{\Lambda}_{\alpha,d} := -4\,\alpha - 2\,d$ *if* $\alpha < -d$ *and* $\widetilde{\Lambda}_{\alpha,d} := \lambda^{\rm cont}_{\alpha,d}$ *if* $\alpha \in [-d, -d/2)$*. If* $d \geq 2$*, for any* $\alpha \in (-\infty, -d)$*, we have*

$$\widetilde{\Lambda}_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 \, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$

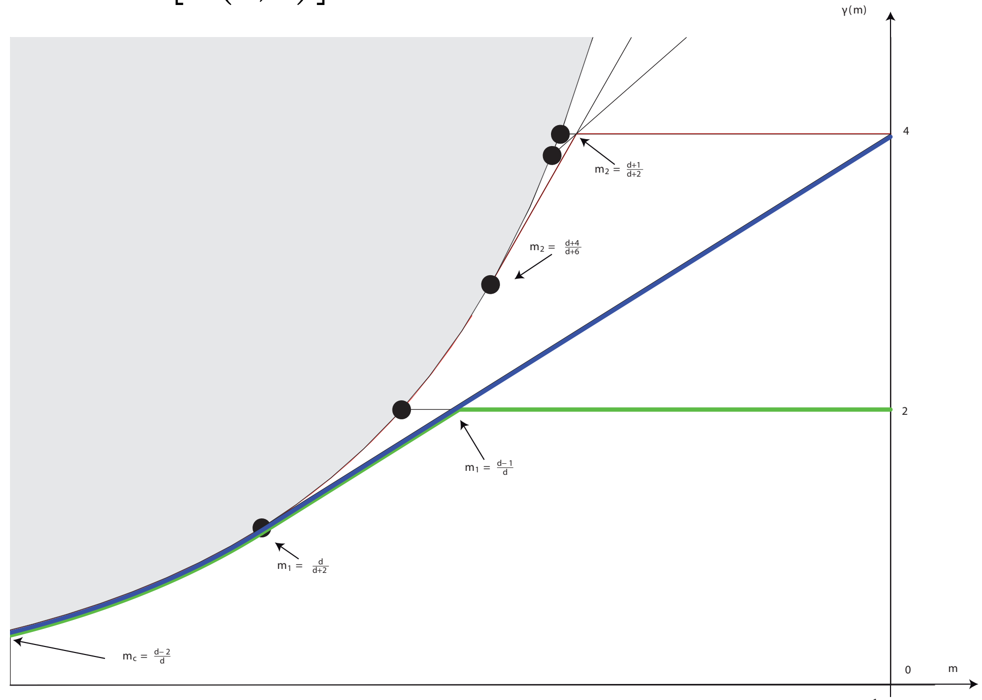
under the conditions $\int_{\mathbb{R}^d} f \, d\mu_{\alpha-1} = 0$ *and* $\int_{\mathbb{R}^d} x\, f \, d\mu_{\alpha-1} = 0$*. The constant* $\widetilde{\Lambda}_{\alpha,d}$ *is sharp*

This covers the range $m \in (m_1, 1)$ with $m_1 = (d-1)/d$

Faster convergence 1

Theorem 11. [Bonforte, J.D., Grillo, Vázquez] *Assume that* $m \in (m_1, 1)$, $d \geq 3$. *Under Assumption* (H1), *if* v *is a solution of* (2) *with initial datum* v_0 *such that* $\int_{\mathbb{R}^d} x\, v_0\, dx = 0$ *and if* D *is chosen so that* $\int_{\mathbb{R}^d} (v_0 - V_D)\, dx = 0$, *then there exists a positive constant* $\widetilde{C}$ *depending only on* m, d, D_0, D_1, D *and* $\mathcal{E}[v_0]$ *such that, with* $\gamma(m) = (1-m)\,\widetilde{\Lambda}_{1/(m-1),d}$, *the relative entropy decays like*

$$\mathcal{E}[v(t,\cdot)] \leq \widetilde{C}\, \mathrm{e}^{-\gamma(m)\, t} \quad \forall\, t \geq 0$$



A variational approach of sharpness

- Recall that $(d-2)/d = m_c < m_1 = (d-1)/d$. The entropy / entropy production inequality

$$\mathcal{F} \leq \frac{1}{2} \mathcal{I}$$

is equivalent to optimal Gagliardo-Nirenberg inequalities [delPino, J.D. (2002)] in the range $m \in [m_1, 1)$. It is sharp: equality is achieved if and only if $v = V_D$

