

Symmetry, symmetry breaking, and phase transitions in interpolation inequalities

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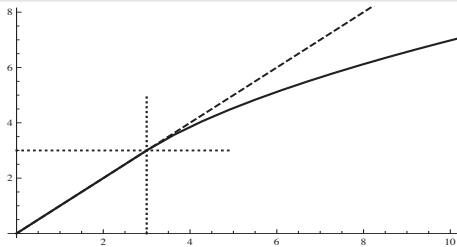
Outline

- 1 GNS inequalities on $\mathbb{S}^d$ and phase transitions
 - Subcritical inequalities and classical bifurcation results
 - Other mechanisms of phase transition
 - Caffarelli-Kohn-Nirenberg inequalities
- 2 Stability results based on entropy methods
 - Subcritical inequalities
 - Sobolev and logarithmic Sobolev inequalities
- 3 Further results on symmetry
 - Symmetry results for spinors in dimension $d = 3$
 - Symmetry results for spinors in dimension $d = 2$
 - A Sobolev inequality for a Dirac operator

Gagliardo-Nirenberg-Sobolev inequalities on the sphere and phase transitions

- ▶ Subcritical inequalities and classical bifurcation results
- ▶ Other mechanisms of phase transition; the *carré du champ* method for the pressure variable
- ▶ Caffarelli-Kohn-Nirenberg inequalities: a proof of symmetry by the parabolic *carré du champ* method

Bifurcation and phase transition in GNS inequalities



$\lambda \mapsto \mu(\lambda)$ on $\mathbb{S}^d$ with $d = 3$

$$\|\nabla u\|_{L^2(\mathbb{S}^d)}^2 + \frac{\lambda}{p-2} \|u\|_{L^2(\mathbb{S}^d)}^2 \geq \frac{\mu(\lambda)}{p-2} \|u\|_{L^p(\mathbb{S}^d)}^2$$

Taylor expansion of $u = 1 + \varepsilon \varphi_1$ as $\varepsilon \rightarrow 0$ with $-\Delta \varphi_1 = d \varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > d$$

▷ The inequality holds with $\mu(\lambda) = \lambda = d$ [Bakry, Emery, 1985]
 [Beckner, 1993], [Bidaud-Véron, Véron, 1991, Corollary 6.1]

GNS as entropy-entropy production inequalities

🟢 (subcritical) Gagliardo-Nirenberg-Sobolev inequality

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 \geq d \mathcal{E}_p[F] := \frac{d}{p-2} \left(\|F\|_{L^p(\mathbb{S}^d)}^2 - \|F\|_{L^2(\mathbb{S}^d)}^2 \right)$$

for any $p \in [1, 2) \cup (2, 2^*)$
 with $2^* := \frac{2d}{d-2}$ if $d \geq 3$ and $2^* = +\infty$ if $d = 1$ or 2

🟢 Limit $p \rightarrow 2$: the logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d} |\nabla F|^2 d\mu \geq \frac{d}{2} \mathcal{E}_2[F] := \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log \left(\frac{F^2}{\|F\|_{L^2(\mathbb{S}^d)}^2} \right) d\mu$$

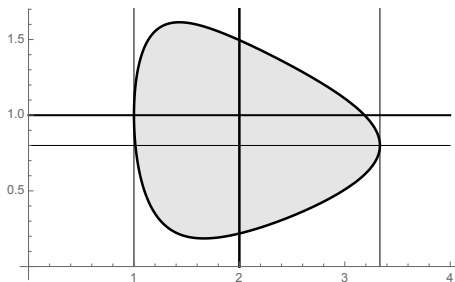
🟢 $p = 1$: Poincaré inequality

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 \geq d \mathcal{E}_1[F] := d \left(\|F\|_{L^2(\mathbb{S}^d)}^2 - \|F\|_{L^1(\mathbb{S}^d)}^2 \right)$$

Carré du champ – admissible parameters on $\mathbb{S}^d$

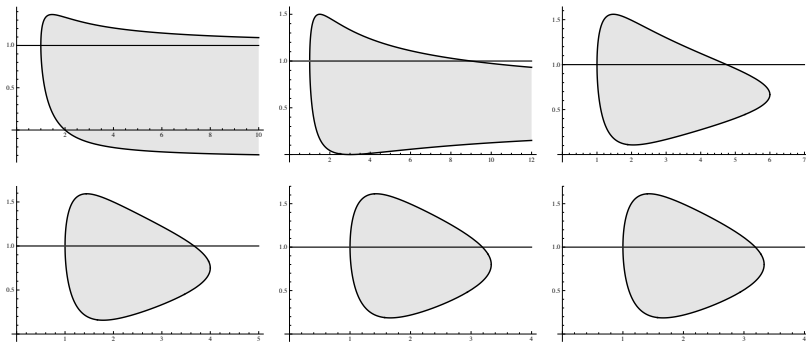
[JD, Esteban, Kowalczyk, Loss] Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p(1-m)} \left(\Delta u + (mp - 1) \frac{|\nabla u|^2}{u} \right)$$



Case $d = 5$: admissible parameters $1 \leq p \leq 2^* = 10/3$ and m (horizontal axis: p , vertical axis: m). Improved inequalities inside !

