Interpolation inequalities and spectral estimates for magnetic operators

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Outline

- **Magnetic rings**
  ▷ Results: a one-dimensional magnetic interpolation inequality
  ▷ Consequences: Keller-Lieb-Thirring estimates, Bohm-Aharonov magnetic fields and a new Hardy inequality in $\mathbb{R}^2$
- **Interpolation inequalities and spectral estimates for magnetic operators in dimensions 2 and 3**
  ▷ Theoretical results
  ▷ Estimates, numerics and a conjecture

Preliminaries: a simple interpolation on the circle

On $(-\pi,\pi] \approx \mathbb{S}^1 \ni s$, let us consider the uniform probability measure $d\sigma = ds/(2\pi)$ and denote by $\|\psi\|_{L^p(\mathbb{S}^1)}$ the corresponding L^p norm
The inequality

$$\|\psi'\|^2_{L^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{L^2(\mathbb{S}^1)} \geq \mu_{0,p}(\alpha)\,\|\psi\|^2_{L^p(\mathbb{S}^1)} \tag{1}$$

holds for some concave function $\alpha \mapsto \mu_{0,p}(\alpha)$ on $(0,+\infty)$

Lemma

- *If $p > 2$ and $0 < \alpha \leq 1/(p-2)$, then $\mu_{0,p}(\alpha) = \alpha$*
- *If $p = -2$ and $\alpha = 1/(p-2) = -1/4$, then $\mu_{0,p}(-1/4) = -1/4$*

In both cases, equality achieved only by constant functions

Case $p = -2$ (Exner, Harrell, Loss, 1998)):

$$\|\psi'\|^2_{L^2(\mathbb{S}^1)} + \frac{1}{4}\,\|\psi\|^2_{L^p(\mathbb{S}^1)} \geq \frac{1}{4}\,\|\psi\|^2_{L^2(\mathbb{S}^1)}$$

Case $p > 2$: Bakry-Emery method applies to *Kolmogorov's inequality*

Carré du champ method

Let $\mathcal{F}[u] := \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{p-2}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\right)$ and consider a positive solution of the parabolic equation

$$\frac{\partial u}{\partial t} = u'' + (p-1)\,\frac{|u'|^2}{u}$$

If $p = -2$ (new application of the *carré du champ* method)

$$-\frac{d}{dt}\mathcal{F}[u(t,\cdot)] = \underbrace{\int_{-\pi}^{\pi}\left(|u''|^2 - |u'|^2\right)\mathrm{d}\sigma}_{\geq 0\,\text{(Poincaré)}} + \int_{-\pi}^{\pi}\frac{|u'|^4}{u^2}\,\mathrm{d}\sigma$$

If $p > 1$, $p \neq 2$, the method is well known (Bakry, Emery, 85)

Magnetic rings

▷ A magnetic interpolation inequality on $\mathbb{S}^1$: with $p > 2$

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \geq \mu_{a,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

▷ Consequences

- A Keller-Lieb-Thirring inequality
- A new Hardy inequality for Bohm-Aharonov magnetic fields in $\mathbb{R}^2$

Magnetic flux, a reduction

Assume that $a:\mathbb{R}\to\mathbb{R}$ is a 2π-periodic function such that its restriction to $(-\pi,\pi]\approx\mathbb{S}^1$ is in $\mathrm{L}^1(\mathbb{S}^1)$ and define the space

$$X_a := \left\{\psi\in C_{\rm per}(\mathbb{R})\,:\,\psi'+i\,a\,\psi\in\mathrm{L}^2(\mathbb{S}^1)\right\}$$

- A standard change of gauge (see *e.g.* (Ilyin, Laptev, Loss, Zelik, 2016))

$$\psi(s)\mapsto e^{\,i\int_{-\pi}^{s}(a(s)-\bar a)\,\mathrm{d}\sigma}\,\psi(s)$$

where $\bar a:=\int_{-\pi}^{\pi}a(s)\,\mathrm{d}\sigma$ is the *magnetic flux*, reduces the problem to

$$\textit{a is a constant function}$$

- For any $k\in\mathbb{Z}$, ψ by $s\mapsto e^{iks}\,\psi(s)$ shows that $\mu_{a,p}(\alpha)=\mu_{k+a,p}(\alpha)$

$$a\in[0,1]$$

- $\mu_{a,p}(\alpha)=\mu_{1-a,p}(\alpha)$ because
$|\psi'+i\,a\,\psi|^2=|\chi'+i\,(1-a)\,\chi|^2=\left|\overline{\psi}'-i\,a\,\overline{\psi}\right|^2$ if $\chi(s)=e^{-is}\,\overline{\psi(s)}$

$$a\in[0,1/2]$$

Optimal interpolation

We want to characterize the *optimal constant* in the inequality

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \geq \mu_{a,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

written for any $p > 2$, $\alpha \in (-a^2, +\infty)$, $\psi \in X_a$

$$\mu_{a,p}(\alpha) := \inf_{\psi \in X_a \setminus \{0\}} \frac{\int_{-\pi}^{\pi} \left(|\psi' + i\,a\,\psi|^2 + \alpha\,|\psi|^2\right) d\sigma}{\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

$p = -2 = 2\,d/(d-2)$ with $d = 1$ (Exner, Harrell, Loss, 1998)
$p = +\infty$ (Galunov, Olienik, 1995) (Ilyin, Laptev, Loss, Zelik, 2016)
$\lim_{\alpha \to -a^2} \mu_{a,p}(\alpha) = 0$ (JD, Esteban, Laptev, Loss, 2016)

Using a Fourier series $\psi(s) = \sum_{k\in\mathbb{Z}} \psi_k\,e^{iks}$, we obtain that

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \sum_{k\in\mathbb{Z}} (a+k)^2\,|\psi_k|^2 \geq a^2\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

$\psi \mapsto \|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$ is coercive for any $\alpha > -a^2$

An interpolation result for the magnetic ring

Theorem

For any $p>2$, $a\in\mathbb{R}$, *and* $\alpha>-a^2$, $\mu_{a,p}(\alpha)$ *is achieved and*