- The inequality has been extended in [Carrillo, Vázquez (2003)] to the range $m \in (m_c, 1)$ using the Bakry-Emery method, with the same constant $1/2$, and again equality is achieved if and only if $v = V_D$
- The optimality of the constant can be reformulated as a variational problem

$$\mathcal{C} = \inf \frac{\mathcal{I}[v]}{\mathcal{E}[v]}$$

where the infimum is taken over the set of all functions such that $v \in \mathcal{D}(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} v \, dx = M$

A variational approach of sharpness

- Rephrasing the sharpness results, we know that $\mathcal{C} = 2$ if $m \in (m_1, 1)$ and $\mathcal{C} \geq 2$ if $m \in (m_c, m_1)$. By taking $v_n = V_D\,(1 + \frac{1}{n}\,f\,V_D^{1-m})$ and letting $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{\mathcal{I}[v_n]}{\mathcal{E}[v_n]} = \frac{\int_{\mathbb{R}^d} |\nabla f|^2\, V_D\, dx}{\int_{\mathbb{R}^d} |f|^2\, V_D^{2-m}\, dx}$$

With the optimal choice for f, the above limit is less or equal than 2. Since we already know that $\mathcal{C} \geq 2$, this shows that $\mathcal{C} = 2$ for any $m > m_c$. It is quite enlightening to observe that optimality in the quotient gives rise to indetermination since both numerator and denominator are equal to zero when $v = V_D$
- This explains why the optimal constant, $\mathcal{C} = 2$, is determined by the linearized problem
- When $m \leq m_c$, the variational approach is less clear since the problem has to be constrained by a uniform estimate

Sharp rates of convergence and conjectures

The precise meaning of Theorem 4 is that

$$\Lambda_{\alpha,d} = \liminf_{h \to 0_+} \inf_{w \in \mathcal{S}_h} \frac{\mathcal{I}[w]}{\mathcal{F}[w]}$$

where the infimum is taken on the set $\mathcal{S}_h$ of smooth, nonnegative bounded functions w such that $\|w-1\|_{L^\infty(dx)} \le h$ and such that $\int_{\mathbb{R}^d} (w-1)\, V_D\, dx$ is zero if $d = 1, 2$ and $m < 1$, or if $d \ge 3$ and $m_* < m < 1$, and it is finite if $d \ge 3$ and $m < m_*$

- *Sharp rate* means the best possible rate, which is uniform in $t \ge 0$
In other words, for any $\gamma > \gamma(m)$, one can find some initial datum in $\mathcal{S}_h$ such that the estimate $\mathcal{F}[w(t,\cdot)] \le \mathcal{F}[w(0,\cdot)]\, \exp(-\gamma\, t)$ is wrong for some $t > 0$
- We did not prove that the rate $\exp(-\gamma(m)\, t)$ is *globally sharp* in the sense that for some special initial data, $\mathcal{F}[w(t,\cdot)]$ decays exactly at this rate, nor that $\liminf_{t\to\infty} \exp(\gamma(m)\, t)\, \mathcal{F}[w(t,\cdot)] > 0$, which is possibly less restrictive
- If $m \in (m_1, 1)$, $m_1 = (d-1)/d$, then $\exp(-\gamma(m)\, t)$ is also a *globally sharp* rate: $u_0(x) = V_D(x + x_0)$, $x_0 \ne 0$

An improvement based on the variance

Heuristics

For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1} \right) = 0$$

The strategy is easy to understand using a time-dependent rescaling and the relative entropy formalism. Define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t, x) \; , \quad R = R(\tau) \; , \quad t = \tfrac{1}{2} \log R \; , \quad x = \frac{y}{R} \qquad (6)$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2\, x \right) \right] = 0 \quad t > 0 \; , \quad x \in \mathbb{R}^d \qquad (7)$$

with (as long as we make no assumption on R)

$$2\, \sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \tfrac{1}{\sigma}\,|x|^2\right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d \tag{8}$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1}\,(v - B_\sigma)\right]\, dx$$

