When do spectral problems determine optimal constants in nonlinear interpolation inequalities ?

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Outline

▷ Interpolation inequalities on the sphere

▷ Caffarelli-Kohn-Nirenberg inequalities: the bifurcation point of view

▷ Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg inequalities: an approach based on nonlinear flows

Interpolation inequalities on the sphere

▷ A spectral point of view on fractional and non-fractional interpolation inequalities

▷ The *bifurcation* point of view

▷ Flows on the sphere

- *Carré du champ*
- Can one prove Sobolev's inequalities with a heat flow ?
- Some open problems: constraints and improved inequalities

[Beckner, 1993], [J.D., Zhang, 2016]
[Bakry, Emery, 1984]
[Bidault-Véron, Véron, 1991], [Bakry, Ledoux, 1996]
[Demange, 2008][J.D., Esteban, Loss, 2014 & 2015]

Non-fractional interpolation inequalities

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ induced by the Lebesgue measure on $\mathbb{R}^{d+1}$

$$1 \leq p < 2 \quad \text{or} \quad 2 < p \leq 2^* := \frac{2\,d}{d-2}$$

if $d \geq 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \setminus \{0\}$$

Optimal interpolation inequalities for fractional operators

- The sharp *Hardy-Littlewood-Sobolev inequality on* $\mathbb{S}^n$ [Lieb, 1983]

$$\iint_{\mathbb{S}^n\times\mathbb{S}^n} F(\zeta)\,|\zeta-\eta|^{-\lambda}\,F(\eta)\,d\mu(\zeta)\,d\mu(\eta) \le \frac{\Gamma(n)\,\Gamma\left(\frac{n-\lambda}{2}\right)}{2^\lambda\,\Gamma\left(\frac{n}{2}\right)\,\Gamma\left(\frac{n}{p}\right)}\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^n)}$$

$\lambda\in(0,n)$, $p=\frac{2\,n}{2\,n-\lambda}\in(1,2)$
$\lambda=\frac{2\,n}{q_\star}$ where $\frac{1}{p}+\frac{1}{q_\star}=1$

- A *subcritical interpolation inequality*

$d\mu$ is the uniform probability measure on $\mathbb{S}^n$
$\mathcal{L}_s$ is the fractional Laplace operator of order $s\in(0,n)$
$q\in[1,2)\cup(2,q_\star]$, $q_\star=\frac{2\,n}{n-s}$

$$\frac{\|F\|^2_{\mathrm{L}^q(\mathbb{S}^n)}-\|F\|^2_{\mathrm{L}^2(\mathbb{S}^n)}}{q-2}\le \mathsf{C}_{q,s}\int_{\mathbb{S}^n} F\,\mathcal{L}_s F\,d\mu\quad\forall\,F\in\mathrm{H}^{s/2}(\mathbb{S}^n)$$

The sharp constants

Theorem

[J.D., Zhang] *Let $n \geq 1$. If either $s \in (0, n]$, $q \in [1,2) \cup (2, q_\star]$, or $s = n$ and $q \in [1,2) \cup (2, \infty)$, then*

$$\mathsf{C}_{q,s} = \frac{n-s}{2\,s}\,\frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right)}$$

$$\mathsf{C}_{q,s}^{-1} = \lambda_1(\mathcal{L}_s) = \inf_{F \in \mathrm{H}^{s/2}(\mathbb{S}^n)\setminus\mathbb{R}} \mathcal{Q}[F]\,, \quad \mathcal{Q}[F] := \frac{(q-2)\int_{\mathbb{S}^n} F\,\mathcal{L}_s F\,d\mu}{\|F\|^2_{\mathrm{L}^q(\mathbb{S}^n)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^n)}}$$

- *Sharp subcritical fractional logarithmic Sobolev inequalities*

Corollary

[J.D., Zhang] *Let $s \in (0, n]$*

$$\int_{\mathbb{S}^n} |F|^2 \log\left(\frac{|F|}{\|F\|_{\mathrm{L}^2(\mathbb{S}^n)}}\right) d\mu \leq \mathsf{C}_{2,s} \int_{\mathbb{S}^n} F\,\mathcal{L}_s F\,d\mu \quad \forall\, F \in \mathrm{H}^{s/2}(\mathbb{S}^n)$$

From HLS to Sobolev

Lieb's approach... Decomposition on spherical harmonics:
$F=\sum_{k=0}^\infty F_{(k)}$
Funk-Hecke formula

$$\iint_{\mathbb{S}^n\times\mathbb{S}^n} F(\zeta)\,|\zeta-\eta|^{-\lambda}\,F(\eta)\,d\mu(\zeta)\,d\mu(\eta)$$
$$=\frac{\Gamma(n)\,\Gamma\big(\frac{n-\lambda}{2}\big)}{2^\lambda\,\Gamma\big(\frac{n}{2}\big)\,\Gamma\big(\frac{n}{p}\big)}\sum_{k=0}^\infty\frac{\Gamma(\frac{n}{p})\,\Gamma(\frac{n}{p'}+k)}{\Gamma(\frac{n}{p'})\,\Gamma(\frac{n}{p}+k)}\int_{\mathbb{S}^n}|F_{(k)}|^2\,d\mu$$

- The *fractional Sobolev inequality*

$$\|F\|^2_{\mathrm{L}^{q_\star}(\mathbb{S}^n)}\le\int_{\mathbb{S}^n}F\,\mathcal{K}_s\,F\,d\mu:=\sum_{k=0}^\infty\gamma_k\big(\tfrac{n}{q_\star}\big)\int_{\mathbb{S}^n}|F_{(k)}|^2\,d\mu$$

is dual of HLS, where $q_\star=\dfrac{2\,n}{n-s}$ is the critical exponent and

$$\gamma_k(x):=\frac{\Gamma(x)\,\Gamma(n-x+k)}{\Gamma(n-x)\,\Gamma(x+k)}=\frac{(n+k-1-x)\,(n+k-2-x)\dots(n-x)}{(k-1+x)\,(k-2+x)\dots x}$$

The subcritical inequalities

$$\mathcal{L}_s := \frac{1}{\kappa_{n,s}}\left(\mathcal{K}_s - \mathrm{Id}\right) \quad \text{with} \quad \kappa_{n,s} := \frac{\Gamma\left(\frac{n}{q_\star}\right)}{\Gamma\left(n-\frac{n}{q_\star}\right)} = \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right)}$$

Subcritical interpolation inequalities: $q \in [1,2) \cup (2, q_\star]$

$$\frac{\|F\|^2_{\mathrm{L}^q(\mathbb{S}^n)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^n)}}{q-2} \le \mathsf{C}_{q,s} \int_{\mathbb{S}^n} F\,\mathcal{L}_s F\,d\mu \quad \forall\, F \in \mathrm{H}^{s/2}(\mathbb{S}^n)$$

Lemma

[J.D., Zhang] *For any $n \ge 1$, the function $q \mapsto \dfrac{\gamma_k\left(\frac{n}{q}\right)-1}{q-2}$ is monotone increasing on $(1,\infty)$ for any $k \ge 2$*

- [Beckner, 1993]: if $q \in (2, q_\star(2)]$, $q_\star = q_\star(2) = 2\,n/(n-2)$, then

$$\delta_k(x) = \frac{1}{\kappa_{n,s}}\left(\gamma_k\left(\tfrac{n}{q}\right) - 1\right) =: \delta_k\left(\tfrac{n}{q}\right) \le \delta_k\left(\tfrac{n}{q_\star}\right) = k\,(k+n-1)$$

results in $\|F\|^2_{\mathrm{L}^q(\mathbb{S}^n)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^n)} \le \frac{q-2}{n}\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^n)}$

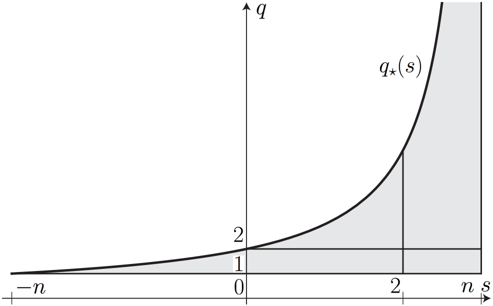


Figure: *The optimal constant $\mathsf{C}_{q,s}$ is independent of q and determined for any given s by the critical case $q = q_\star(s)$ which corresponds to the Hardy-Littlewood-Sobolev inequality if $s \in (-n, 0)$ and to the Sobolev inequality if $s \in (0, n)$*
The case $s = 0$ corresponds to the critical fractional logarithmic Sobolev inequality if $s = 0$ [Beckner, 1993] and the subcritical fractional logarithmic Sobolev inequality if $s \in (0, n]$.