Admissible parameters



$d = 1, 2, 3$ (first line) and $d = 4, 5$ and 10 (second line)
 the curves $p \mapsto m_{\pm}(p)$ determine the admissible parameters (p, m)
 [JD, Esteban, Kowalczyk, Loss 2014] [JD, Esteban, 2019]

$$m_{\pm}(d, p) := \frac{1}{(d+2)p} \left(d p + 2 \pm \sqrt{d(p-1)(2d - (d-2)p)} \right)$$

Another *Gagliardo-Nirenberg-Sobolev* inequality

[JD, Esteban]

$$\left(\|\nabla u\|_{L^2(\mathbb{S}^d)}^2 + \frac{\lambda}{p-2} \|u\|_{L^2(\mathbb{S}^d)}^2 \right)^\theta \|u\|_{L^2(\mathbb{S}^d)}^{2(1-\theta)} \geq \left(\frac{\mu(p, \theta, \lambda)}{p-2} \right)^\theta \|u\|_{L^p(\mathbb{S}^d)}^2$$

- *Symmetry* holds if $\mu(p, \theta, \lambda) = \lambda$, optimal functions are constant
- *Symmetry breaking* if $\lambda > d\theta$: take $u_\varepsilon := 1 + \varepsilon \varphi$, $\Delta \varphi + d\varphi = 0$

Bakry-Emery exponent : $2^\# := +\infty$ if $d = 1$, $2^\# := (2d^2 + 1)/(d - 1)^2$ if $d \geq 2$

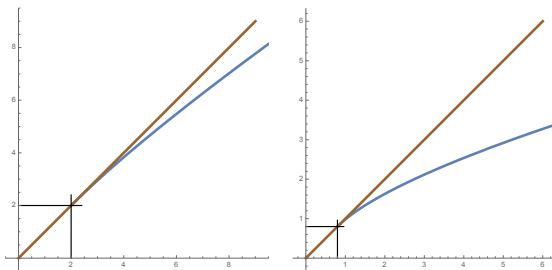
and take $p \in (2, 2^\#]$

$$\theta^\# := 3 \frac{p-2}{4p-7} \quad \text{if } d = 1, \quad \theta^\# := 1 + \frac{(p-1)(2^\# - p)}{p-2} \left(\frac{d-1}{d+2} \right)^2 \quad \text{if } d \geq 2$$

Proposition

Let $d \geq 1$, $p \in (2, 2^\#)$, and $\theta \geq \theta^\#$. The function $\lambda \mapsto \mu(p, \theta, \lambda)$ is monotone increasing, concave and $\mu(p, \theta, \lambda) < \lambda$ if and only if $\lambda > d\theta$

Second order phase transition



$d = 1, p = 5$: $\theta = 2$ (left) $\theta = 0.8$ (right). Bifurcation at $\lambda = \mu = d\theta$

Parameter range

Theorem (Bou Dagher, JD)

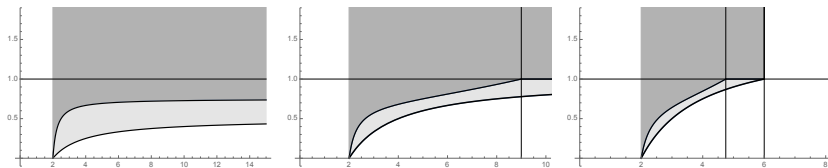
Let $d \geq 1$, $p \in (2, 2^*)$ and $\theta > \theta_* := d(p-2)/(2p)$

The function $\lambda \mapsto \mu(p, \theta, \lambda)$ is monotone increasing, concave

$$\mu(p, \theta, \lambda) \sim \kappa \lambda^{1-\theta_*/\theta} \quad \text{as } \lambda \rightarrow +\infty$$

$$\mu(p, \theta, \lambda) \leq \lambda \text{ and } \mu(p, \theta, \lambda) < \lambda \text{ if } \lambda > d\theta$$

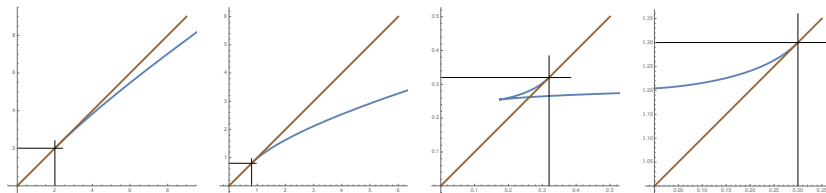
$$\mu(p, \theta, \lambda) = \lambda \text{ if } \lambda \leq d\theta, \theta \geq \theta^\# , p \in (2, 2^\#] \text{ or } p > 2 \text{ if } d = 1$$



horizontal axis: p , vertical axis: θ

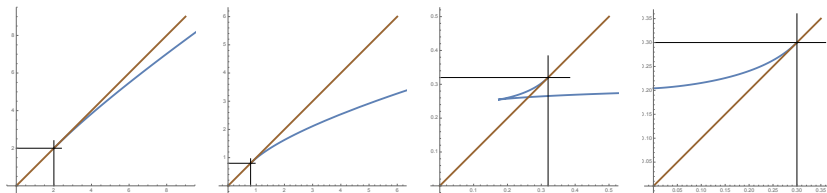
in dimensions $d = 1$, $d = 2$ and $d = 3$ (from left to right)

