Inequalities and transport

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Optimal transport

[Cordero-Erausquin, Gangbo, Houdré, Nazaret, Villani],

[Agueh, Ghoussoub, Kang]

- Sobolev inequality: $\|f\|_{L^{2^*}} \leq S\,\|\nabla f\|_{L^2}$

- (Standard) logarithmic Sobolev inequality

- Logarithmic Sobolev inequality in $W^{1,p}(\mathbb{R}^N)$

Sobolev inequalities

$$\|f\|_{L^{2*}} \leq S\,\|\nabla f\|_{L^2}$$

$N \geq 3$. Optimal function: $f(x) = (\sigma + |x|^2)^{-(N-2)/2}$.

A proof based on mass transportation:

$$\inf\left\{\frac{1}{2\lambda^2}\int_{\mathbf{R}^N} |\nabla f|^2\,dx \ :\ \int_{\mathbf{R}^N} |f|^{2^*}\,dx = 1\right\}$$

$$= \tfrac{n(n-2)}{2(n-1)}\ \sup\left\{\textstyle\int_{\mathbf{R}^N} |g|^{2^*(1-\frac{1}{n})}dy - \frac{\lambda^2}{2}\int_{\mathbf{R}^N} |y|^2\,|g|^{2^*}dy \ :\ \int_{\mathbf{R}^N} |g|^{2^*}dy = 1\right\}$$

Mass transportation: basic results

μ and ν two Borel probability measures on $\mathbb{R}^N$. $T : \mathbb{R}^N \to \mathbb{R}^N$
$T\#\mu = \nu \Longleftrightarrow \nu(A) = \mu(T^{-1}(A))$ for any Borel measurable set A.

Theorem 1 (Brenier, McCann) $\exists T = \nabla\phi$ *such that* $T\#\mu = \nu$ *and* ϕ *is convex.*

$\mu = F(x)\,dx$, $\nu = G(x)\,dx$, $\int_{\mathbb{R}^N} F(x)\,dx = \int_{\mathbb{R}^N} G(y)\,dy = 1$

$$\forall\, b \in C(\mathbb{R}^N, \mathbb{R}^+) \quad \int_{\mathbb{R}^N} b(y)G(y)\,dy = \int_{\mathbb{R}^N} b(\nabla\phi(x))F(x)\,dx$$

Under technical assumptions: $\phi \in C^2$, supp(F) or supp(G) is convex... [Caffarelli] ϕ solves the Monge-Ampère equation

$$G(\nabla\phi)\det\mathrm{Hess}(\phi) = F$$

A PROOF OF THE SOBOLEV INEQUALITY

$$G(\nabla\phi)^{-\frac{1}{n}} = (\,\det \mathrm{Hess}(\phi))^{\frac{1}{n}} F^{-\frac{1}{n}} \leq \frac{1}{n}\,\Delta\phi\, F^{-\frac{1}{n}}$$

$$\int G(y)^{1-\frac{1}{n}} dy \leq \frac{1}{n} \int G(\nabla\phi(x))^{1-\frac{1}{n}} (\,\det \mathrm{Hess}(\phi))^{\frac{1}{n}} \Delta\phi\; dx$$

$$= \frac{1}{n} \int F^{1-\frac{1}{n}} \Delta\phi\, dx = -\frac{1}{n} \int \nabla(F^{1-\frac{1}{n}}) \cdot \nabla\phi\, dx$$

by the arithmetic-geometric inequality. $F = |f|^{2^*}$, $G = |g|^{2^*}$

$$\int |g|^{2^*(1-\frac{1}{n})} dy \leq -\frac{2(n-1)}{n(n-2)} \int (f^{\frac{n}{n-2}}) \nabla f \cdot \nabla\phi\, dx$$

$$\frac{n(n-2)}{2(n-1)} \int |g|^{2^*(1-\frac{1}{n})} dy \leq \frac{2}{\lambda^2} \int |\nabla f|^2 dx + \frac{\lambda^2}{2} \int |f|^{2^*} |\nabla\phi|^2 dx$$

by Young's inequality. Use: $\int F\, |\nabla\phi|^2\, dx = \int G|y|^2 dy$

A proof of the standard logarithmic Sobolev inequality

$G(y) = e^{-|y|^2/2}$, $F(x) = f(x)\, e^{-|x|^2/2}$, $\nabla\phi \# F\, dx = G\, dy$.

$$e^{-|\nabla\phi|^2/2} \det \mathrm{Hess}(\phi) = f(x)\, e^{-|x|^2/2}$$

$\theta(x) = \phi(x) - \frac{1}{2}|x|^2$

$$f(x)\, e^{-|x|^2/2} = \det(\mathrm{Id} + \mathrm{Hess}(\theta))\, e^{-|x+\nabla\theta(x)|^2/2}$$

$$\begin{aligned} \log f - |x|^2/2 &= -|x+\nabla\theta(x)|^2/2 + \log\Big[\det(\mathrm{Id} + \mathrm{Hess}(\theta))\Big] \\ &\le -|x+\nabla\theta(x)|^2/2 + \Delta\theta \end{aligned}$$

(use $\log(1+t) \le t$). Let $d\mu(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx$.

$$\log f \le -\frac{1}{2}|\nabla\theta|^2 - x\cdot\nabla\theta + \Delta\theta$$

$$\int f \log f\, d\mu \le -\frac{1}{2}\int \left|\sqrt{f}\,\nabla\theta + \frac{\nabla f}{\sqrt{f}}\right|^2 d\mu + \frac{1}{2}\int \frac{|\nabla f|^2}{f}\, d\mu \le \frac{1}{2}\int \frac{|\nabla f|^2}{f}\, d\mu$$

LOGARITHMIC SOBOLEV INEQUALITY IN $W^{1,p}(\mathbb{R}^N)$

$G(y) = c_{p,n}\, e^{-\frac{p}{p-1}\,|y|^{p/(p-1)}} =: f_\infty(y)$, $F(x) = f(x)\, c_{p,n}\, e^{-\frac{p}{p-1}\,|x|^{p/(p-1)}}$
$\nabla\phi \# F\, dx = G\, dy$, $d\mu(x) = f_\infty^p(x)\, dx$

$$f(x)\, e^{-\frac{p}{p-1}\,|x|^{p/(p-1)}} = \det(\mathrm{Hess}(\phi))\, e^{-\frac{p}{p-1}\,|x+\nabla\theta(x)|^{p/(p-1)}}$$

$$f^p(x) = f_\infty^p(\nabla\phi)\det(\mathrm{Id} + \mathrm{Hess}(\phi))$$

$$\int f^p \log f^p\, d\mu = \int f^p \log f_\infty^p\, d\mu + \int (\Delta\phi - n) f^p d\mu$$

$$\int \Delta\phi\, f^p d\mu = -p \int f^{p-1} \nabla f \cdot \nabla\phi\, d\mu \le \frac{\lambda^{-q}}{q} \int |f|^p\, |\nabla\phi|^{p/(p-1)} + \frac{\lambda^p}{p} \int |\nabla f|^p d\mu$$

using Young's inequality: $X = f^{p-1}\nabla\phi$, $Y = \nabla f$

$$\int X \cdot Y\, d\mu \le \frac{\lambda^{-q}}{q}\, \|X\|_q^q + \frac{\lambda^p}{p}\, \|Y\|_p^p$$