Sketch of the proof

$q \mapsto \gamma_k(n/q)$ is strictly *convex* with respect to q iff

$$x\,\gamma_k'' + 2\,\gamma_k' > 0 \quad \forall\, x \in (0,n)$$

$\Longleftrightarrow \alpha_k(x) := -\frac{\gamma_k'(x)}{\gamma_k(x)} = \sum_{j=0}^{k-1} \beta_j(x)$ with $\beta_j(x) = \frac{1}{n+j-x} + \frac{1}{j+x}$ solves

$$\alpha_k^2 - \alpha_k' - \tfrac{2}{x}\,\alpha_k > 0$$

▷ A proof by induction: from

$$\alpha_k^2 \geq 2\,\beta_0 \sum_{j=1}^{k-1} \beta_j + \sum_{j=0}^{k-1} \beta_j^2$$

$$\beta_0^2 - \beta_0' - \tfrac{2}{x}\,\beta_0 = 0$$

and

$$2\,\beta_0\,\beta_j + \beta_j^2 - \beta_j' - \tfrac{2}{x}\,\beta_j = \frac{2\,(n+j)\,(n+2\,j)}{(n-x)\,(n+j-x)\,(j+x)^2}$$

we deduce that

$$\alpha_k^2 - \alpha_k' - \tfrac{2}{x}\,\alpha_k \geq \sum_{j=1}^{k-1} \frac{2\,(n+j)\,(n+2\,j)}{(n-x)\,(j+n-x)\,(j+x)^2} \quad \forall\, k \geq 2$$

The non-fractional interpolation inequalities (again)

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \ge 1$, $p \neq 2$, is such that

$$p \le 2^* := \frac{2\,d}{d-2}$$

if $d \ge 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2}\int_{\mathbb{S}^d} |u|^2 \,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}$$

The Bakry-Emery method

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and compute $\frac{d}{dt}\mathcal{E}_p[\rho] = -\mathcal{I}_p[\rho]$ and $\frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

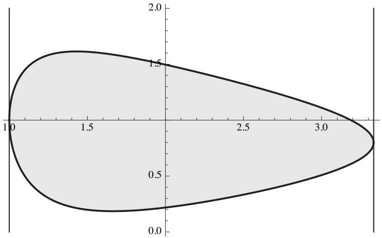
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m \,. \tag{1}$$

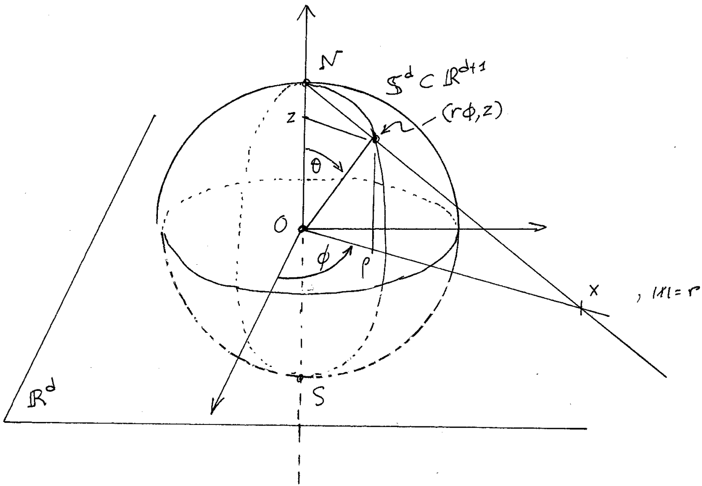
[Demange], [J.D., Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) *admissible region*, $d = 5$

Cylindrical coordinates, Schwarz symmetrization, stereographic projection...



... and the ultra-spherical operator

Change of variables $z=\cos\theta$, $v(\theta)=f(z)$, $d\nu_d := \nu^{\frac d2-1}\,dz/Z_d$, $\nu(z):=1-z^2$

The self-adjoint *ultraspherical* operator is

$$\mathcal L\,f := (1-z^2)\,f''-d\,z\,f' = \nu\,f''+\frac d2\,\nu'\,f'$$

which satisfies $\langle f_1,\mathcal L\,f_2\rangle = -\int_{-1}^1 f_1'\,f_2'\;\nu\;d\nu_d$

Proposition

Let $p\in[1,2)\cup(2,2^*]$, $d\ge1$. *For any* $f\in\mathrm H^1([-1,1],d\nu_d)$,

$$-\langle f,\mathcal L\,f\rangle = \int_{-1}^1|f'|^2\;\nu\;d\nu_d \ge d\,\frac{\|f\|^2_{\mathrm L^p(\mathbb S^d)}-\|f\|^2_{\mathrm L^2(\mathbb S^d)}}{p-2}$$

The heat equation $\frac{\partial g}{\partial t} = \mathcal{L}\, g$ for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f, \mathcal{L}\, f\rangle = \langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\left\langle \frac{|f'|^2}{f}\,\nu, \mathcal{L}\, f\right\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\,d\,\mathcal{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$

$$= -\,2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} = 2^{\#} < \frac{2\,d}{d-2} = 2^*$$

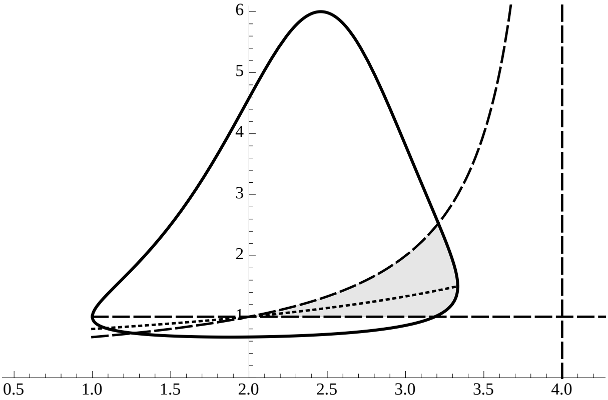
Improved functional inequalities

- The range $2^\# < p \le 2^*$ is covered using the adapted fast diffusion eq.

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

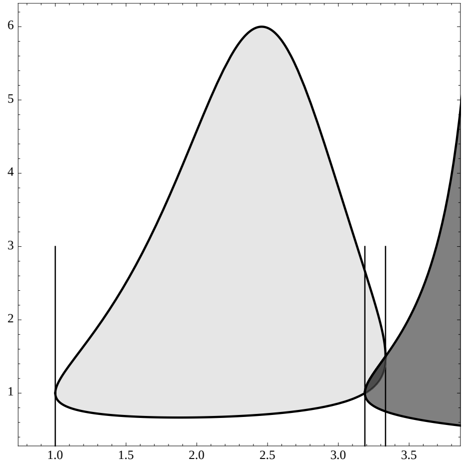
$$\rho = |u|^{\beta p}$$

$$m = 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right)$$



(p, β) representation of the admissible range of parameters when $d = 5$ [J.D., Esteban, Kowalczyk, Loss]

Can one prove Sobolev's inequalities with a heat flow ?

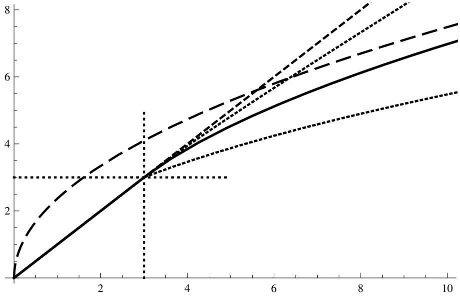


(p, β) *representation when* $d = 5$. *In the dark grey area, the functional is not monotone under the action of the heat flow [J.D., Esteban, Loss]*

The bifurcation point of view

$\mu(\lambda)$ is the optimal constant in the functional inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda \, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda) \, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall \, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$



Here $d = 3$ *and* $p = 4$

- A critical point of $u \mapsto \mathcal{Q}_\lambda[u] := \dfrac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}$ solves

$$-\Delta u + \lambda\, u = |u|^{p-2}\, u \qquad \text{(EL)}$$

up to a multiplication by a constant (and a conformal transformation if $p = 2^*$)

- The best constant $\mu(\lambda) = \inf_{u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}} \mathcal{Q}_\lambda[u]$ is such that $\mu(\lambda) < \lambda$ if $\lambda > \frac{d}{p-2}$, and $\mu(\lambda) = \lambda$ if $\lambda \le \frac{d}{p-2}$ so that

$$\frac{d}{p-2} = \min\{\lambda > 0 \,:\, \mu(\lambda) < \lambda\}$$

- *Rigidity* : the unique positive solution of (EL) is $u = \lambda^{1/(p-2)}$ if $\lambda \le \frac{d}{p-2}$

Constraints and improvements

- Taylor expansion:

$$d = \inf_{u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \setminus \{0\}} \frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

is achieved in the limit as $\varepsilon \to 0$ with $u = 1 + \varepsilon\,\varphi_1$ such that

$$-\Delta\varphi_1 = d\,\varphi_1$$

▷ This suggest that improved inequalities can be obtained under appropriate orthogonality constraints...

Integral constraints

With the heat flow...

Proposition

For any $p \in (2, 2^\#)$, the inequality

$$\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{\lambda}{p-2} \|f\|_2^2 \geq \frac{\lambda}{p-2} \|f\|_p^2$$

$$\forall \, f \in \mathrm{H}^1((-1,1), d\nu_d) \ s.t. \int_{-1}^{1} z \, |f|^p \, d\nu_d = 0$$

holds with

$$\lambda \geq d + \frac{(d-1)^2}{d\,(d+2)} \, (2^\# - p) \, (\lambda^\star - d)$$

... and with a nonlinear diffusion flow ?

Antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x) = u(x) \quad \forall\, x \in \mathbb{S}^d$$

Theorem

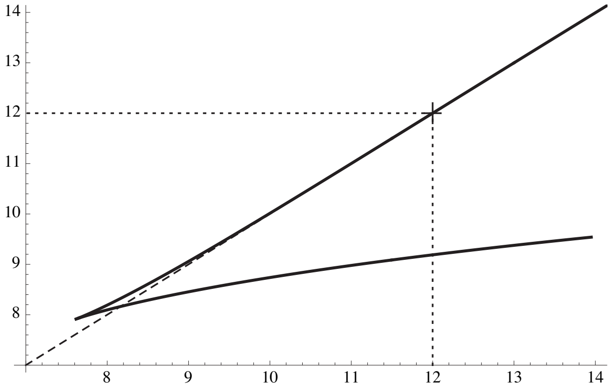
If $p \in (1,2) \cup (2, 2^*)$, *we have*

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{d}{p-2} \left[1 + \frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1} \right] \left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ *with antipodal symmetry. The limit case* $p = 2$ *corresponds to the improved logarithmic Sobolev inequality*

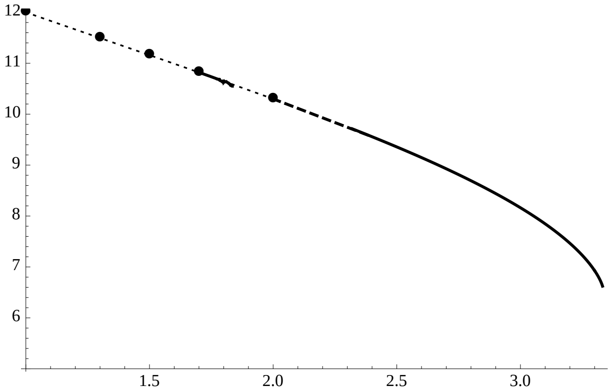
$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{d}{2} \frac{(d+3)^2}{(d+1)^2} \int_{\mathbb{S}^d} |u|^2 \log \left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu$$

The larger picture: branches of antipodal solutions



Case $d = 5$*,* $p = 3$*: values of the shooting parameter* a *as a function of* λ

The optimal constant in the antipodal framework



Numerical computation of the optimal constant when $d=5$ *and* $1 \le p \le 10/3 \approx 3.33$*. The limiting value of the constant is numerically found to be equal to* $\lambda_\star = 2^{1-2/p}\,d \approx 6.59754$ *with* $d=5$ *and* $p=10/3$

Symmetries, symmetry breaking and bifurcations in Caffarelli-Kohn-Nirenberg inequalities

- ▷ Symmetry, symmetry breaking and branches of solutions
- ▷ The sharp result on symmetry
- ▷ Bifurcation and branches

Critical Caffarelli-Kohn-Nirenberg inequalities

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

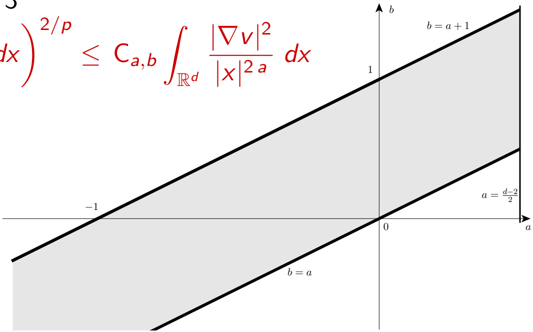
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d = 3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a \le b \le a+1$ if $d \ge 3$

$a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$

and $a < a_c := (d-2)/2$

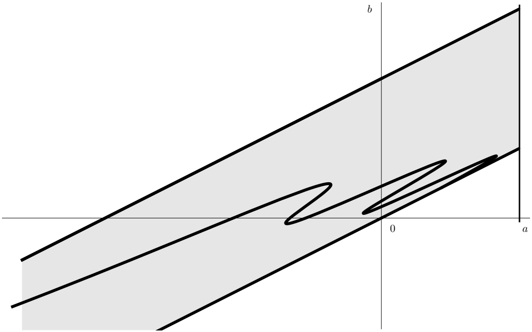
$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

[Glaser, Martin, Grosse, Thirring (1976)]

[F. Catrina, Z.-Q. Wang (2001)]

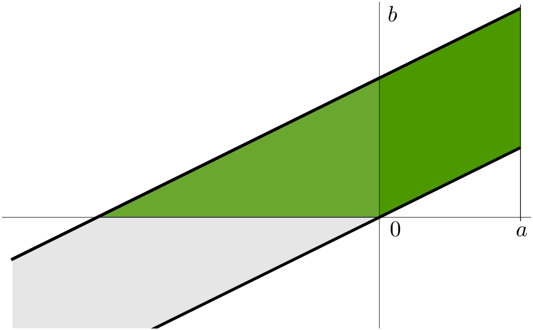
Proving symmetry breaking

[F. Catrina, Z.-Q. Wang], [V. Felli, M. Schneider (2003)]



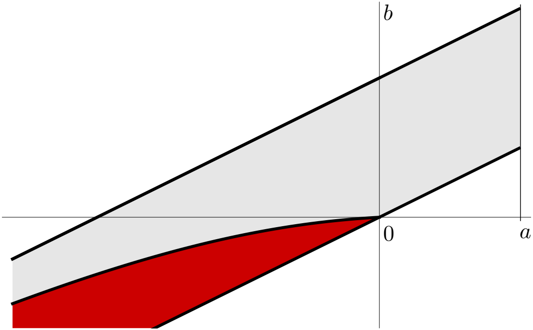
[J.D., Esteban, Loss, Tarantello, 2009] There is a curve which separates the symmetry region from the symmetry breaking region, which is parametrized by a function $p \mapsto a + b$

Moving planes and symmetrization techniques



[Chou, Chu], [Horiuchi]
[Betta, Brock, Mercaldo, Posteraro]
+ Perturbation results: [CS Lin, ZQ Wang], [Smets, Willem], [J.D., Esteban, Tarantello 2007], [J.D., Esteban, Loss, Tarantello, 2009]

Linear instability of radial minimizers: the Felli-Schneider curve

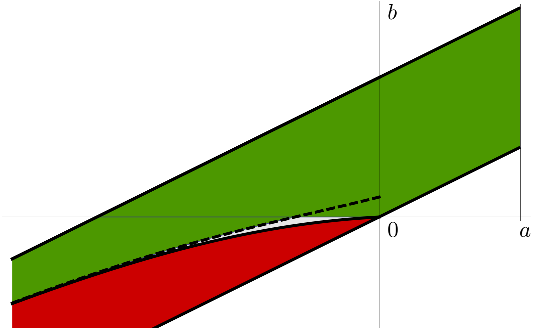


[Catrina, Wang], [Felli, Schneider] The functional

$$C^\star_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx-\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

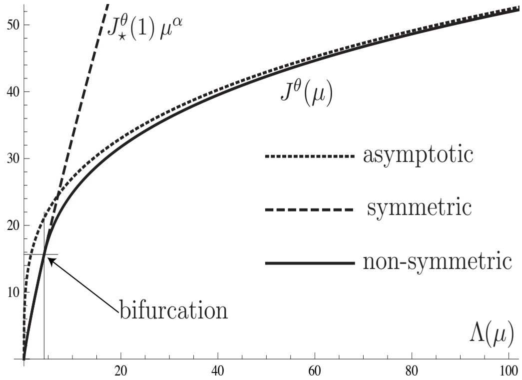
is linearly instable at $v=v_\star$

Direct spectral estimates



[J.D., Esteban, Loss, 2011]: sharp interpolation on the sphere and a Keller-Lieb-Thirring spectral estimate on the line

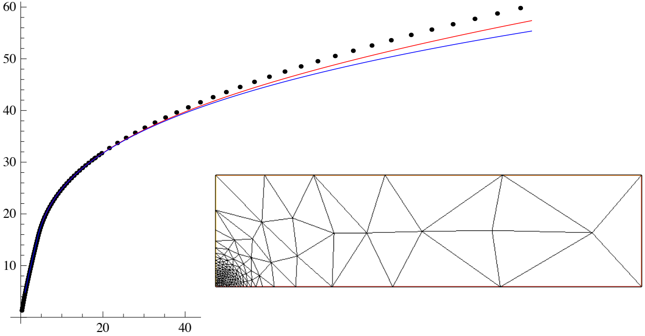
Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as predicted by F. Catrina and Z.-Q. Wang*

Other evidences

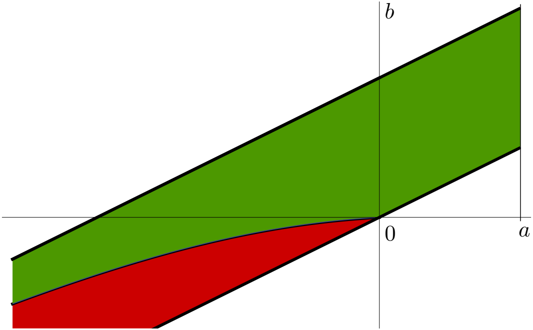
- Further numerical results [J.D., Esteban, 2012] (coarse / refined / self-adaptive grids)



- Formal commutation of the non-symmetric branch near the bifurcation point [J.D., Esteban, 2013]
- Asymptotic energy estimates [J.D., Esteban, 2013]

Symmetry *versus* symmetry breaking: the sharp result

A result based on entropies and nonlinear flows



[J.D., Esteban, Loss, 2015]: http://arxiv.org/abs/1506.03664

The symmetry result

The Felli & Schneider curve is defined by

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$

Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, Caffarelli-Kohn-Nirenberg inequalities on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega) = r^{a-a_c}\,\varphi(s,\omega) \quad \text{with} \quad r=|x|\,, \quad s=-\log r \quad \text{and} \quad \omega=\frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} \ge \mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})} \quad \forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda := (a_c-a)^2$, $\mathcal{C}=\mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda)=\frac{1}{\mathsf{C}_{a,b}} \quad \text{with} \quad a=a_c\pm\sqrt{\Lambda} \quad \text{and} \quad b=\frac{d}{p}\pm\sqrt{\Lambda}$$

Generalized Caffarelli-Kohn-Nirenberg inequalities (CKN)

Let $2^* = \infty$ if $d=1$ or $d=2$, $2^* = 2d/(d-2)$ if $d \ge 3$ and define

$$\vartheta(p,d) := \frac{d\,(p-2)}{2\,p}$$

[Caffarelli-Kohn-Nirenberg-84] Let $d \ge 1$. For any $\theta \in [\vartheta(p,d),1]$, with $p = \frac{2\,d}{d-2+2\,(b-a)}$, there exists a positive constant $\mathsf{C}_{\rm CKN}(\theta,p,a)$ such that

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \le \mathsf{C}_{\rm CKN}(\theta,p,a) \left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right)^{\theta} \left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