Second and first order phase transitions

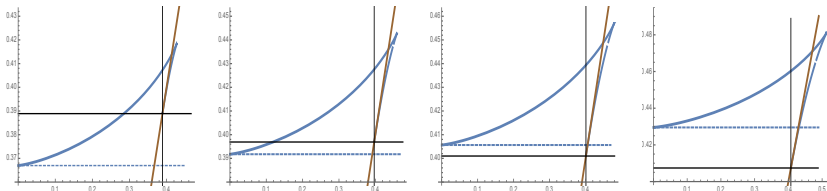


$d = 1, p = 5, \theta = 2$: $\theta = 0.8, \theta = 0.32$ and $\theta = \theta_* = 0.3$

Second and first order phase transitions



$d = 1, p = 5, \theta = 2$: $\theta = 0.8, \theta = 0.32$ and $\theta = \theta_\star = 0.3$



Critical case: $d = 1, \theta = \theta_\star$, for $p = 9.0, 9.7, 10.1$ and 10.8

Reparametrization and consequences

Euler-Lagrange equation for an optimal function (with $\theta = 1$)

$$-\Delta u + \frac{\Lambda}{p-2} u = u^{p-1} \quad (\text{EL}_{1,\Lambda})$$

Theorem (Bou Dagher, JD)

Let $d \geq 1$, $p \in (2, 2^*)$, $\theta \geq \theta_*$

A solution u of $(\text{EL}_{1,\Lambda})$ also solves $(\text{EL}_{\theta,\lambda})$ for $\lambda = \lambda(\theta, \Lambda)$ with

$$\lambda(\theta, \Lambda) := \frac{1}{\theta} \left(\Lambda + (1 - \theta)(p - 2) \frac{\|\nabla u\|_{L^2(\mathbb{S}^d)}^2}{\|u\|_{L^2(\mathbb{S}^d)}^2} \right)$$

- For $\lambda > 0$ small enough, we have $\mu(\theta, \lambda) = \lambda$
- For $\theta - \theta_* > 0$ small enough, symmetry breaking occurs for $\lambda < d\theta$

Symmetry breaking with $\lambda < d\theta$ means *first order phase transition*

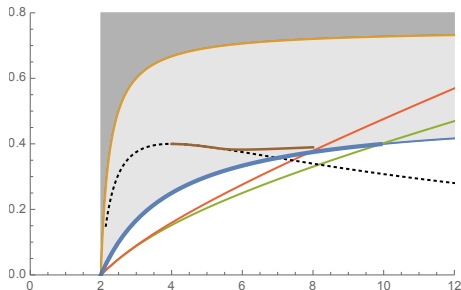
More qualitative properties

Proposition (Bou Dagher, JD)

$$\text{Let } \theta_0 := \frac{(d+2)(d+3)(p-2)}{2(p^2+2p-6)+d(p^2+6p-12)-d^2(p-2)^2}$$

Assuming that the curve $\mathcal{C} : [d, d + \epsilon) \rightarrow (\mathbb{R}^+)^2$ is smooth enough:

- If $\theta \neq \theta_0$, the curve $\mathcal{C}$ bifurcates from (d, d) tangentially to $\mu = \lambda$
- The curve $\mathcal{C}$ is concave and below $\mu = \lambda$ (on the right) if $\theta > \theta_0$
- The curve $\mathcal{C}$ is convex and above the line $\mu = \lambda$ (on the left) if $\theta < \theta_0$



- blue curve: $p \mapsto \theta_*(p)$
- yellow curve: if $\theta \geq \theta^\#(p)$, the phase transition is of second order
- red curve: if it is below $p \mapsto \theta_*(p)$, the phase transition is of first order for $\theta - \theta_*(p) > 0$ small (Gaussian test functions)
- green curve: if $\kappa(p, \theta_*) < \theta_*(p)$, the phase transition is of first order for $\theta - \theta_*(p) > 0$ small (comparison with GNS on $\mathbb{R}^d$)
- black, dotted curve: $p \mapsto \theta_0(p)$ (at the bifurcation point)
- brown curve $p \mapsto \theta_\bullet(p)$: a numerical approximation of the threshold between first / second order phase transitions

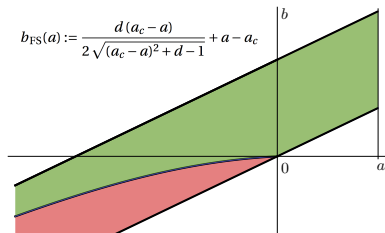
References

[Gidas, Spruck], [Bidaut-Véron, Véron]
[Bakry, Emery]; [Demange]
[Obata]
[Toscani et al.], [Arnold et al.],...