(i) *if* $a\in[0,1/2]$ *and* $a^2\,(p+2)+\alpha\,(p-2)\le 1$, *then* $\mu_{a,p}(\alpha)=a^2+\alpha$ *and equality in* (1) *is achieved only by the constant functions*

(ii) *if* $a\in[0,1/2]$ *and* $a^2\,(p+2)+\alpha\,(p-2)>1$, *then* $\mu_{a,p}(\alpha)<a^2+\alpha$ *and equality in* (1) *is not achieved by the constant functions*

If $\alpha>-a^2$, $a\mapsto\mu_{a,p}(\alpha)$ *is monotone increasing on* $(0,1/2)$

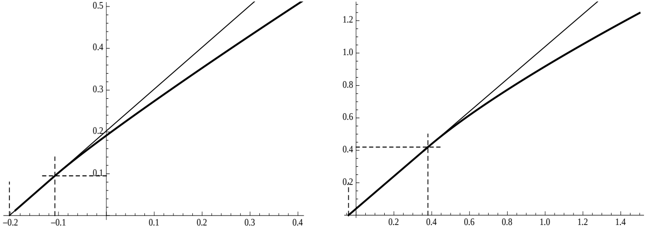

Figure: $\alpha\mapsto\mu_{a,p}(\alpha)$ with $p=4$ and (left) $a=0.45$ or (right) $a=0.2$

Reformulations of the interpolation problem (1/3)

Any minimizer $\psi \in X_a$ of $\mu_{a,p}(\alpha)$ satisfies the Euler-Lagrange equation

$$(H_a + \alpha)\,\psi = |\psi|^{p-2}\,\psi\,, \quad H_a\psi = -\left(\frac{d}{ds} + i\,a\right)^2 \psi$$

up to a multiplication by a constant and $v(s) = \psi(s)\,e^{ias}$ satisfies the condition

$$v(s+2\pi) = e^{2i\pi a}\,v(s) \quad \forall\, s \in \mathbb{R} \tag{2}$$

Hence

$$\mu_{a,p}(\alpha) = \min_{v \in Y_a \setminus \{0\}} \mathsf{Q}_{p,\alpha}[v]$$

where $Y_a := \left\{v \in C(\mathbb{R}) \,:\, v' \in \mathrm{L}^2(\mathbb{S}^1)\,,\ (2) \text{ holds}\right\}$ and

$$\mathsf{Q}_{p,\alpha}[v] := \frac{\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|v\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|v\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

Reformulations of the interpolation problem (2/3)

With $v = u\,e^{i\phi}$ the boundary condition becomes

$$u(\pi) = u(-\pi)\,, \quad \phi(\pi) = 2\pi\,(a+k) + \phi(-\pi) \tag{3}$$

for some $k \in \mathbb{Z}$, and $\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$
Hence

$$\mu_{a,p}(\alpha) = \min_{(u,\phi)\in Z_a\setminus\{0\}} \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

where $Z_a := \left\{(u,\phi) \in C(\mathbb{R})^2\,:\, u',\, u\,\phi' \in \mathrm{L}^2(\mathbb{S}^1)\,,\ (3)\ \text{holds}\right\}$

Reformulations of the interpolation problem (3/3)

We use the Euler-Lagrange equations

$$-\,u''+|\phi'|^2\,u+\alpha\,u=|u|^{p-2}\,u\quad\text{and}\quad(\phi'\,u^2)'=0$$

Integrating the second equation, and *assuming that u never vanishes*, we find a constant L such that $\phi'=L/u^2$. Taking (3) into account, we deduce from

$$L\int_{-\pi}^{\pi}\frac{\mathrm{d}s}{u^2}=\int_{-\pi}^{\pi}\phi'\,\mathrm{d}s=2\pi\,(a+k)$$

that

$$\|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}=L^2\int_{-\pi}^{\pi}\frac{\mathrm{d}\sigma}{u^2}=\frac{(a+k)^2}{\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}$$

Hence

$$\phi(s)-\phi(0)=\frac{a+k}{\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}\int_{-\pi}^{s}\frac{\mathrm{d}s}{u^2}$$

Let us define

$$\mathcal{Q}_{a,p,\alpha}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

Lemma

For any $a \in (0,1/2)$, $p > 2$, $\alpha > -a^2$,

$$\mu_{a,p}(\alpha) = \min_{u \in \mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\alpha}[u]$$

is achieved by a function $u > 0$

Proofs

- The existence proof is done on the original formulation of the problem using the diamagnetic inequality
- $\psi(s)\,e^{ias}=v_1(s)+i\,v_2(s)$, solves

$$-v_j''+\alpha\,v_j=(v_1^2+v_2^2)^{\frac{p}{2}-1}\,v_j\,,\quad j=1\,,\,2$$

and the Wronskian $w=(v_1\,v_2'-\,v_1'\,v_2)$ is constant so that $\psi(s)=0$ is incompatible with the twisted boundary condition

- if $a^2\,(p+2)+\alpha\,(p-2)\le 1$, then $\mu_{a,p}(\alpha)=a^2+\alpha$ because

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)}+\alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}=(1-4\,a^2)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+4\,a^2\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\tfrac14\,\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right)$$

if $a^2\,(p+2)+\alpha\,(p-2)>1$, the test function $u_\varepsilon:=1+\varepsilon\,w_1$

$$\mathcal{Q}_{a,p,\alpha}[u_\varepsilon]=a^2+\alpha+\big(1-a^2\,(p+2)-\,\alpha\,(p-2)\big)\,\varepsilon^2+o(\varepsilon^2)$$

proves the linear instability of the constants and $\mu_{a,p}(\alpha)<a^2+\alpha$

$$\mathcal{Q}_{a,p,\alpha}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}\,,$$

$$\mu_{a,p}(\alpha) = \min_{u\in\mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\alpha}[u]$$

$$\mathsf{Q}_{p,\alpha}[u] = \mathcal{Q}_{a=0,p,\alpha}[u]\,,\quad \nu_p(\alpha) := \inf_{v\in\mathrm{H}^1_0(\mathbb{S}^1)\setminus\{0\}} \mathsf{Q}_{p,\alpha}[v]$$