In the radial case, with $\Lambda = (a-a_c)^2$, the best constant when the inequality is restricted to radial functions is $\mathsf{C}^*_{\rm CKN}(\theta,p,a)$ and

$$\mathsf{C}_{\rm CKN}(\theta,p,a) \ge \mathsf{C}^*_{\rm CKN}(\theta,p,a) = \mathsf{C}^*_{\rm CKN}(\theta,p)\,\Lambda^{\frac{p-2}{2p}-\theta}$$

$$\mathsf{C}^*_{\rm CKN}(\theta,p) = \left[\frac{2\,\pi^{d/2}}{\Gamma(d/2)}\right]^{2\,\frac{p-1}{p}} \left[\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right]^{\frac{p-2}{2\,p}} \left[\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right]^{\theta} \left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}} \left[\frac{\Gamma\left(\frac{2}{p-2}+\frac12\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}}$$

The method of Catrina-Wang / Felli-Schneider

Among functions $w \in H^1(\mathcal{C})$ which depend only on s, the minimum of

$$\mathcal{J}[w] := \int_{\mathcal{C}} \left(|\nabla w|^2 + \tfrac{1}{4}\,(d-2-2\,a)^2\,|w|^2 \right)\,dx - [\mathsf{C}^*(\theta,p,a)]^{-\frac{1}{\theta}}\,\frac{\left(\int_{\mathcal{C}} |w|^p\,dx\right)^{\frac{2}{p\,\theta}}}{\left(\int_{\mathcal{C}} |w|^2\,dx\right)^{\frac{1-\theta}{\theta}}}$$

is achieved by $\overline{w}(y) := \left[\cosh(\lambda\,s)\right]^{-\frac{2}{p-2}}$, $y=(s,\omega)\in\mathbb{R}\times\mathbb{S}=\mathcal{C}$ with $\lambda := \frac{1}{4}\,(d-2-2\,a)\,(p-2)\,\sqrt{\frac{p+2}{2\,p\,\theta-(p-2)}}$ as a solution of

$$\lambda^2\,(p-2)^2\,w'' - 4\,w + 2\,p\,|w|^{p-2}\,w = 0$$

Spectrum of $\mathcal{L} := -\Delta + \kappa\,\overline{w}^{p-2} + \mu$ is given for $\sqrt{1+4\,\kappa/\lambda^2} \ge 2\,j+1$ by $\lambda_{i,j} = \mu + i\,(d+i-2) - \frac{\lambda^2}{4}\left(\sqrt{1+4\,\kappa/\lambda^2} - (1+2\,j)\right)^2 \quad \forall\, i\,,\,j\in\mathbb{N}$

- The eigenspace of $\mathcal{L}$ corresponding to $\lambda_{0,0}$ is generated by $\overline{w}$
- The eigenfunction $\phi_{(1,0)}$ associated to $\lambda_{1,0}$ is not radially symmetric and such that $\int_{\mathcal{C}} \overline{w}\,\phi_{(1,0)}\,dx = 0$ and $\int_{\mathcal{C}} \overline{w}^{p-1}\,\phi_{(1,0)}\,dx = 0$
- If $\lambda_{1,0} < 0$, *optimal functions for (CKN) cannot be radially symmetric* and $\mathsf{C}(\theta,p,a) > \mathsf{C}^*(\theta,p,a)$

A parametrization of the solutions

All optimal functions can be computed

▷ by solving (numerically)

$$-\Delta u + \mu\, u = u^{p-1}$$

▷ by computing $\Lambda = \Lambda^\theta(\mu)$ (reparametrization)
and the corresponding optimal constant is given by

$$J^\theta(\mu) := \mathcal{Q}^\theta_{\Lambda^\theta(\mu)}[u_\mu]$$

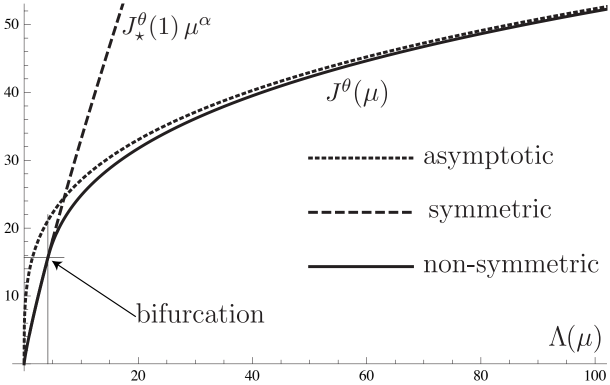
where

$$\mathcal{Q}^\theta_\Lambda[u] := \frac{\left(\|\nabla u\|^2_{\mathcal{L}^2(\mathcal{C})} + \Lambda\,\|u\|^2_{\mathcal{L}^2(\mathcal{C})}\right)^\theta\ \|u\|^{2\,(1-\theta)}_{\mathcal{L}^2(\mathcal{C})}}{\|u\|^2_{\mathcal{L}^p(\mathcal{C})}}$$

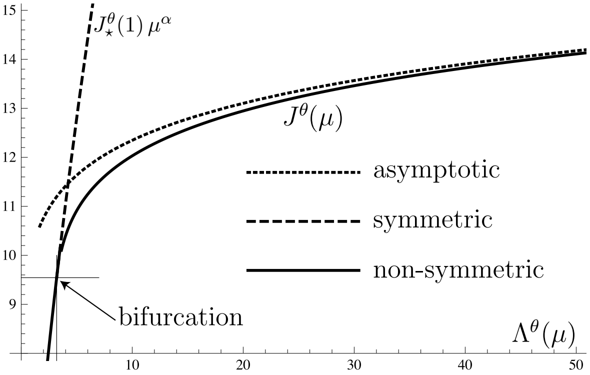
If u_μ is symmetric, then

$$J^\theta(\mu) = J^\theta(1)\,\mu^\alpha$$

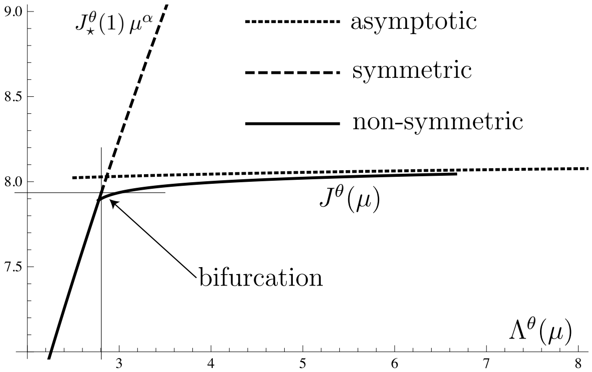
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 1$



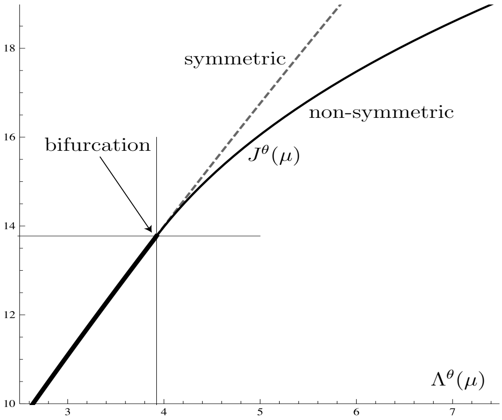
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 0.8$



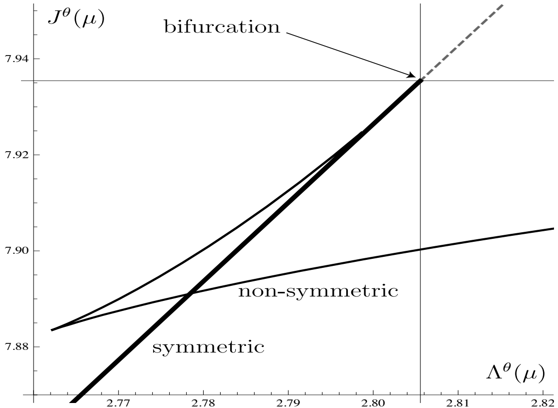
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 0.72$



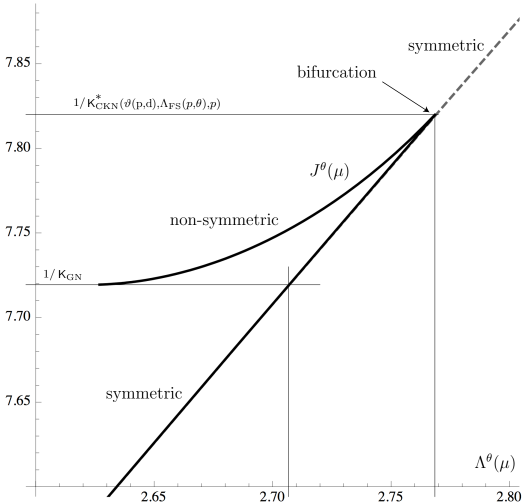
Enlargement for $p = 2.8$, $d = 5$, $\theta = 0.95$



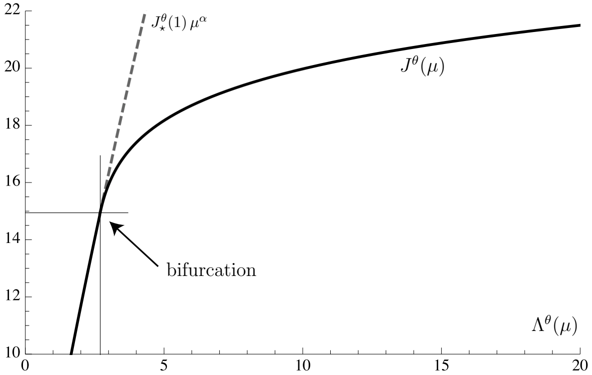
Enlargement for $p = 2.8$, $d = 5$, $\theta = 0.72$