Joint paper with E. Bou Dagher
***Interpolation inequalities on the sphere and phase transition:
rigidity, symmetry and symmetry breaking***
[arXiv:2210.16878](#)

Joint paper with M. Bonforte, B. Nazaret and N. Simonov
***Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows,
regularity and the entropy method***
[arXiv:2007.03674](#), *Memoirs of the AMS* 308 (1554), 2025

The critical Caffarelli-Kohn-Nirenberg inequality



$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} dx$$

$$a \leq b \leq a + 1, a < a_c, d \geq 3$$

$$p = \frac{2d}{d-2+2(b-a)} > 0, a_c = \frac{1}{2}(d-2)$$

▷ A radial optimal function:
 $v_*(x) = (1 + |x|^{(p-2)(a_c-a)})^{-2/(p-2)}$
 among radially symmetric functions

Theorem (JD, Esteban, Loss, 2015)

There is *symmetry*, i.e., $C_{a,b} = C_{a,b}^*$, and all optimal functions are radially symmetric if $b_{FS(a)} \leq b < a + 1$. If $a < b < b_{FS(a)}$, then there is *symmetry breaking*, $C_{a,b} > C_{a,b}^*$, and optimal functions are not radially symmetric.

[Caffarelli, Kohn, Nirenberg (1984)], [F. Catrina, Z.-Q. Wang (2001)]
 [Smets, Willem], [Catrina, Wang], [Felli, Schneider]
 [Bonforte, JD, Nazaret, Muratori]

A new proof: rewriting of CKN

1) **Change of variables:** $v(r, \omega) = u(r^\alpha, \omega)$, $D_\alpha u = (\alpha \partial_r u, \nabla_\omega u)$

$$\int_{\mathbb{R}^d} |D_\alpha u|^2 |x|^{n-d} dx \geq C_{\alpha,n} \left(\int_{\mathbb{R}^d} |u|^p |x|^{n-d} dx \right)^{2/p}$$

with $n = 2p/(p-2)$. Symmetry means that the Aubin-Talenti function $u_*(x) := (1 + |x|^2)^{-(n-2)/2}$ realizes the equality case

2) **Relative measure:** with $w = u/u_*$ and $d\mu_q(x) = |u_*(x)|^q |x|^{n-d} dx$

$$\int_{\mathbb{R}^d} |D_\alpha w|^2 d\mu_2 dx + \frac{1}{4} \alpha^2 n(n-2) \int_{\mathbb{R}^d} |w|^2 d\mu_p dx \geq C_{\alpha,n} \left(\int_{\mathbb{R}^d} |w|^p d\mu_p dx \right)^{2/p}$$

3) **Stereographic projection:** $w(x) = f(z, \omega)$ with $z = \frac{1-|x|^2}{1+|x|^2}$, $\omega = \frac{2x}{1+|x|^2}$

$$\begin{aligned} \int_{\mathbb{S}^d} \left(\alpha^2 (1-z^2) |f'|^2 + \frac{|\nabla_\omega f|^2}{1-z^2} \right) d\sigma_n + \frac{\alpha^2}{4} n(n-2) \int_{\mathbb{S}^d} |f|^2 d\sigma_n \\ \geq K_{\alpha,n} \left(\int_{\mathbb{S}^d} |f|^p d\sigma_n \right)^{2/p} \end{aligned}$$

$$d\sigma_n = Z_n^{-1} (1-z^2)^{(n-2)/2} dz d\omega, \quad z \in [-1, +1], \quad \omega \in \mathbb{S}^{d-1}$$

A new proof: fast diffusion equation and *carré du champ*

Let $'$ and ∇ denote the derivatives with respect to $z \in [-1, 1]$ and $\omega \in \mathbb{S}^{d-1}$, $\Delta = \nabla \cdot \nabla$ and

$$\mathbf{D}v := \left(\alpha \sqrt{1-z^2} v', \frac{1}{\sqrt{1-z^2}} \nabla v \right), \quad \mathbf{L}v := \mathbf{D} \cdot \mathbf{D}v$$

$$\mathbf{L}v = \alpha^2 \mathcal{L}v + \frac{1}{1-z^2} \Delta v, \quad \mathcal{L}v := (1-z^2) v'' - n z v'$$

Weighted fast diffusion equation

$$\frac{\partial v}{\partial t} = \mathbf{L}v^m = -\mathbf{D} \cdot (v \mathbf{D}P), \quad P = \frac{m}{1-m} v^{m-1}, \quad m = \frac{n-1}{n}, \quad p = \frac{2n}{n-2}$$

$$v = u^p \quad \text{and} \quad \mathcal{D}(t) := \int_{\mathbb{S}^d} |\mathbf{D}u(t, \cdot)|^2 d\sigma_n + \frac{n\alpha^2}{p-2} \int_{\mathbb{S}^d} |u(t, \cdot)|^2 d\sigma_n$$