Proposition

$\forall\, p > 2,\ \alpha > -a^2$, *we have* $\mu_{a,p}(\alpha) < \mu_{1/2,p}(\alpha) \le \nu_p(\alpha) = \mu_{1/2,p}(\alpha)$

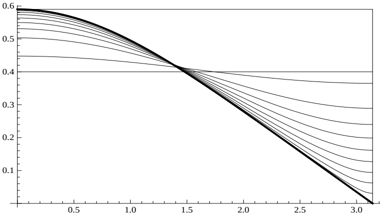

Figure: $p = 4$, $\alpha = 0$, $a = 0.40,\ 0.41,\ldots 0.49$; $u'' + u^{p-1} = 0$

A Keller-Lieb-Thirring inequality

Magnetic Schrödinger operator $H_a - \varphi = -\left(\frac{d}{ds} + i\,a\right)^2 \psi - \varphi$

- The function $\alpha \mapsto \mu_{a,p}(\alpha)$ is monotone increasing, concave, and therefore has an inverse, denoted by $\alpha_{a,p} : \mathbb{R}^+ \to (-a^2, +\infty)$, which is monotone increasing, and convex

Corollary

Let $p > 2$, $a \in [0, 1/2]$, $q = p/(p-2)$ and assume that φ is a non-negative function in $\mathrm{L}^q(\mathbb{S}^1)$. Then

$$\lambda_1(H_a - \varphi) \geq -\,\alpha_{a,p}\left(\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\right)$$

and $\alpha_{a,p}(\mu) = \mu - a^2$ iff $4\,a^2 + \mu\,(p-2) \leq 1$ (optimal φ is constant)

Equality is achieved

Bohm-Aharonov magnetic fields

On the two-dimensional Euclidean space $\mathbb{R}^2$, let us introduce the polar coordinates $(r,\vartheta)\in[0,+\infty)\times\mathbb{S}^1$ of $\mathbf{x}\in\mathbb{R}^2$ and consider a magnetic potential $\mathbf{a}$ in a transversal (Poincaré) gauge, or Poincaré gauge

$$(\mathbf{a},\mathbf{e}_r)=0 \quad \text{and} \quad (\mathbf{a},\mathbf{e}_\vartheta)=a_\vartheta(r,\vartheta)$$

Magnetic Schrödinger energy

$$\int_{\mathbb{R}^2}|(i\,\nabla+\mathbf{a})\,\Psi|^2\,dx=\int_0^{+\infty}\int_{-\pi}^{\pi}\left(|\partial_r\Psi|^2+\frac{1}{r^2}\,|\,\partial_\vartheta\Psi+i\,r\,a_\vartheta\,\Psi|^2\right)r\,d\vartheta\,dr$$

Bohm-Aharonov magnetic fields: $a_\vartheta(r,\vartheta)=a/r$ for some constant $a\in\mathbb{R}$ (a is the *magnetic flux*), with magnetic field $b=\operatorname{curl}\mathbf{a}$

$$\int_{\mathbb{R}^2}|(i\,\nabla+\mathbf{a})\,\Psi|^2\,dx\ge\tau\int_{\mathbb{R}^2}\frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\,|\Psi|^2\,d\mathbf{x}\quad\forall\,\varphi\in\mathrm{L}^q(\mathbb{S}^1)\,,\quad q\in(1,+\infty)$$

$$\Longrightarrow\tau=\tau\left(a,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\right)\ ?$$

Hardy inequalities

(Hoffmann-Ostenhof, Laptev, 2015) proved Hardy's inequality

$$\int_{\mathbb{R}^d} |\nabla \Psi|^2 \, d\mathbf{x} \geq \tau \int_{\mathbb{R}^d} \frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} |\Psi|^2 \, d\mathbf{x}$$

where the constant τ depends on the value of $\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^{d-1})}$ and $d \geq 3$

Bohm-Aharonov vector potential in dimension $d = 2$

$$\mathbf{a}(\mathbf{x}) = a \left(\frac{x_2}{|\mathbf{x}|^2}, \frac{-x_1}{|\mathbf{x}|^2} \right), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \,, \quad a \in \mathbb{R}$$

and recall the inequality (Laptev, Weidl, 1999)

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2 \, d\mathbf{x} \geq \min_{k \in \mathbb{Z}} (a-k)^2 \int_{\mathbb{R}^2} \frac{|\Psi|}{|\mathbf{x}|^2} \, d\mathbf{x}$$

A new Hardy inequality

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2\,d\mathbf{x} \geq \tau \int_{\mathbb{R}^2} \frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\,|\Psi|^2\,\mathrm{d}\mathbf{x} \quad \forall\,\varphi \in \mathrm{L}^q(\mathbb{S}^1)\,, \quad q \in (1,+\infty)$$

Corollary

Let $p > 2$, $a \in [0,1/2]$, $q = p/(p-2)$ and assume that φ is a non-negative function in $\mathrm{L}^q(\mathbb{S}^1)$. Then the inequality holds with $\tau > 0$ given by

$$\alpha_{a,p}\left(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\right) = 0$$

Moreover, $\tau = a^2/\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$ if $4\,a^2 + \|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2) \leq 1$

For any $a \in (0,1/2)$, by taking φ constant, small enough in order that $4\,a^2 + \|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2) \leq 1$, we recover the inequality

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2\,\mathrm{d}\mathbf{x} \geq a^2 \int_{\mathbb{R}^2} \frac{|\Psi|^2}{|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}$$

Proofs (Keller-Lieb-Thirring inequality)

Hölder's inequality

$$\|\psi'+i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}-\int_{-\pi}^{\pi}\varphi\,|\psi|^2\,d\sigma\geq\|\psi'+i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}-\mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

where $\mu=\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$ and $\frac{1}{q}+\frac{2}{p}=1$: choose $\mu_{a,p}(\alpha)=\mu$

$$\|\psi'+i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}-\mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\geq-\alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

Proofs (Hardy inequality)