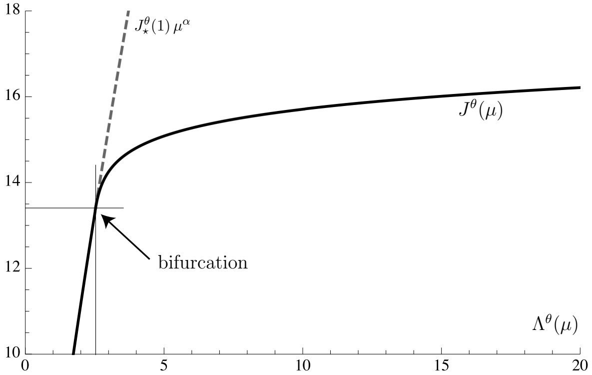
Critical case $\theta = \vartheta(p, d)$



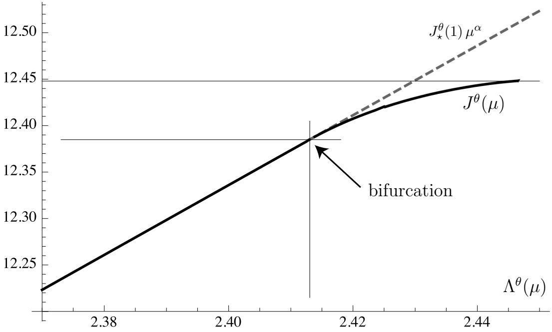
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = 1$



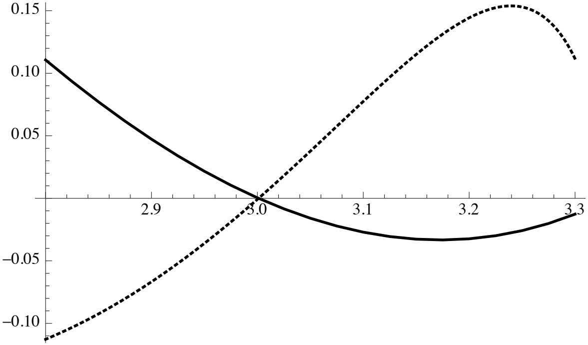
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = 0.95$



Case $p = 3.15$, $d = 5$, $\theta = \vartheta(3.15, 5) \approx 0.9127$



Local and asymptotic criteria for $\theta = \vartheta(p, d)$ – *numerical*



- *Local criterion: based on an expansion of the solutions near the bifurcation point, it decides whether the branch goes to the right to to the left.*
- *Asymptotic criterion: based on the energy of the branch as* $\Lambda \to +\infty$ *and an analysis in a semi-classical regime*

Fast diffusion equations, large time asymptotics, spectrum

▷ Weighted fast diffusion equations
- The equation and the self-similar solutions
- Without weights
- A perturbation result
- Symmetry breaking
- More on symmetry

▷ Large time asymptotics

▷ Linearization and optimality

Most recent results: joint work with M. Bonforte, M. Muratori and B. Nazaret,
and with M.J. Esteban and M. Loss

Fast diffusion equations with weights: self-similar solutions

Let us consider the *fast diffusion equation with weights*

$$u_t + |x|^\gamma \nabla \cdot \left(|x|^{-\beta} \, u \, \nabla u^{m-1} \right) = 0 \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$$

Here β and γ are two real parameters, and $m \in [m_1, 1)$ with

$$m_1 := \frac{2\,d-2-\beta-\gamma}{2\,(d-\gamma)}$$

Generalized *Barenblatt self-similar solutions*

$$u_\star(\rho\,t,x) = t^{-\rho\,(d-\gamma)}\,\mathfrak{B}_{\beta,\gamma}\left(t^{-\rho}\,x\right)\,, \quad \mathfrak{B}_{\beta,\gamma}(x) = \left(1+|x|^{2+\beta-\gamma}\right)^{\frac{1}{m-1}}$$

where $1/\rho = (d-\gamma)\,(m-m_c)$ with $m_c := \frac{d-2-\beta}{d-\gamma} < m_1 < 1$

Self-similar solutions are known to govern the asymptotic behavior of the solutions when $(\beta,\gamma) = (0,0)$

- Mass conservation

$$\frac{d}{dt}\int_{\mathbb{R}^d} u\,\frac{dx}{|x|^\gamma}=0$$

and self-similar solutions suggest to introduce the

- Time-dependent rescaling

$$u(t,x)=R^{\gamma-d}\,v\left((2+\beta-\gamma)^{-1}\,\log R,\frac{x}{R}\right)$$

with $R=R(t)$ defined by

$$\frac{dR}{dt}=(2+\beta-\gamma)\,R^{(m-1)(\gamma-d)-(2+\beta-\gamma)+1}\,,\quad R(0)=1$$

$$R(t)=\left(1+\tfrac{2+\beta-\gamma}{\rho}\,t\right)^\rho$$

with $1/\rho=(1-m)\,(\gamma-d)+2+\beta-\gamma=(d-\gamma)\,(m-m_c)$

- A *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[\,|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0$$

with initial condition $v(t=0,\cdot)=u_0$

Without weights: time-dependent rescaling, free energy

- *Time-dependent rescaling:* Take $u(\tau,y) = R^{-d}(\tau)\, v\,(t, y/R(\tau))$ where

$$\frac{dR}{d\tau} = R^{d(1-m)-1}\,, \quad t = \frac{1}{2}\log R$$

- The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \tfrac{m-1}{m}\,\Delta v^m + \nabla\cdot(x\,v)$$

- [Ralston, Newman, 1984] Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{F}[v] := \int_{\mathbb{R}^d}\left(-\frac{v^m}{m} + |x|^2 v\right)\,dx - \mathcal{F}_0$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{F}[v] = -\mathcal{I}[v]\,, \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v\left|\nabla v^{m-1} + 2\,x\right|^2\,dx$$

Without weights: relative entropy, entropy production

- *Stationary solution:* choose C such that $\|v_\infty\|_{\mathrm{L}^1} = \|u\|_{\mathrm{L}^1} = M > 0$

$$v_\infty(x) := \left(C + |x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{F}_0$ so that $\mathcal{F}[v_\infty] = 0$

- *Entropy – entropy production inequality*

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$$

Corollary

A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[u_0]\,e^{-\,4\,t}$

More simple facts...

▷ The entropy – entropy production inequality is equivalent to the Gagliardo-Nirenberg inequality
[del Pino, J.D.] With $1 < p \le \frac{d}{d-2}$ (fast diffusion case) and $d \ge 3$

$$\|w\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \le \mathcal{C}^{\mathrm{GN}}_{p,d}\,\|\nabla w\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|w\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)}$$

Proofs: variational methods [del Pino, J.D.], or *carré du champ method* (Bakry-Emery): [Carrillo, Toscani], [Carrillo, Vázquez], [CJMTU]

▷ Sharp asymptotic rates are determined by the spectral gap in the linearized entropy – entropy production (Hardy–Poincaré) inequality [Blanchet, Bonforte, J.D., Grillo, Vázquez]

▷ Higher order matching asymptotics can be achieved by *best matching* methods: [Bonforte, J.D., Grillo, Vázquez], [J.D., Toscani]

...

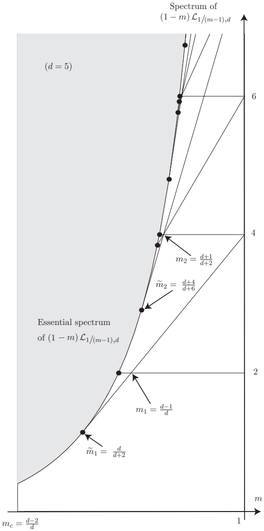
▷ *Improved entropy – entropy production inequalities* $\varphi(\mathcal{F}[v]) \leq \mathcal{I}[v]$ can be proved [J.D., Toscani], [Carrillo, Toscani]

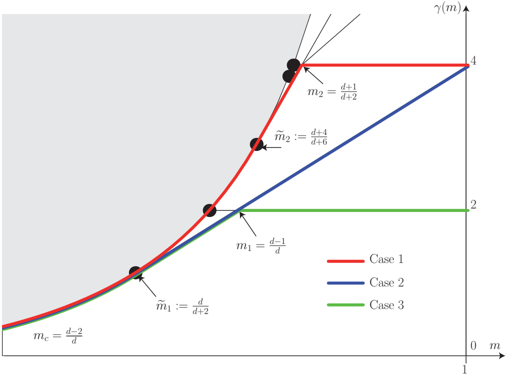
▷ *Rényi entropy powers:* concavity, asymptotic regime (self-similar solutions) and Gagliardo-Nirenberg inequalities in scale invariant form [Savaré, Toscani], [J.D., Toscani]

▷ Concavity of second moment estimates and *delays* [J.D., Toscani]

▷ *Stability* of entropy – entropy production inequalities (scaling methods), and improved rates of convergence [Carrillo, Toscani], [J.D., Toscani]

Spectrum of the linearized operator





With one weight: a perturbation result

On the space of smooth functions on $\mathbb{R}^d$ with compact support

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_\gamma\,\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\vartheta}\,\|w\|_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}^{1-\vartheta}$$

where $\vartheta := \frac{2^*_\gamma\,(p-1)}{2\,p\big(2^*_\gamma-p-1\big)} = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+2-2\,\gamma-p\,(d-2)\big)}$ and

$$\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \Big(\int_{\mathbb{R}^d}|w|^q\,|x|^{-\gamma}\,dx\Big)^{1/q} \quad \text{and} \quad \|w\|_{\mathrm{L}^{q}(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$$

and $d\ge3$, $\gamma\in(0,2)$, $p\in\big(1,2^*_\gamma/2\big)$ with $2^*_\gamma := 2\,\frac{d-\gamma}{d-2}$

Theorem

[J.D., Muratori, Nazaret] *Let $d\ge3$. For any $p\in(1,d/(d-2))$, there exists a positive γ^* such that equality holds for all $\gamma\in(0,\gamma^*)$ with*

$$w_\star(x) := \big(1+|x|^{2-\gamma}\big)^{-\frac{1}{p-1}} \quad \forall\,x\in\mathbb{R}^d$$