Proposition (Bou Dagher, JD)

$$\mathcal{D}'(t) \leq 0 \text{ if } \alpha \leq \alpha_{\text{FS}} := \sqrt{\frac{d-1}{n-1}}$$

Nonlinear *carré du champ* techniques and Felli & Schneider (FS)

[JD, Zhang, 2021]: weights $(1 + \varepsilon - z^2) \implies$ First parabolic proof

Details

$$\mathcal{D}'(t) = - \frac{8}{(p+2)^2} \int_{\mathbb{S}^d} v^m (\mathbf{K}[P] - m n \alpha^2 |\mathbf{D}P|^2) d\sigma_n$$

$$\text{with } \mathbf{K}[P] := \frac{1}{2} \mathbf{L} (|\mathbf{D}P|^2) - \mathbf{D}P \cdot \mathbf{D}(\mathbf{L}P) - \frac{1}{n} (\mathbf{L}P)^2$$

$$\begin{aligned} \mathbf{K}[P] = m & \left| \alpha^2 (1 - z^2) P'' - \frac{\Delta P}{(n-1)(1-z^2)} \right|^2 + 2 \alpha^2 \left| \nabla P' + \frac{z \nabla P}{1-z^2} \right|^2 \\ & + \alpha^2 (n-1) |\mathbf{D}P|^2 \\ & + (1 - z^2)^{-2} \left(\frac{1}{2} \Delta(|\nabla P|^2) - \nabla P \cdot \nabla \Delta P - \frac{(\Delta P)^2}{n-1} - (n-2) \alpha^2 |\nabla P|^2 \right) \end{aligned}$$

Corollary (Bou Dagher, JD)

If $n > d \geq 3$, $m = (n-1)/n$ and $p = 2n/(n-2)$, then

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \rho^q \left(\frac{1}{2} \Delta(|\nabla P|^2) - \nabla P \cdot \nabla \Delta P - \frac{(\Delta P)^2}{n-1} \right) d\omega \\ & = a \int_{\mathbb{S}^{d-1}} \rho^q \left\| \mathbf{L}P - \frac{b}{a} \mathbf{M}P \right\|^2 d\omega + \left(c - \frac{b^2}{a} \right) \int_{\mathbb{S}^{d-1}} \rho^q \frac{|\nabla P|^4}{p^2} d\omega \\ & \quad + (n-2) \frac{d-1}{n-1} \int_{\mathbb{S}^{d-1}} \rho^q |\nabla P|^2 d\omega \end{aligned}$$

Regularization as in [JD, Zhang]

Stability results based on entropy methods

- ▷ Subcritical Gagliardo-Nirenberg inequalities on $\mathbb{S}^d$
- ▷ Sobolev inequality: the Bianchi-Egnell stability estimate made constructive
- ▷ The Gaussian logarithmic Sobolev inequality seen as an infinite dimensional limit

Gagliardo-Nirenberg inequalities: stability

An improved inequality under orthogonality constraint (Π_1 is a projection on some positive spherical harmonic functions) and the stability inequality arising from the *carré du champ* method can be combined *in the subcritical case* as follows

Theorem (Brigati, JD, Simonov)

Let $d \geq 1$ and $p \in (1, 2^*)$. For any $F \in H^1(\mathbb{S}^d, d\mu)$, we have

$$\begin{aligned} \int_{\mathbb{S}^d} |\nabla F|^2 d\mu - d \mathcal{E}_p[F] \\ \geq \mathcal{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|_{L^2(\mathbb{S}^d)}^4}{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 + \|F\|_{L^2(\mathbb{S}^d)}^2} + \|\nabla(\text{Id} - \Pi_1) F\|_{L^2(\mathbb{S}^d)}^2 \right) \end{aligned}$$

for some explicit stability constant $\mathcal{S}_{d,p} > 0$

▷ The result holds true for the logarithmic Sobolev inequality ($p = 2$), again with an explicit constant $\mathcal{S}_{d,2}$, for any finite dimension d

▷ The *far away* regime: use an improved interpolation inequality

If $\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 / \|F\|_{L^p(\mathbb{S}^d)}^2 \geq \vartheta_0 > 0$, by the convexity of ψ

$$\begin{aligned} \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[F] &\geq d \|F\|_{L^p(\mathbb{S}^d)}^2 \psi \left(\frac{1}{d} \frac{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2}{\|F\|_{L^p(\mathbb{S}^d)}^2} \right) \\ &\geq \frac{d}{\vartheta_0} \psi \left(\frac{\vartheta_0}{d} \right) \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 \end{aligned}$$

▷ The *local* case: $\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 < \vartheta_0 \|F\|_{L^p(\mathbb{S}^d)}^2$