Let us briefly sketch the strategy of our method before giving all details

The time derivative of this relative entropy is

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt}\left(\frac{d}{d\sigma}\,\mathcal{F}_\sigma[v]\right)_{|\sigma=\sigma(t)}}_{\text{choose it} = 0} + \frac{m}{m-1}\int_{\mathbb{R}^d}\left(v^{m-1} - B_{\sigma(t)}^{m-1}\right)\frac{\partial v}{\partial t}\, dx \tag{9}$$

First step: choice of the scaling parameter

Lemma 12. *For any given* $v \in L^1_+(\mathbb{R}^d)$ *such that* v^m *and* $|x|^2\, v$ *are both integrable, if* $m \in (\widetilde{m}_1, 1)$, *there is a unique* $\sigma = \sigma^* > 0$ *which minimizes* $\sigma \mapsto \mathcal{F}_\sigma[v]$, *and it is explicitly given by*

$$\sigma^* = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2\, v\, dx$$

For $\sigma = \sigma^*$, *the Barenblatt profile* B_{σ^*} *satisfies*

$$\int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx = \int_{\mathbb{R}^d} |x|^2\, v\, dx$$

The condition $m > \widetilde{m}_1$ guarantees that B_σ^m is integrable and $K_M = \int_{\mathbb{R}^d} |x|^2\, B_1\, dx$ is finite

Proof: we have to minimize

$$h(\sigma) := (1-m) \int_{\mathbb{R}^d} B_\sigma^m\, dx + m \int_{\mathbb{R}^d} B_\sigma^{m-1}\, v\, dx$$

Second step: the entropy / entropy production estimate

According to the definition of B_σ, we know that $2\,x = \sigma^{\frac{d}{2}(m-m_c)}\,\nabla B_\sigma^{m-1}$
Using (7), we know that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} v\left|\nabla\left[v^{m-1}-B_{\sigma(t)}^{m-1}\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[v] = \frac{m}{1-m}\int_{\mathbb{R}^d}\left[w-1-\frac{1}{m}\left(w^m-1\right)\right]B_\sigma^m\,dx$$

Define the *relative Fisher information* by

$$\mathcal{I}_\sigma[v] := \int_{\mathbb{R}^d}\left|\frac{1}{m-1}\nabla\left[\left(w^{m-1}-1\right)B_\sigma^{m-1}\right]\right|^2 B_\sigma\,w\,dx$$

so that $$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\,m\,(1-m)\,\sigma(t)\,\mathcal{I}_{\sigma(t)}[v(t,\cdot)] \quad \forall\, t>0$$

Third step: orthogonality conditions

Lemma 13. *Let* u *be a solution of* (7) *and* $f = B_\sigma^{m-1}\,(u/B_\sigma - 1)$ *where* $\sigma = \sigma(t)$ *is optimizes* $\mathcal{F}_\sigma[v]$. *With these notations, the function* f *has the following properties, for any* $t > 0$ *:*

- *(i)* Mass conservation: $\int_{\mathbb{R}^d} f(t,x)\,B_\sigma^{2-m}\,dx = 0$ *if* $m > m_c$
- *(ii) Position of the* center of mass: $\int_{\mathbb{R}^d} x\,f(t,x)\,B_\sigma^{2-m}\,dx = 0$ *if* $m > (d-1)/(d+1)$
- *(iii) Conservation of the second moment:* $\int_{\mathbb{R}^d} |x|^2\,f(t,x)\,B_\sigma^{2-m}\,dx = 0$ *if* $m > \widetilde{m}_1$

Improved Hardy-Poincaré inequalities (2)

Corollary 14 (Sharp Hardy-Poincaré inequalities)**.** *Let $d \geq 2$, $\alpha < -(d+2)/2$. For any $\alpha \in (-\infty, 0) \setminus \{\alpha_*\}$, there is a positive constant $\Lambda_{\alpha,d}$ such that*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 \, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_\alpha \quad \forall \, f \in L^2 \left(d\mu_{\alpha-1} \right)$$

if $f \in L^2 \left(d\mu_{\alpha-1} \right)$ is such that

$$\int_{\mathbb{R}^d} f \, d\mu_{\alpha-1} = 0 \, , \quad \int_{\mathbb{R}^d} x \, f \, d\mu_{\alpha-1} = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, f \, d\mu_{\alpha-1} = 0$$

with $d\mu_\alpha := h_\alpha \, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$. Moreover

$$\Lambda_{\alpha,d} = \begin{cases} \frac{1}{4} \, (d - 2 + 2 \, \alpha)^2 & \text{if } \alpha \in \left[-\frac{d+6}{2}, -\frac{d+2}{2} \right) \\ -\, 8 \, \alpha - 4 \, (d+2) & \text{if } \alpha \in \left[-(d+2) \, , -\frac{d+6}{2} \right] \\ -\, 4 \, \alpha & \text{if } \alpha \in (-\infty, -(d+2)] \end{cases}$$