Caffarelli-Kohn-Nirenberg inequalities (with two weights)

Norms: $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q \, |x|^{-\gamma} \, dx\right)^{1/q}$, $\|w\|_{\mathrm{L}^{q}(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$
(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p} \, \|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)} \, \|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \qquad \text{(CKN)}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \ge 2\,, \quad \gamma - 2 < \beta < \tfrac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with } p_\star := \tfrac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

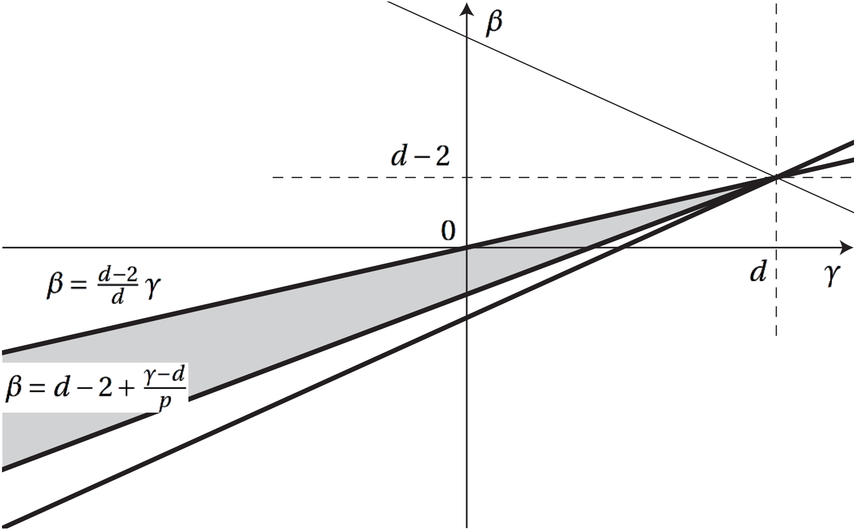
$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$$

- Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d \quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions?

Range of the parameters



CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\tfrac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{F}[v]\le\mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}$. Here the *free energy* and the *relative Fisher information* are defined by

$$\mathcal{F}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\,\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\,\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\,\frac{dx}{|x|^\beta}\,.$$

If v solves the *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0 \qquad \text{(WFDE-FP)}$$

then

$$\frac{d}{dt}\mathcal{F}[v(t,\cdot)]=-\,\frac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$$

Proposition

Let $m=\frac{p+1}{2\,p}$ and consider a solution to (WFDE-FP) with nonnegative initial datum $u_0\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$ such that $\|u_0^m\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}$ and $\int_{\mathbb{R}^d}u_0\,|x|^{2+\beta-2\gamma}\,dx$ are finite. Then

$$\mathcal{F}[v(t,\cdot)]\le\mathcal{F}[u_0]\,e^{-(2+\beta-\gamma)^2\,t}\quad\forall\,t\ge0$$

if one of the following two conditions is satisfied:
(i) either u_0 is a.e. radially symmetric
(ii) or symmetry holds in (CKN)

With two weights: a symmetry breaking result

Let us define

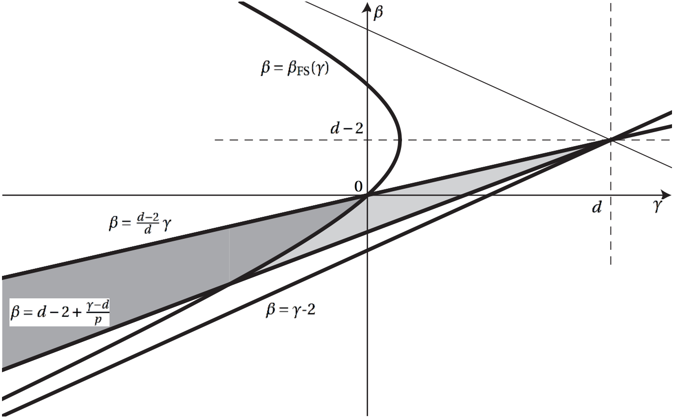
$$\beta_{\rm FS}(\gamma) := d - 2 - \sqrt{(d-\gamma)^2 - 4\,(d-1)}$$

Theorem

Symmetry breaking holds in (CKN) if

$$\gamma < 0 \quad \textit{and} \quad \beta_{\rm FS}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$$

In the range $\beta_{\rm FS}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$, $w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$ *is not* optimal.



The grey area corresponds to the admissible cone. The light grey area is the region of symmetry, while the dark grey area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$$

A useful change of variables

With

$$\alpha=1+\frac{\beta-\gamma}{2}\quad\text{and}\quad n=2\,\frac{d-\gamma}{\beta+2-\gamma}\,,$$

(CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x)=w(x)$ as

$$\|v\|_{\mathrm L^{2p,d-n}(\mathbb R^d)}\le\mathsf K_{\alpha,n,p}\,\|\mathfrak D_\alpha v\|_{\mathrm L^{2,d-n}(\mathbb R^d)}^{\vartheta}\,\|v\|_{\mathrm L^{p+1,d-n}(\mathbb R^d)}^{1-\vartheta}$$

with the notations $s=|x|$, $\mathfrak D_\alpha v=\left(\alpha\,\frac{\partial v}{\partial s},\frac1s\,\nabla_\omega v\right)$. Parameters are in the range

$$d\ge2\,,\quad\alpha>0\,,\quad n>d\quad\text{and}\quad p\in(1,p_\star]\,,\quad p_\star:=\frac{n}{n-2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x):=\left(1+|x|^2\right)^{-1/(p-1)}\quad\forall\,x\in\mathbb R^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha>\alpha_{\rm FS}\quad\text{with}\quad\alpha_{\rm FS}:=\sqrt{\frac{d-1}{n-1}}$$

The second variation

$$\mathcal{J}[v] := \vartheta\,\log\left(\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\right) + (1-\vartheta)\,\log\left(\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}\right) + \log \mathsf{K}_{\alpha,n,p} - \log\left(\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\right)$$

Let us define $d\mu_\delta := \mu_\delta(x)\,dx$, where $\mu_\delta(x) := (1+|x|^2)^{-\delta}$. Since $v_\star$ is a critical point of $\mathcal{J}$, a Taylor expansion at order ε^2 shows that

$$\|\mathfrak{D}_\alpha v_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\mathcal{J}\left[v_\star + \varepsilon\,\mu_{\delta/2}\,f\right] = \tfrac12\,\varepsilon^2\,\vartheta\,\mathcal{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2\,p}{p-1}$ and

$$\mathcal{Q}[f] = \int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\,|x|^{n-d}\,d\mu_\delta - \frac{4\,p\,\alpha^2}{p-1}\int_{\mathbb{R}^d} |f|^2\,|x|^{n-d}\,d\mu_{\delta+1}$$

We assume that $\int_{\mathbb{R}^d} f\,|x|^{n-d}\,d\mu_{\delta+1} = 0$ (mass conservation)

Symmetry breaking: the proof

Proposition (Hardy-Poincaré inequality)

Let $d \ge 2$, $\alpha \in (0,+\infty)$, $n > d$ and $\delta \ge n$. If f has 0 average, then

$$\int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2 \, |x|^{n-d} \, d\mu_\delta \ge \Lambda \int_{\mathbb{R}^d} |f|^2 \, |x|^{n-d} \, d\mu_{\delta+1}$$

with optimal constant $\Lambda = \min\{2\,\alpha^2\,(2\,\delta - n), 2\,\alpha^2\,\delta\,\eta\}$ where η is the unique positive solution to $\eta\,(\eta + n - 2) = (d-1)/\alpha^2$. The corresponding eigenfunction is not radially symmetric if $\alpha^2 > \frac{(d-1)\,\delta^2}{n\,(2\,\delta-n)\,(\delta-1)}$.

$\mathcal{Q} \ge 0$ iff $\frac{4\,p\,\alpha^2}{p-1} \le \Lambda$ and symmetry breaking occurs in (CKN) if

$$2\,\alpha^2\,\delta\,\eta < \frac{4\,p\,\alpha^2}{p-1} \iff \eta < 1$$

$$\iff \frac{d-1}{\alpha^2} = \eta\,(\eta + n - 2) < n - 1 \iff \alpha > \alpha_{\rm FS}$$

Fast diffusion equations with weights: a symmetry result

- Rényi entropy powers
- The symmetry result
- The strategy of the proof

Joint work with M.J. Esteban, M. Loss in the critical case $\beta = d - 2 + \frac{\gamma - d}{p}$

Joint work with M.J. Esteban, M. Loss and M. Muratori in the subcritical case $d - 2 + \frac{\gamma - d}{p} < \beta < \frac{d-2}{d}\,\gamma$

Rényi entropy powers

[Savaré, Toscani] We consider the flow $\frac{\partial u}{\partial t} = \Delta u^m$ and the Gagliardo-Nirenberg inequalities (GN)

$$\|w\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \le \mathcal{C}_{p,d}^{\mathrm{GN}}\, \|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\, \|w\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

where $u = w^{2p}$, that is, $w = u^{m-1/2}$ with $p = \frac{1}{2\,m-1}$. Straightforward computations show that (GN) can be brought into the form

$$\left(\int_{\mathbb{R}^d} u\, dx\right)^{(\sigma+1)\,m-1} \le C\,\mathcal{I}\,\mathcal{E}^{\sigma-1} \quad \text{where} \quad \sigma = \frac{2}{d\,(1-m)} - 1$$

where $\mathcal{E} := \int_{\mathbb{R}^d} u^m\, dx$ and $\mathcal{I} := \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\, dx$, $\mathsf{P} = \frac{m}{1-m}\, u^{m-1}$ is the *pressure variable*. If $\mathcal{F} = \mathcal{E}^\sigma$ is the *Rényi entropy power* and $\sigma = \frac{2}{d}\,\frac{1}{1-m} - 1$, then $\mathcal{F}''$ is proportional to

$$-2\,(1-m)\,\left\langle \mathrm{Tr}\left(\left(\mathrm{Hess}\,\mathsf{P} - \tfrac{1}{d}\,\Delta \mathsf{P}\,\mathrm{Id}\right)^2\right)\right\rangle + (1-m)^2\,(1-\sigma)\,\left\langle \left(\Delta \mathsf{P} - \langle \Delta \mathsf{P}\rangle\right)^2\right\rangle$$

where we have used the notation $\langle A\rangle := \int_{\mathbb{R}^d} u^m\, A\, dx \Big/ \int_{\mathbb{R}^d} u^m\, dx$

The symmetry result

▷ critical case: [J.D., Esteban, Loss; Inventiones]

▷ subcritical case: [J.D., Esteban, Loss, Muratori; CR Math.]