Take $\|F\|_{L^p(\mathbb{S}^d)} = 1$, assume that $\frac{d \vartheta_0}{d - (p-2) \vartheta_0} > 0$ and deduce from the Poincaré inequality that

$$1 - \frac{\vartheta}{d} < \left(\int_{\mathbb{S}^d} F d\mu \right)^2 \leq 1$$

+ a Taylor expansion using a partial decomposition
 on spherical harmonics

Large dimensional limit

... based on the Maxwell-Poincaré lemma [McKean, 1973]
Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1, 2]$

$$\|\nabla u\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 \geq \frac{d}{p-2} \left(\|u\|_{L^p(\mathbb{S}^d, d\mu_d)}^2 - \|u\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 \right)$$

Theorem (Brigati, JD, Simonov)

Let $v \in H^1(\mathbb{R}^n, dx)$ with compact support, $d \geq n$ and

$$u_d(\omega) = v\left(\omega_1/\sqrt{d}, \omega_2/\sqrt{d}, \dots, \omega_n/\sqrt{d}\right)$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. With $d\gamma(y) := (2\pi)^{-n/2} e^{-\frac{1}{2}|y|^2} dy$,

$$\begin{aligned} \lim_{d \rightarrow +\infty} d \left(\|\nabla u_d\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 - \frac{d}{2-p} \left(\|u_d\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 - \|u_d\|_{L^p(\mathbb{S}^d, d\mu_d)}^2 \right) \right) \\ = \|\nabla v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \frac{1}{2-p} \left(\|v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \|v\|_{L^p(\mathbb{R}^n, d\gamma)}^2 \right) \end{aligned}$$

An explicit stability result for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 \geq S_d \|f\|_{L^{2^*}(\mathbb{R}^d)}^2 \quad \forall f \in \dot{H}^1(\mathbb{R}^d) = \mathcal{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c (a + |x - b|^2)^{-\frac{d-2}{2}}, \quad a \in (0, \infty), \quad b \in \mathbb{R}^d, \quad c \in \mathbb{R}$$

Theorem (JD, Esteban, Figalli, Frank, Loss)

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in H^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f\|_{L^{2^*}(\mathbb{R}^d)}^2 \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^d)}^2$$

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \rightarrow +\infty$

Stability for the Sobolev inequality: the history

▷ [Rodemich, 1969], [Aubin, 1976], [Talenti, 1976]

In the inequality $\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 \geq S_d \|f\|_{L^{2^*}(\mathbb{R}^d)}^2$, the optimal constant is

$$S_d = \frac{1}{4} d (d-2) |\mathbb{S}^d|^{1-2/d}$$

with equality on the manifold $\mathcal{M} = \{g_{a,b,c}\}$ of the *Aubin-Talenti functions*

▷ [Lions] a qualitative stability result

$$\text{if } \lim_{n \rightarrow \infty} \|\nabla f_n\|_2^2 / \|f_n\|_{2^*}^2 = S_d, \text{ then } \lim_{n \rightarrow \infty} \inf_{g \in \mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 / \|\nabla f_n\|_2^2 = 0$$

▷ [Brezis, Lieb, 1985] a quantitative stability result ?

▷ [Bianchi, Egnell, 1991] there is some non-explicit $C_{BE} > 0$ such that

$$\|\nabla f\|_2^2 \geq S_d \|f\|_{2^*}^2 + C_{BE} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

🟢 The strategy of Bianchi & Egnell involves two steps:

– a local (spectral) analysis: the *neighbourhood* of $\mathcal{M}$

– a local-to-global extension based on concentration-compactness :

🟢 The constant C_{BE} is not explicit the *far away regime*

A stability result for the logarithmic Sobolev inequality

- Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\begin{aligned} & \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 - \frac{1}{4} d(d-2) \left(\|F\|_{L^{2^*}(\mathbb{S}^d)}^2 - \|F\|_{L^2(\mathbb{S}^d)}^2 \right) \\ & \geq \frac{\beta}{d} \inf_{G \in \mathcal{M}(\mathbb{S}^d)} \left(\|\nabla F - \nabla G\|_{L^2(\mathbb{S}^d)}^2 + \frac{1}{4} d(d-2) \|F - G\|_{L^2(\mathbb{S}^d)}^2 \right) \end{aligned}$$

- Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d \rightarrow +\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi|x|^2} dx$

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta > 0$ as in the result for the Sobolev inequality

$$\begin{aligned} & \|\nabla u\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \pi \int_{\mathbb{R}^n} u^2 \log \left(\frac{|u|^2}{\|u\|_{L^2(\mathbb{R}^n, d\gamma)}^2} \right) d\gamma \\ & \geq \frac{\beta \pi}{2} \inf_{a \in \mathbb{R}^n, c \in \mathbb{R}} \int_{\mathbb{R}^n} |u - c e^{a \cdot x}|^2 d\gamma \end{aligned}$$