Let $\tau \geq 0$, $\mathbf{x} = (r, \vartheta) \in \mathbb{R}^2$ be polar coordinates in $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} \left(|(i\,\nabla + \mathbf{a})\,\Psi|^2 - \tau\,\frac{\varphi}{|x|^2}\,|\Psi|^2 \right) d\mathbf{x}$$

$$= \int_0^\infty \int_{\mathbb{S}^1} \Big(\underbrace{r\,|\partial_r \Psi|^2}_{\geq 0} + \frac{1}{r}\,|\partial_\vartheta \Psi + i\,a\,\Psi|^2 - \tau\,\frac{\varphi}{r}\,|\Psi|^2 \Big) d\vartheta\,dr$$

$$\geq \lambda_1\,(H_a - \tau\,\varphi) \int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r}\,|\Psi|^2\,d\vartheta\,dr$$

$$\geq -\,\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}) \int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r}\,|\Psi|^2\,d\vartheta$$

- If $\tau = 0$, then $\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}) = \alpha_{a,p}(0) = -\,a^2$
- $\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}) > 0$ for τ large

$\Longrightarrow$ $\exists!\,\tau > 0$ such that $\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}) = 0$

Comments

▷ The region $a^2\,(p+2)+\alpha\,(p-2)<1$ is exactly the set where the constant functions are linearly stable critical points

▷ The proof of the *rigidity result* is based
- neither on the *carré du champ* method, at least directly
- nor on a Fourier representation of the operator as it was the case in earlier proofs ($p=+\infty$, or $p>2$ and $\alpha=0$)

▷ Magnetic rings: see (Bonnaillie-Noël, Hérau, Raymond, 2017)

▷ Deducing *Hardy's inequality* applied with *Bohm-Aharonov* magnetic fields from a *Keller-Lieb-Thirring inequality* is an extension of (Hoffmann-Ostenhof, Laptev, 2015) to the magnetic case

▷ Our results are not limited to the semi-classical regime

Magnetic interpolation inequalities in the Euclidean space

▷ Three interpolation inequalities and their dual forms

▷ Estimates in dimension $d = 2$ for constant magnetic fields

- Lower estimates
- Upper estimates and numerical results
- A linear stability result (numerical) and an open question

- Warning: assumptions are not repeated
- Estimates are given only in the case $p > 2$ but similar estimates hold in the other cases

Magnetic Laplacian and spectral gap

In dimensions $d=2$ and $d=3$: the *magnetic Laplacian* is

$$-\Delta_{\mathbf{A}}\,\psi=-\Delta\,\psi-\,2\,i\,\mathbf{A}\cdot\nabla\psi+|\mathbf{A}|^2\psi-\,i\,(\operatorname{div}\mathbf{A})\,\psi$$

where the magnetic potential (resp. field) is $\mathbf{A}$ (resp. $\mathbf{B}=\operatorname{curl}\mathbf{A}$) and

$$\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d):=\left\{\psi\in\mathrm{L}^2(\mathbb{R}^d)\,:\,\nabla_{\mathbf{A}}\psi\in\mathrm{L}^2(\mathbb{R}^d)\right\}\,,\quad\nabla_{\mathbf{A}}:=\nabla+\,i\,\mathbf{A}$$

Spectral gap inequality

$$\|\nabla_{\mathbf{A}}\psi\|_2^2\ge\Lambda[\mathbf{B}]\,\|\psi\|_2^2\quad\forall\,\psi\in\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)\tag{4}$$

- Λ depends only on $\mathbf{B}=\operatorname{curl}\mathbf{A}$
- Assumption: *equality in* (4) *holds for some* $\psi\in\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$
- If $\mathbf{B}$ is a constant magnetic field, $\Lambda[\mathbf{B}]=|\mathbf{B}|$
- If $d=2$, $\operatorname{spec}(-\Delta_{\mathbf{A}})=\{(2j+1)\,|\mathbf{B}|\,:\,j\in\mathbb{N}\}$ is generated by the *Landau levels*. The *Lowest Landau Level* corresponds to $j=0$

Magnetic interpolation inequalities

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 \ge \mu_{\mathbf{B}}(\alpha)\,\|\psi\|_p^2 \quad \forall\,\psi\in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) \tag{5}$$

for any $\alpha\in(-\Lambda[\mathbf{B}],+\infty)$ and any $p\in(2,2^*)$,

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \beta\,\|\psi\|_p^2 \ge \nu_{\mathbf{B}}(\beta)\,\|\psi\|_2^2 \quad \forall\,\psi\in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) \tag{6}$$

for any $\beta\in(0,+\infty)$ and any $p\in(1,2)$

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 \ge \gamma\int_{\mathbb{R}^d}|\psi|^2\,\log\left(\frac{|\psi|^2}{\|\psi\|_2^2}\right)dx+\xi_{\mathbf{B}}(\gamma)\,\|\psi\|_2^2 \quad \forall\,\psi\in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) \tag{7}$$

(limit case corresponding to $p=2$) for any $\gamma\in(0,+\infty)$

$$\mathsf{C}_p := \begin{cases} \min_{u\in\mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}}\frac{\|\nabla u\|_2^2+\|u\|_2^2}{\|u\|_p^2} & \text{if}\quad p\in(2,2^*)\\ \min_{u\in\mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}}\frac{\|\nabla u\|_2^2+\|u\|_p^2}{\|u\|_2^2} & \text{if}\quad p\in(1,2)\end{cases}$$

$\mu_{\mathbf{0}}(1)=\mathsf{C}_p$ if $p\in(2,2^*)$, $\nu_{\mathbf{0}}(1)=\mathsf{C}_p$ if $p\in(1,2)$
$\xi_{\mathbf{0}}(\gamma)=\gamma\,\log\left(\pi\,e^2/\gamma\right)$ if $p=2$