Theorem

Assume that $\beta \le \beta_{\rm FS}(\gamma)$. Then all positive solutions in $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ of

$$-\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w\right) = |x|^{-\gamma}\left(w^{2p-1} - w^p\right) \quad \text{in} \quad \mathbb{R}^d\setminus\{0\}$$

are radially symmetric and, up to a scaling and a multiplication by a constant, equal to $w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$

The strategy of the proof (1/3)

The first step is based on a change of variables which amounts to rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise to consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$v(|x|^{\alpha-1}\,x) = w(x)\,,\quad \alpha = 1 + \frac{\beta-\gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta+2-\gamma}$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|^{\vartheta}_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\|v\|^{1-\vartheta}_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)} \quad \forall\, v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$ and

$$d \ge 2\,,\quad \alpha > 0\,,\quad n > d \quad \text{and} \quad p \in (1, p_\star]\,.$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d$$

The strategy of the proof (2/3): concavity of the Rényi entropy power

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathfrak{D}_\alpha \mathsf{P}|^2 \, d\mu$$

where $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d} \, dx$ and the pressure is

$$\mathsf{P} := \frac{m}{1-m} \, u^{m-1}$$

Proving the symmetry in the inequality amounts to

proving the monotonicity of $\mathcal{G}[u]$

along a well chosen fast diffusion flow

With $\mathcal{L}_\alpha = -\mathcal{D}_\alpha^*\,\mathfrak{D}_\alpha = \alpha^2\left(u'' + \frac{n-1}{s}\,u'\right) + \frac{1}{s^2}\,\Delta_\omega\,u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{L}_\alpha\,u^m$$

in the subcritical range $1 - 1/n < m < 1$. The key computation is the proof that

$$\begin{aligned}
&-\tfrac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\textstyle\int_{\mathbb{R}^d} u^m\,d\mu\right)^{1-\sigma}\\
&\geq (1-m)\,(\sigma-1)\int_{\mathbb{R}^d} u^m\left|\mathcal{L}_\alpha \mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathfrak{D}_\alpha \mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d} u^m\,d\mu}\right|^2 d\mu\\
&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\tfrac{1}{n}\right)\left|\mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega \mathsf{P}' - \frac{\nabla_\omega \mathsf{P}}{s}\right|^2\right) u^m\,d\mu\\
&+2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)|\nabla_\omega \mathsf{P}|^2 + c(n,m,d)\,\frac{|\nabla_\omega \mathsf{P}|^4}{\mathsf{P}^2}\right) u^m\,d\mu =: \mathcal{H}[u]
\end{aligned}$$

for some numerical constant $c(n,m,d) > 0$. Hence if $\alpha \leq \alpha_{\rm FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result.

The strategy of the proof (3/3): integrations by parts

This method has a hidden difficulty: integrations by parts ! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings... to deduce decay estimates
- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \, \nabla \cdot \Big[|x|^{-\beta} \, v \, \nabla \big(v^{m-1} - |x|^{2+\beta-\gamma} \big) \Big] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

Relative uniform convergence

$$\zeta := 1 - \left(1 - \tfrac{2-m}{(1-m)\,q}\right)\left(1 - \tfrac{2-m}{1-m}\,\theta\right)$$

$\theta := \frac{(1-m)\,(2+\beta-\gamma)}{(1-m)\,(2+\beta)+2+\beta-\gamma}$ is in the range $0 < \theta < \frac{1-m}{2-m} < 1$

Theorem

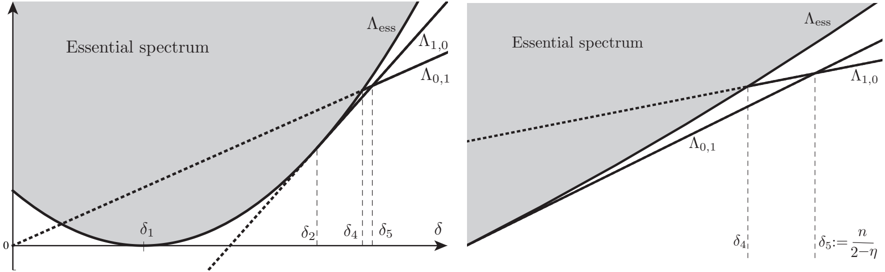
For "good" initial data, there exist positive constants $\mathcal{K}$ and t_0 such that, for all $q \in \left[\frac{2-m}{1-m}, \infty\right]$, the function $w = v/\mathfrak{B}$ satisfies

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,\zeta\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \in (0,d)$, and

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \le 0$



The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\rm ess}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions

Main steps of the proof:

- Existence of weak solutions, $\mathrm{L}^{1,\gamma}$ contraction, Comparison Principle, conservation of relative mass
- Self-similar variables and the Ornstein-Uhlenbeck equation in relative variables: the ratio $w(t,x) := v(t,x)/\mathfrak{B}(x)$ solves

$$\begin{cases} |x|^{-\gamma}\, w_t = -\frac{1}{\mathfrak{B}}\,\nabla\cdot\Big(|x|^{-\beta}\,\mathfrak{B}\,w\,\nabla\left(\left(w^{m-1}-1\right)\mathfrak{B}^{m-1}\right)\Big) & \text{in } \mathbb{R}^+\times\mathbb{R}^d \\ w(0,\cdot) = w_0 := v_0/\mathfrak{B} & \text{in } \mathbb{R}^d \end{cases}$$

- *Regularity*, relative uniform convergence (without rates) and asymptotic rates (linearization)
- The relative free energy and the relative Fisher information: linearized free energy and linearized Fisher information
- A Duhamel formula and a bootstrap

Regularity (1/2): Harnack inequality and Hölder regularity

We change variables: $x \mapsto |x|^{\alpha-1}\,x$ and adapt the ideas of F. Chiarenza and R. Serapioni to

$$u_t + \mathsf{D}_\alpha^*\Big[\mathsf{a}\,(\mathsf{D}_\alpha\,u + \mathsf{B}\,u)\Big] = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d$$

Proposition (A parabolic Harnack inequality)

Let $d \geq 2$, $\alpha > 0$ and $n > d$. If u is a bounded positive solution, then for all $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $r > 0$ such that $Q_r(t_0, x_0) \subset \mathbb{R}^+ \times B_1$, we have

$$\sup_{Q_r^-(t_0,x_0)} u \leq H \inf_{Q_r^+(t_0,x_0)} u$$

The constant $H > 1$ depends only on the local bounds on the coefficients a, B *and on d, α, and n*

By adapting the classical method *à la De Giorgi* to our weighted framework: Hölder regularity at the origin

Regularity (1/2): from local to global estimates

Lemma

If w is a solution of the the Ornstein-Uhlenbeck equation with initial datum w_0 bounded from above and from below by a Barenblatt profile (+ relative mass condition) = "good solutions", then there exist $\nu \in (0,1)$ and a positive constant $\mathcal{K} > 0$, depending on d, m, β, γ, C, C_1, C_2 such that:

$$\|\nabla v(t)\|_{\mathrm{L}^\infty(B_{2\lambda}\setminus B_\lambda)} \le \frac{Q_1}{\lambda^{\frac{2+\beta-\gamma}{1-m}+1}} \quad \forall\, t \ge 1\,, \quad \forall\, \lambda > 1\,,$$

$$\sup_{t\ge 1} \|w\|_{C^k((t,t+1)\times B_\varepsilon^c)} < \infty \quad \forall\, k \in \mathbb{N}\,,\ \forall\, \varepsilon > 0$$

$$\sup_{t\ge 1} \|w(t)\|_{C^\nu(\mathbb{R}^d)} < \infty$$

$$\sup_{\tau\ge t} |w(\tau) - 1|_{C^\nu(\mathbb{R}^d)} \le \mathcal{K} \sup_{\tau\ge t} \|w(\tau) - 1\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \quad \forall\, t \ge 1$$

Asymptotic rates of convergence

Corollary

Assume that $m \in (0,1)$, with $m \neq m_*$ with $m_* :=$. Under the relative mass condition, for any "good solution" v there exists a positive constant $\mathcal{C}$ such that

$$\mathcal{F}[v(t)] \le \mathcal{C}\, e^{-\,2\,(1-m)\,\Lambda\,t} \quad \forall\, t \ge 0\,.$$

- With Csiszár-Kullback-Pinsker inequalities, these estimates provide a rate of convergence in $\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$
- Improved estimates can be obtained using "best matching techniques"

From asymptotic to global estimates

When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{F}[v] \le \frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{F}[v(t)] \le \mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t} \quad \forall\,t\ge0 \quad \text{with} \quad \Lambda_\star := \frac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{F}[v] \le \mathcal{I}[v] \quad \forall\,v\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d) \quad \text{such that} \quad \|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)} = M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M) := \frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{F}[v(t)] \le \mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t} \quad \forall\,t\ge0$$