Stability for the logarithmic Sobolev inequality

- ▷ [Gross, 1975] *Gaussian logarithmic Sobolev inequality* for $n \geq 1$

$$\|\nabla u\|_{L^2(\mathbb{R}^n, d\gamma)}^2 \geq \pi \int_{\mathbb{R}^n} u^2 \log \left(\frac{|u|^2}{\|u\|_{L^2(\mathbb{R}^n, d\gamma)}^2} \right) d\gamma$$

- ▷ [Weissler, 1979] scale invariant (but dimension-dependent) version of the Euclidean form of the inequality

- ▷ [Stam, 1959], [Federbush, 1969], [Costa, 1985] Cf. [Villani, 2008]

- ▷ [Bakry, Emery, 1984], [Carlen, 1991] equality iff

$$u \in \mathcal{M} := \{w_{a,c} : (a, c) \in \mathbb{R}^d \times \mathbb{R}\} \quad \text{where} \quad w_{a,c}(x) = c e^{a \cdot x} \quad \forall x \in \mathbb{R}^n$$

- [Carlen, 1991] reinforcement of the inequality (Wiener transform)

- ▷ [McKean, 1973], [Beckner, 92] (LSI) as a large d limit of Sobolev

- ▷ [Bobkov, Gozlan, Roberto, Samson, 2014], [Indrei et al., 2014-23]

stability in Wasserstein distance, in $W^{1,1}$, etc.

- ▷ [JD, Toscani, 2016] Comparison with Weissler's form, a (dimension dependent) improved inequality

- ▷ [Fathi, Indrei, Ledoux, 2016] improved inequality assuming a Poincaré inequality (Mehler formula)

The global and the local problem

$$d(u, v)^2 := q[u - v] \quad \text{where} \quad q[w] := \|\nabla w\|_{L^2(\mathbb{S}^d)}^2 + \frac{d}{p-2} \|w\|_{L^2(\mathbb{S}^d)}^2$$

• *deficit* : $\delta[u] := \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 + \frac{d}{p-2} \left(\|u\|_{L^2(\mathbb{S}^d)}^2 - \|u\|_{L^p(\mathbb{S}^d)}^2 \right), \quad p = \frac{2d}{d-2}$

• *distance* to the set $\mathcal{M}$ of the Aubin-Talenti (optimal) functions

$$d(u, \mathcal{M}) := \inf_{v \in \mathcal{M}} d(u, v)$$

$\lim_{t \rightarrow +\infty} d(u(t, \cdot), \mathcal{M}) = 0$ and $\delta[u(t, \cdot)]$ is monotone non-increasing if

$$\frac{\partial u}{\partial t} = m u^{(m-1)p} \left(\Delta u + (mp - 1) \frac{|\nabla u|^2}{u} \right)$$

For a given $\varepsilon \in (0, 1)$, u is in the **far away** regime if

$$d(u, \mathcal{M})^2 > \varepsilon q[u]$$

and in the neighbourhood of $\mathcal{M}$ if $d(u, \mathcal{M})^2 \leq \varepsilon q[u]$

local stability : $\mathcal{I}(\varepsilon) := \inf \left\{ \frac{\delta[u]}{d(u, \mathcal{M})^2} : u \in H^1(\mathbb{S}^d, d\sigma), d(u, \mathcal{M})^2 \leq \varepsilon q[u] \right\}$

A new proof for the global to local reduction

[Bonforte, JD, Esteban, Figalli, Frank, Loss] on an idea by Christ. If we start in the *far away* regime, which means

$$d(u|_{t=0}, \mathcal{M})^2 > \varepsilon q[u|_{t=0}]$$

using $d(u|_{t=0}, \mathcal{M}) \leq d(u|_{t=0}, 0) = q[u|_{t=0}]$, $\|u(t, \cdot)\|_{L^p(\mathbb{S}^d)} = 1$ we obtain

$$\frac{\delta[u|_{t=0}]}{d(u|_{t=0}, \mathcal{M})^2} \geq \frac{q[u|_{t=0}] - \frac{d}{p-2}}{q[u|_{t=0}]} \geq 1 - \frac{\frac{d}{p-2}}{q[u(t, \cdot)]} = \frac{\delta[u(t, \cdot)]}{q[u(t, \cdot)]}$$

We know that

$$\lim_{t \rightarrow +\infty} q[u(t, \cdot)] = \frac{d}{p-2} \quad \text{and} \quad \lim_{t \rightarrow +\infty} d(u(t, \cdot), \mathcal{M})^2 = 0$$

so that for some $t_* > 0$ we have

$$q[u(t_*, \cdot)] = \frac{1}{\varepsilon} d(u(t_*, \cdot), \mathcal{M})^2$$

$$\frac{\delta[u|_{t=0}]}{d(u|_{t=0}, \mathcal{M})^2} \geq \frac{\delta[u(t_*, \cdot)]}{q[u(t_*, \cdot)]} = \varepsilon \frac{\delta[u(t_*, \cdot)]}{d(u(t_*, \cdot), \mathcal{M})^2} \geq \varepsilon \mathcal{I}(\varepsilon)$$