Technical assumptions

$\mathbf{A} \in \mathrm{L}^\alpha_{\mathrm{loc}}(\mathbb{R}^d)$, $\alpha > 2$ if $d=2$ or $\alpha = 3$ if $d=3$ and

$$\lim_{\sigma \to +\infty} \sigma^{d-2} \int_{\mathbb{R}^d} |\mathbf{A}(x)|^2 \, e^{-\sigma\,|x|} \, dx = 0 \quad \text{if} \quad p \in (2, 2^*)$$

$$\lim_{\sigma \to +\infty} \frac{\sigma^{\frac{d}{2}-1}}{\log \sigma} \int_{\mathbb{R}^d} |\mathbf{A}(x)|^2 \, e^{-\sigma\,|x|^2} \, dx = 0 \quad \text{if} \quad p = 2$$

$$\lim_{\sigma \to +\infty} \sigma^{d-2} \int_{|x|<1/\sigma} |\mathbf{A}(x)|^2 \, dx \quad \text{if} \quad p \in (1, 2)$$

A statement

Theorem

$p \in (2, 2^*)$: $\mu_{\mathbf{B}}$ is monotone increasing on $(-\Lambda[\mathbf{B}], +\infty)$, concave and

$$\lim_{\alpha \to (-\Lambda[\mathbf{B}])_+} \mu_{\mathbf{B}}(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to +\infty} \mu_{\mathbf{B}}(\alpha)\, \alpha^{\frac{d-2}{2} - \frac{d}{p}} = \mathsf{C}_p$$

$p \in (1, 2)$: $\nu_{\mathbf{B}}$ is monotone increasing on $(0, +\infty)$, concave and

$$\lim_{\beta \to 0_+} \nu_{\mathbf{B}}(\beta) = \Lambda[\mathbf{B}] \quad \text{and} \quad \lim_{\beta \to +\infty} \nu_{\mathbf{B}}(\beta)\, \beta^{-\frac{2\,p}{2\,p + d\,(2-p)}} = \mathsf{C}_p$$

$\xi_{\mathbf{B}}$ is continuous on $(0, +\infty)$, concave, $\xi_{\mathbf{B}}(0) = \Lambda[\mathbf{B}]$ and

$$\xi_{\mathbf{B}}(\gamma) = \tfrac{d}{2}\, \gamma\, \log\left(\tfrac{\pi\, e^2}{\gamma}\right)(1 + o(1)) \quad \text{as} \quad \gamma \to +\infty$$

Constant magnetic fields: equality is achieved
Nonconstant magnetic fields: only partial answers are known

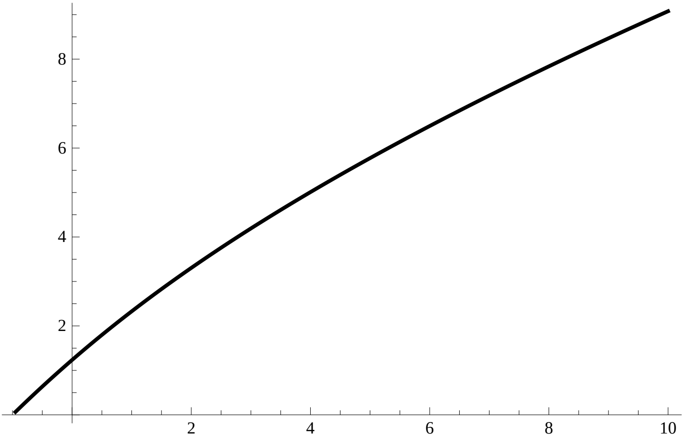

Figure: Case $d = 2$, $p = 3$, $B = 1$: plot of $\alpha \mapsto (2\,\pi)^{\frac{2}{p}-1}\,\mu_{\mathbf{B}}(\alpha)$

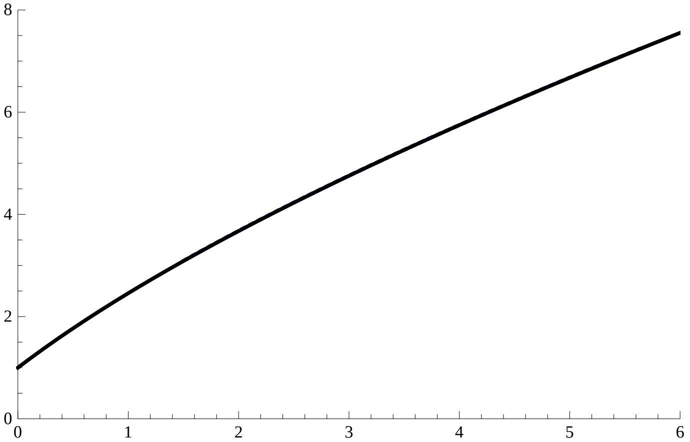

Figure: Case $d = 2$, $p = 1.4$, $B = 1$: plot of $\beta \mapsto \nu_{\mathbf{B}}(\beta)$. The horizontal axis is measured in units of $(2\,\pi)^{1-\frac{2}{p}}\,\beta$

Magnetic Keller-Lieb-Thirring inequalities

$\lambda_{\mathbf{A},V}$ is the principal eigenvalue of $-\Delta_{\mathbf{A}}+V$
$\alpha_{\mathbf{B}}:(0,+\infty)\to(-\Lambda,+\infty)$ is the inverse function of $\alpha\mapsto\mu_{\mathbf{B}}(\alpha)$

Corollary

(i) *For any* $q=p/(p-2)\in(d/2,+\infty)$ *and any potential* $0\ge V\in\mathrm{L}^q(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},V}\ge-\alpha_{\mathbf{B}}(\|V\|_q)$$

$\lim_{\mu\to0_+}\alpha_{\mathbf{B}}(\mu)=\Lambda$ *and* $\lim_{\mu\to+\infty}\alpha_{\mathbf{B}}(\mu)\,\mu^{\frac{2\,(q+1)}{d-2-2\,q}}=-\,\mathsf{C}_p^{\frac{2\,(q+1)}{d-2-2\,q}}$

(ii) *For any* $q=p/(2-p)\in(1,+\infty)$ *and any* $0<W^{-1}\in\mathrm{L}^q(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},W}\ge\nu_{\mathbf{B}}\left(\|W^{-1}\|_q^{-1}\right)$$