Symmetry breaking and global entropy – entropy production inequalities

Proposition

- *In the symmetry breaking range of (CKN), for any $M>0$, we have*
$$0<\mathcal{K}(M)\le\frac{2}{m}\,(1-m)^2\,\Lambda_{0,1}$$
- *If symmetry holds in (CKN) then*
$$\mathcal{K}(M)\ge\frac{1-m}{m}\,(2+\beta-\gamma)^2$$

Corollary

Assume that $m\in[m_1,1)$

(i) For any $M>0$, if $\Lambda(M)=\Lambda_\star$ then $\beta=\beta_{\rm FS}(\gamma)$

(ii) If $\beta>\beta_{\rm FS}(\gamma)$ then $\Lambda_{0,1}<\Lambda_\star$ and $\Lambda(M)\in(0,\Lambda_{0,1}]$ for any $M>0$

(iii) For any $M>0$, if $\beta<\beta_{\rm FS}(\gamma)$ and if symmetry holds in (CKN), then
$$\Lambda(M)>\Lambda_\star$$

Linearization and optimality

Joint work with M.J. Esteban and M. Loss

Linearization and scalar products

With u_ε such that

$$u_\varepsilon = \mathcal{B}_\star \left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\varepsilon\, dx = M_\star$$

at first order in $\varepsilon \to 0$ we obtain that f solves

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f \quad \text{where} \quad \mathcal{L}\, f := (1-m)\, \mathcal{B}_\star^{m-2}\, |x|^\gamma\, \mathsf{D}_\alpha^* \left(|x|^{-\beta}\, \mathcal{B}_\star\, \mathsf{D}_\alpha f\right)$$

Using the scalar products

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1\, f_2\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx \quad \text{and} \quad \langle\!\langle f_1, f_2 \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D}_\alpha f_1 \cdot \mathsf{D}_\alpha f_2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx$$

we compute

$$\frac{1}{2}\,\frac{d}{dt}\,\langle f, f \rangle = \langle f, \mathcal{L}\, f \rangle = \int_{\mathbb{R}^d} f\, (\mathcal{L}\, f)\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx = -\int_{\mathbb{R}^d} |\mathsf{D}_\alpha f|^2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx =$$

for any f smooth enough, and

$$\frac{1}{2}\,\frac{d}{dt}\langle\!\langle f, f \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D}_\alpha f \cdot \mathsf{D}_\alpha (\mathcal{L}\, f)\, u\, |x|^{-\beta}\, dx = -\,\langle\!\langle f, \mathcal{L}\, f \rangle\!\rangle$$

Linearization of the flow, eigenvalues and spectral gap

Now let us consider an eigenfunction associated with the smallest positive eigenvalue λ_1 of $\mathcal{L}$

$$-\mathcal{L}\,f_1 = \lambda_1\,f_1$$

so that f_1 realizes the equality case in the *Hardy-Poincaré inequality*

$$\langle\!\langle g, g \rangle\!\rangle = -\,\langle f, \mathcal{L}\,f \rangle \ge \lambda_1\,\|g - \bar{g}\|^2\,, \quad \bar{g} := \langle g, 1 \rangle \,/\, \langle 1, 1 \rangle$$

$$-\,\langle\!\langle g, \mathcal{L}\,g \rangle\!\rangle \ge \lambda_1\,\langle\!\langle g, g \rangle\!\rangle$$

Proof: expansion of the square :
$-\,\langle\!\langle (g - \bar{g}), \mathcal{L}\,(g - \bar{g}) \rangle\!\rangle = \langle \mathcal{L}\,(g - \bar{g}), \mathcal{L}\,(g - \bar{g}) \rangle = \|\mathcal{L}\,(g - \bar{g})\|^2$

- Key observation:

$$\lambda_1 \ge 4 \quad \Longleftrightarrow \quad \alpha \le \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$$

Symmetry breaking in CKN inequalities

- Symmetry holds in (CKN) if $\mathcal{J}[w] \geq \mathcal{J}[w_\star]$ with

$$\mathcal{J}[w] := \vartheta \log\left(\|\mathsf{D}_\alpha w\|_{\mathrm{L}^{2,\delta}(\mathbb{R}^d)}\right) + (1-\vartheta) \log\left(\|w\|_{\mathrm{L}^{p+1,\delta}(\mathbb{R}^d)}\right) - \log\left(\|w\|_{\mathrm{L}^{2p,\delta}(\mathbb{R}^d)}\right)$$

with $\delta := d - n$ and

$$\mathcal{J}[w_\star + \varepsilon\, g] = \varepsilon^2\, \mathcal{Q}[g] + o(\varepsilon^2)$$

where

$$\frac{2}{\vartheta}\,\|\mathsf{D}_\alpha w_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\mathcal{Q}[g]$$
$$= \|\mathsf{D}_\alpha g\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)} + \frac{p\,(2+\beta-\gamma)}{(p-1)^2}\left[d-\gamma-p\,(d-2-\beta)\right]\int_{\mathbb{R}^d}|g|^2\,\frac{|x|^{n-d}}{1+|x|^2}\,dx$$
$$- p\,(2\,p-1)\,\frac{(2+\beta-\gamma)^2}{(p-1)^2}\int_{\mathbb{R}^d}|g|^2\,\frac{|x|^{n-d}}{(1+|x|^2)^2}\,dx$$

is a nonnegative quadratic form if and only if $\alpha \leq \alpha_{\rm FS}$

- Symmetry breaking holds if $\alpha > \alpha_{\rm FS}$

Information – production of information inequality

Let $\mathcal{K}[u]$ be such that

$$\frac{d}{d\tau}\mathcal{I}[u(\tau,\cdot)] = -\mathcal{K}[u(\tau,\cdot)] = -\,(\text{sum of squares})$$

If $\alpha \le \alpha_{\rm FS}$, then $\lambda_1 \ge 4$ and

$$u \mapsto \frac{\mathcal{K}[u]}{\mathcal{I}[u]} - 4$$

is a nonnegative functional whose minimizer is achieved by $u = \mathcal{B}_\star$. With $u_\varepsilon = \mathcal{B}_\star\,\big(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\big)$, we observe that

$$4 \le \mathcal{C}_2 := \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} \le \liminf_{\varepsilon\to 0}\, \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \inf_f \frac{\langle\!\langle f, \mathcal{L}\, f\rangle\!\rangle}{\langle\!\langle f, f\rangle\!\rangle} = \frac{\langle\!\langle f_1, \mathcal{L}\, f_1\rangle\!\rangle}{\langle\!\langle f_1, f_1\rangle\!\rangle} = \lambda_1$$

- if $\lambda_1 = 4$, that is, if $\alpha = \alpha_{\rm FS}$, then $\inf \mathcal{K}/\mathcal{I} = 4$ is achieved in the asymptotic regime as $u \to \mathcal{B}_\star$ and determined by the spectral gap of $\mathcal{L}$
- if $\lambda_1 > 4$, that is, if $\alpha < \alpha_{\rm FS}$, then $\mathcal{K}/\mathcal{I} > 4$

Symmetry in Caffarelli-Kohn-Nirenberg inequalities

If $\alpha \le \alpha_{\rm FS}$, the fact that $\mathcal{K}/\mathcal{I} \ge 4$ has an important consequence. Indeed we know that

$$\frac{d}{d\tau}\left(\mathcal{I}[u(\tau,\cdot)] - 4\,\mathcal{F}[u(\tau,\cdot)]\right) \le 0$$

so that

$$\mathcal{I}[u] - 4\,\mathcal{F}[u] \ge \mathcal{I}[\mathcal{B}_\star] - 4\,\mathcal{F}[\mathcal{B}_\star] = 0$$

This inequality is equivalent to $\mathcal{J}[w] \ge \mathcal{J}[w_\star]$, which establishes that optimality in (CKN) is achieved among symmetric functions. In other words, the linearized problem shows that for $\alpha \le \alpha_{\rm FS}$, the function

$$\tau \mapsto \mathcal{I}[u(\tau,\cdot)] - 4\,\mathcal{F}[u(\tau,\cdot)]$$

is monotone decreasing

- this explains why the method based on nonlinear flows provides the *optimal range for symmetry*

Entropy – production of entropy inequality

Using $\frac{d}{d\tau}\left(\mathcal{I}[u(\tau,\cdot)] - \mathcal{C}_2\,\mathcal{F}[u(\tau,\cdot)]\right) \le 0$, we know that

$$\mathcal{I}[u] - \mathcal{C}_2\,\mathcal{F}[u] \ge \mathcal{I}[\mathcal{B}_\star] - \mathcal{C}_2\,\mathcal{F}[\mathcal{B}_\star] = 0$$

As a consequence, we have that

$$\mathcal{C}_1 := \inf_u \frac{\mathcal{I}[u]}{\mathcal{F}[u]} \ge \mathcal{C}_2 = \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]}$$

With $u_\varepsilon = \mathcal{B}_\star\,\left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right)$, we observe that

$$\mathcal{C}_1 \le \lim_{\varepsilon\to0} \inf_f \frac{\mathcal{I}[u_\varepsilon]}{\mathcal{F}[u_\varepsilon]} = \inf_f \frac{\langle f, \mathcal{L}\, f\rangle}{\langle f, f\rangle} = \frac{\langle f_1, \mathcal{L}\, f_1\rangle}{\langle f_1, f\rangle_1} = \lambda_1 = \lim_{\varepsilon\to0} \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]}$$

- If $\lim_{\varepsilon\to0} \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \mathcal{C}_2$, then $\mathcal{C}_1 = \mathcal{C}_2 = \lambda_1$

This happens if $\alpha = \alpha_{\rm FS}$ and in particular in the case without weights (Gagliardo-Nirenberg inequalities)

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Thank you for your attention !