Corollary 15. *Let $M > 0$ and f be a function in $L^2(B_\sigma^{2-m}\,dx)$ such that*

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f\, B_\sigma^{2-m}\, dx = (0,0,0) \quad \textit{and} \quad \nabla f \in L^2(B_\sigma\, dx)$$

Then

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, B_\sigma^{2-m}\, dx \leq \sigma^{\frac{d}{2}(m-m_c)} \int_{\mathbb{R}^d} |\nabla f|^2\, B_\sigma\, dx$$

The proof relies on a simple change of variables

Estimates on the second moment

Up to now, we have not determined the behaviour of $R(\tau)$ as $\tau \to \infty$, nor the fact that σ has a finite, positive limit as $t \to \infty$

Using the results of [Bonforte, J.D., Grillo,Vázquez] we find that

Lemma 16. *With the above notations,*

$$R(\tau) \sim \tau^{\frac{1}{d\,(m-m_c)}} \quad \textit{as} \quad \tau \to \infty$$

and, as a function of t, $t \mapsto \sigma(t)$ is positive, decreasing, with

$$\lim_{t\to\infty} \sigma(t) =: \sigma_\infty > 0$$

Notice that the value of σ_∞ is not known.

Rates of convergence

Known estimates

$$\mathcal{F}_\sigma[v] \le \frac{h^{2-m}\,[1+X(h)]}{2\left[\Lambda_{\alpha,d} - \sigma\,Y(h)\right]}\, m\,\sigma^{\frac{d}{2}(m-m_c)}\,\mathcal{I}_\sigma[v]$$

as soon as $0 < h < h_* := \min\{h > 0\,:\,\Lambda_{\alpha,d} - \sup_{t\in\mathbb{R}^+}\sigma(t)\,Y(h) \ge 0\}$, and

$$0 \le h(t) - 1 \le \mathsf{C}\,\mathcal{F}_{\sigma(t)}[v(t,\cdot)]^{\frac{1-m}{d+2-(d+1)m}}$$

Summarizing $\limsup_{t\to\infty}\,\mathrm{e}^{2\,\gamma(m)\,t}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] < \infty$ because

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] \le -\underbrace{2\,\frac{\Lambda_{\alpha,d} - \sigma(t)\,Y(h)}{\left[1+X(h)\right]\,h^{2-m}}\,(1-m)}_{\sim 2\,\gamma(m):=2\,(1-m)\,\Lambda_{1/(m-1),d}}\,\mathcal{F}_{\sigma(t)}[v(t,\cdot)]$$

Notice that for some constant $c > 0$, $\lim_{t\to\infty} e^{c\,t}(h(t)-1) = 0$, so that the rate is exactly $\gamma(m)$

Improved rates of convergence

Theorem 17. *Assume that* $m \in (\widetilde{m}_1, 1)$*,* $d \geq 2$*. Let* $v_0 \in L^1_+(\mathbb{R}^d)$ *be such that* v_0^m *and* $|y|^2\, v_0$ *are integrable*

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,\gamma(m)\,t} \quad \forall\, t \geq 0$$

where

$$\gamma(m) = \begin{cases} \frac{((d-2)\,m-(d-4))^2}{4\,(1-m)} & \text{if } m \in (\widetilde{m}_1, \widetilde{m}_2] \\ 4\,(d+2)\,m - 4\,d & \text{if } m \in [\widetilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$

Once a relative entropy estimate is known, it is possible to control the rate of decay of $u - B_{\sigma(t)}$ in various norms, for instance in C^k or in $L^q(\mathbb{R}^d,\, dx)$ for

$$q \geq \max\left\{1,\, \frac{d\,(1-m)}{2\,(2-m)+d\,(1-m)}\right\}$$

A graphical summary $(d = 5)$