(iii) *For any* $\gamma>0$ *and any* $W\ge0$ *s.t.* $e^{-W/\gamma}\in\mathrm{L}^1(\mathbb{R}^d)$

$$\lambda_{\mathbf{A},W}\ge\xi_{\mathbf{B}}(\gamma)-\gamma\log\left(\int_{\mathbb{R}^d}e^{-W/\gamma}\,dx\right)$$

A more general duality result

Proposition

Let $d=2$ or 3. Let $\phi \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ be an arbitrary potential

(i) *If $q>d/2$, $p=\frac{2\,q}{q-1}$, we have*

$$\lambda_{\mathbf{A},\phi} \geq -\left(\alpha_{\mathbf{B}}\left(\|(\lambda-\phi)\|_{q,+}\right)-\lambda\right)$$

(ii) *If $q \in (1,+\infty)$, $p=\frac{2\,q}{q+1}$, we have*

$$\lambda_{\mathbf{A},\phi} \geq \lambda+\nu_{\mathbf{B}}\left(\|(\phi-\lambda)^{-1}\|_q^{-1}\right)$$

These estimates hold for any $\lambda \in \mathbb{R}$ such that all above norms are well defined, with the additional condition that $\phi \geq \lambda$ a.e. in Case (ii)

Preliminaries: interpolation without magnetic field

Assume that $p>2$ and let C_p denote the best constant in

$$\|\nabla u\|_2^2+\|u\|_2^2\ge\mathsf{C}_p\,\|u\|_p^2\quad\forall\,u\in\mathrm{H}^1(\mathbb{R}^d)$$

By scaling, if we test the inequality by $u\big(\cdot/\lambda\big)$, we find that

$$\|\nabla u\|_2^2+\lambda^2\,\|u\|_2^2\ge\mathsf{C}_p\,\lambda^{2-d\,(1-\frac{2}{p})}\,\|u\|_p^2\quad\forall\,u\in\mathrm{H}^1(\mathbb{R}^d)\quad\forall\,\lambda>0$$

An optimization on $\lambda>0$ shows that the best constant in the scale-invariant inequality

$$\|\nabla u\|_2^{d\,(1-\frac{2}{p})}\,\|u\|_2^{2-d\,(1-\frac{2}{p})}\ge\mathsf{S}_p\,\|u\|_p^2\quad\forall\,u\in\mathrm{H}^1(\mathbb{R}^d)$$

is given by

$$\mathsf{S}_p=\tfrac{1}{2\,p}\,\big(2\,p-d\,(p-2)\big)^{1-d\,\frac{p-2}{2\,p}}\,\big(d\,(p-2)\big)^{\frac{d\,(p-2)}{2\,p}}\,\mathsf{C}_p$$

... and with magnetic field

Proposition

Let $d=2$ or 3. For any $p\in(2,+\infty)$, any $\alpha>-\Lambda=-\Lambda[\mathbf{B}]<0$

$$\mu_{\mathbf{B}}(\alpha)\ge\mu_{\rm interp}(\alpha):=\begin{cases}\mathsf{S}_p\,(\alpha+\Lambda)\,\Lambda^{-d\,\frac{p-2}{2\,p}} & \text{if}\,\alpha\in\left[-\Lambda,\frac{\Lambda\,(2\,p-d\,(p-2))}{d\,(p-2)}\right]\\ \mathsf{C}_p\,\alpha^{1-d\,\frac{p-2}{2\,p}} & \text{if}\,\alpha\ge\frac{\Lambda\,(2\,p-d\,(p-2))}{d\,(p-2)}\end{cases}$$

Let $t\in[0,1]$. From the diamagnetic inequality $\|\nabla|\psi|\|_2\le\|\nabla_{\mathbf{A}}\psi\|_2$ and from the inequality with $\lambda=\frac{\alpha+\Lambda\,t}{1-t}$, we deduce that

$$\|\nabla_{\mathbf{A}}\psi\|_2^2+\alpha\,\|\psi\|_2^2\ge t\,\left(\|\nabla_{\mathbf{A}}\psi\|_2^2-\Lambda\,\|\psi\|_2^2\right)$$
$$+(1-t)\,\left(\|\nabla|\psi|\|_2+\frac{\alpha+\Lambda\,t}{1-t}\,\|\psi\|_2^2\right)$$
$$\ge\mathsf{C}_p\,(1-t)^{\frac{d\,(p-2)}{2\,p}}\,(\alpha+t\,\Lambda)^{1-d\,\frac{p-2}{2\,p}}\,\|\psi\|_p^2$$

and optimize on $t\in[\max\{0,-\alpha/\Lambda\},1]$

Lower estimates: $d = 2$, constant magnetic field

Assume that $\mathbf{B} = (0, B)$ is constant, $d = 2$ and choose

$$\mathbf{A}_1 = \tfrac{B}{2} x_2\,, \quad \mathbf{A}_2 = -\tfrac{B}{2} x_1 \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2$$

Proposition

(Loss, Thaller, 1997) Consider a constant magnetic field with field strength B in two dimensions. For every $c \in [0,1]$, we have

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2\,dx \ge \left(1 - c^2\right) \int_{\mathbb{R}^2} |\nabla \psi|^2\,dx + c\,B \int_{\mathbb{R}^2} \psi^2\,dx$$

and equality holds with $\psi = u\,e^{iS}$ and $u > 0$ if and only if

$$\left(-\partial_2 u^2,\, \partial_1 u^2\right) = \frac{2\,u^2}{c}\,(\mathbf{A} + \nabla S)$$

... a computation ($d = 2$, constant magnetic field)

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2\,dx = \int_{\mathbb{R}^2} |\nabla u|^2\,dx + \int_{\mathbb{R}^2} |\mathbf{A} + \nabla S|^2\,u^2\,dx$$

$$= (1 - c^2) \int_{\mathbb{R}^2} |\nabla u|^2\,dx + \underbrace{\int_{\mathbb{R}^2} \left(c^2\,|\nabla u|^2 + |\mathbf{A} + \nabla S|^2\,u^2\right) dx}_{\geq \int_{\mathbb{R}^2} 2\,c\,|\nabla u|\,|\mathbf{A}+\nabla S|\,u\,dx}$$

with equality only if $c\,|\nabla u| = |\mathbf{A} + \nabla S|\,u$

$$2\,|\nabla u|\,|\mathbf{A} + \nabla S|\,u = |\nabla u^2|\,|\mathbf{A} + \nabla S| \geq \left(\nabla u^2\right)^{\perp} \cdot (\mathbf{A} + \nabla S)$$

where $\left(\nabla u^2\right)^{\perp} := \left(-\partial_2 u^2,\, \partial_1 u^2\right)$
Equality case: $\left(-\partial_2 u^2,\, \partial_1 u^2\right) = \gamma\,(\mathbf{A} + \nabla S)$ for $\gamma = 2\,u^2/c$
Integration by parts yields

$$\int_{\mathbb{R}^2} \left(c^2\,|\nabla u|^2 + |\mathbf{A} + \nabla S|^2\,u^2\right) dx \geq B\,c \int_{\mathbb{R}^2} u^2\,dx$$

Lower estimate ($d=2$, constant magnetic field): a result

Proposition

Consider a constant magnetic field with field strength B in two dimensions. Given any $p\in(2,+\infty)$, and any $\alpha>-B$, we have

$$\mu_{\mathbf{B}}(\alpha)\ge \mathsf{C}_p\left(1-c^2\right)^{1-\frac{2}{p}}\,(\alpha+c\,B)^{\frac{2}{p}}=:\mu_{\mathrm{LT}}(\alpha)$$

with

$$c=c(p,\eta)=\frac{\sqrt{\eta^2+p-1}-\eta}{p-1}=\frac{1}{\eta+\sqrt{\eta^2+p-1}}\in(0,1)$$

and $\eta=\alpha\,(p-2)/(2\,B)$

Upper estimate (1): $d = 2$, constant magnetic field

For every integer $k \in \mathbb{N}$ we introduce the special symmetry class

$$\psi(x) = \left(\tfrac{x_2 + i\,x_1}{|x|}\right)^k v(|x|) \quad \forall\, x = (x_1, x_2) \in \mathbb{R}^2 \qquad (\mathcal{C}_k)$$

(Esteban, Lions, 1989): if $\psi \in \mathcal{C}_k$, then

$$\frac{1}{2\,\pi} \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2\,dx = \int_0^{+\infty} |v'|^2\,r\,dr + \int_0^{+\infty} \left(\tfrac{k}{r} - \tfrac{B\,r}{2}\right)^2 |v|^2\,r\,dr$$

and optimality is achieved in $\mathcal{C}_k$

Test function $v_\sigma(r) = e^{-\,r^2/(2\,\sigma)}$: an optimization on $\sigma > 0$ provides an explicit expression of $\mu_{\mathrm{Gauss}}(\alpha)$ such that

Proposition

If $p > 2$, then

$$\mu_{\mathbf{B}}(\alpha) \le \mu_{\mathrm{Gauss}}(\alpha) \quad \forall\, \alpha > -\Lambda[\mathbf{B}]$$

This estimate is not optimal because v_σ does not solve the Euler-Lagrange equations

Upper estimate (2): $d=2$, constant magnetic field

A more numerical point of view. The Euler-Lagrange equation in $\mathcal{C}_0$ is

$$-v''-\frac{v'}{r}+\left(\frac{B^2}{4}\,r^2+\alpha\right)v=\mu_{\rm EL}(\alpha)\left(\int_0^{+\infty}|v|^p\,r\,dr\right)^{\frac{2}{p}-1}|v|^{p-2}\,v$$

We can restrict the problem to positive solutions such that

$$\mu_{\rm EL}(\alpha)=\left(\int_0^{+\infty}|v|^p\,r\,dr\right)^{1-\frac{2}{p}}$$

and then we have to solve the reduced problem

$$-v''-\frac{v'}{r}+\left(\frac{B^2}{4}\,r^2+\alpha\right)v=|v|^{p-2}\,v$$

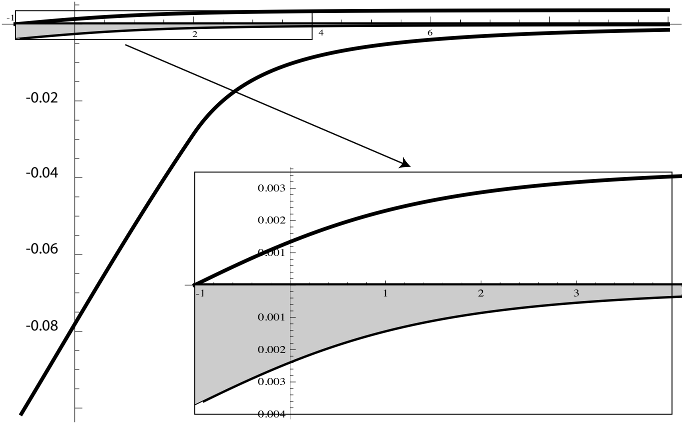

Figure: Case $d = 2$, $p = 3$, $B = 1$: comparison of the upper estimates $\alpha \mapsto \mu_{\text{Gauss}}(\alpha)$ and $\alpha \mapsto \mu_{\text{EL}}(\alpha)$ with the lower estimates $\alpha \mapsto \mu_{\text{interp}}(\alpha)$ and $\alpha \mapsto \mu_{\text{LT}}(\alpha)$
Plots represent the curves $\log_{10}(\mu_{\text{Gauss}}/\mu_{\text{EL}})$, $\log_{10}(\mu_{\text{LT}}/\mu_{\text{EL}})$ and $\log_{10}(\mu_{\text{interp}}/\mu_{\text{EL}})$ so that $\alpha \mapsto \mu_{\text{EL}}(\alpha)$ corresponds to a straight line at level 0. The exact value associated with $\mu_{\mathbf{B}}$ lies in the grey area

Asymptotics (1): *Lowest Landau Level*

Proposition

Let $d = 2$ and consider a constant magnetic field with field strength B. If ψ_α is a minimizer for $\mu_{\mathbf{B}}(\alpha)$ such that $\|\psi_\alpha\|_p = 1$, then there exists a non trivial $\varphi_\alpha \in \mathrm{LLL}$ such that

$$\lim_{\alpha\to(-B)_+} \|\psi_\alpha - \varphi_\alpha\|_{\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2)} = 0$$

Let $\psi_\alpha \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2)$ be an optimal function for (5) such that $\|\psi_\alpha\|_p = 1$ and let us decompose it as $\psi_\alpha = \varphi_\alpha + \chi_\alpha$, where $\varphi_\alpha \in \mathrm{LLL}$ and χ_α is in the orthogonal of LLL

$$\mu_{\mathbf{B}}(\alpha) \ge (\alpha+B)\,\|\varphi_\alpha\|_2^2 + (\alpha+3B)\,\|\chi_\alpha\|_2^2 \ge (\alpha+3B)\,\|\chi_\alpha\|_2^2 \sim 2B\,\|\chi_\alpha\|_2^2$$

as $\alpha \to (-B)_+$ because $\|\nabla\chi_\alpha\|_2^2 \ge 3B\,\|\chi_\alpha\|_2^2$
Since $\lim_{\alpha\to(-B)_+} \mu_{\mathbf{B}}(\alpha) = 0$, $\lim_{\alpha\to(-B)_+} \|\chi_\alpha\|_2 = 0$ and

$$\mu_{\mathbf{B}}(\alpha) = (\alpha+B)\,\|\varphi_\alpha\|_2^2 + \|\nabla_{\mathbf{A}}\,\chi_\alpha\|_2^2 + \alpha\,\|\chi_\alpha\|_2^2 \ge \tfrac{2}{3}\,\|\nabla_{\mathbf{A}}\,\chi_\alpha\|_2^2$$

concludes the proof

Asymptotics (2): *semi-classical regime*

Let us consider the small magnetic field regime. We assume that the magnetic potential is given by

$$\mathbf{A}_1=\tfrac{B}{2}x_2\,,\quad \mathbf{A}_2=-\tfrac{B}{2}x_1\quad\forall\,x=(x_1,x_2)\in\mathbb{R}^2$$

if $d=2$. In dimension $d=3$, we choose $\mathbf{A}=\frac{B}{2}(-x_2,x_1,0)$ and observe that the constant magnetic field is $\mathbf{B}=(0,0,B)$, while the spectral gap in (4) is $\Lambda[\mathbf{B}]=B$.

Proposition

Let $d=2$ or 3 and consider a constant magnetic field $\mathbf{B}$ of intensity B with magnetic potential $\mathbf{A}$
For any $p\in(2,2^*)$ and any fixed α and $\mu>0$, we have

$$\lim_{\varepsilon\to0_+}\mu_{\varepsilon\,\mathbf{B}}(\alpha)=\mathsf{C}_p\,\alpha^{\frac{d}{p}-\frac{d-2}{2}}$$

Consider any function $\psi\in\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$ and let $\psi(x)=\chi(\sqrt{\varepsilon}\,x)$, $\sqrt{\varepsilon}\,\mathbf{A}\big(x/\sqrt{\varepsilon}\big)=\mathbf{A}(x)$ with our conventions on $\mathbf{A}$

Numerical stability of radial optimal functions

Let us denote by ψ_0 an optimal function in $(\mathcal{C}_0)$ such that

$$-\psi_0''-\frac{\psi_0'}{r}+\left(\tfrac{B^2}{4}\,r^2+\alpha\right)\psi_0=|\psi_0|^{p-2}\,\psi_0$$

and consider the test function

$$\psi_\varepsilon=\psi_0+\varepsilon\,e^{i\,\theta}\,v$$

where $v=v(r)$ and $e^{i\,\theta}=(x_1+i\,x_2)/r$
As $\varepsilon\to 0_+$, the leading order term is

$$2\,\pi\left[\int_{\mathbb{R}^2}|v'|^2\,dx+\int_{\mathbb{R}^2}\left(\left(\tfrac{1}{r}-\tfrac{B\,r}{2}\right)^2+\alpha\right)|v|^2\,dx-\tfrac{p}{2}\int_0^{+\infty}|\psi_0|^{p-2}\,v^2\,r\,dr\right]\varepsilon^2$$

and we have to solve the eigenvalue problem

$$-v''-\frac{v'}{r}+\left(\left(\tfrac{1}{r}-\tfrac{B\,r}{2}\right)^2+\alpha\right)v-\tfrac{p}{2}\,|\psi_0|^{p-2}\,v=\mu\,v$$

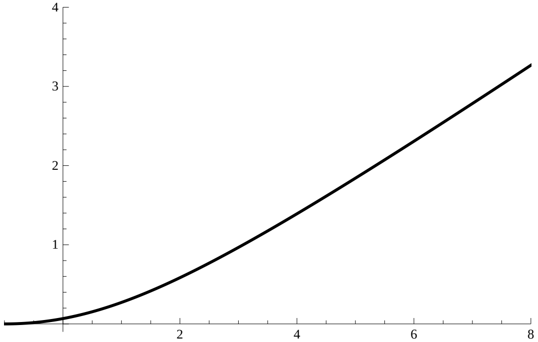

Figure: Case $p = 3$ and $B = 1$: plot of the eigenvalue μ as a function of α

A careful investigation shows that μ is always positive, including in the limiting case as $\alpha \to (-B)_+$, thus proving the numerical stability of the optimal function in $\mathcal{C}_0$ with respect to perturbations in $\mathcal{C}_1$

An open question of symmetry

- (Bonheure, Nys, Van Schaftingen, 2016) for a fixed $\alpha > 0$ and for $\mathbf{B}$ small enough, the optimal functions are radially symmetric functions, *i.e.*, belong to $\mathcal{C}_0$
This regime is equivalent to the regime as $\alpha \to +\infty$ for a given $\mathbf{B}$, at least if the magnetic field is constant
- Numerically our upper and lower bounds are (in dimension $d=2$, for a constant magnetic field) numerically extremely close
- The optimal function in $\mathcal{C}_0$ with respect to perturbations in $\mathcal{C}_1$

Prove that the optimality case is achieved among radial function if $d=2$ and $\mathbf{B}$ is a constant magnetic field

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Thank you for your attention !