References

Joint papers with M.J. Esteban, A. Figalli, R. Frank, M. Loss
Sharp stability for Sobolev and log-Sobolev inequalities, with
optimal dimensional dependence

[arXiv: 2209.08651](#), Cambridge J. Math. 2025

A short review on improvements and stability for some
interpolation inequalities

[arXiv: 2402.08527](#), Proc. ICIAM 2023

Joint work with G. Brigati and N. Simonov
Logarithmic Sobolev inequalities: a review on stability and
instability results; logarithmic uncertainty principle

[arXiv: 2504.08658](#), La Matematica, to appear

Stability results for Sobolev, logarithmic Sobolev,
and related inequalities

Proceedings of the Summer School “Direct and Inverse Problems with
Applications, and Related Topics” August 19-23, 2024

[arXiv: 2411.13271](#)

Further results (and conjectures) on symmetry

- ▶ Caffarelli-Kohn-Nirenberg inequalities for spinor (complex) valued functions in dimension $d = 3$
- ▶ Caffarelli-Kohn-Nirenberg inequalities for spinor (complex) valued functions in dimension $d = 2$
- ▶ A Sobolev inequality for Dirac operators

Symmetry results for spinors in dimension $d = 3$

We consider 2 -spinors, which are $\mathbb{C}^2$ -valued function

$$\mathbb{R}^3 \ni x \mapsto \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \in \mathbb{C}^2$$

Caffarelli-Kohn-Nirenberg inequalities for spinors

$$\int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \psi(x)|^2}{|x|^{2\alpha}} dx \geq C_{\alpha,\beta} \left(\int_{\mathbb{R}^3} \frac{|\psi(x)|^p}{|x|^{\beta p}} dx \right)^{2/p} \quad (\text{SCKN})$$

where $\partial_j = \partial_{x_j}$ and the gradient term is defined by

$$\sigma \cdot \nabla \psi = \sum_{j=1}^3 \sigma_j \partial_j \psi$$

and $\sigma = (\sigma_j)_{j=1,2,3}$ is the family of the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\alpha \leq \beta \leq \alpha + 1$, $p = 6/(1 - 2\alpha + 2\beta)$, $C_{\alpha,\beta} \geq 0$ is the best constant

Symmetry for spinors

Proposition (JD, Esteban, Frank, Loss)

Let $\Lambda := \{k - \frac{1}{2} : k \in \mathbb{Z} \setminus \{0\}\}$

If $\alpha \in \Lambda$, then $\mathcal{C}_{\alpha,\beta} = 0$ for all $\alpha \leq \beta \leq \alpha + 1$

If $\alpha \notin \Lambda$, then $\mathcal{C}_{\alpha,\beta} > 0$ for all $\alpha \leq \beta \leq \alpha + 1$

Angular decomposition in eigenspaces of $\sigma \cdot L$

$$L^2(\mathbb{S}, \mathbb{C}^2; d\omega) = \bigoplus_{k \in \mathbb{Z} \setminus \{-1\}} \mathcal{H}_k$$

where $L := \omega \wedge (-i \nabla)$ is the *angular momentum* operator

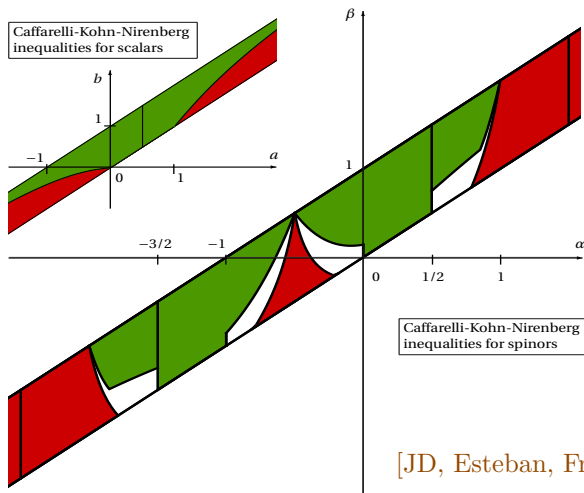
Definition

A spinor ψ on $\mathbb{R}^3$ is *symmetric* if there is a constant $\chi_0 \in \mathbb{C}^2$ and a complex-valued function f on $\mathbb{R}_+$ such that

$$\psi(x) = f(r) \chi_0 \quad \text{or} \quad \psi(x) = f(r) \sigma \cdot \omega \chi_0, \quad r = |x|, \quad \omega = x/r$$

i.e., $\psi \in \mathcal{H}_0$ or $\mathcal{H}_{-2}$

Results



[JD, Esteban, Frank, Loss]

Symmetry regions: green; symmetry breaking regions: red

The ingredients of the proof

- Existence of optimizers
- A Hardy inequality case: $C_{\alpha, \alpha+1} = \min_{k \in \mathbb{Z} \setminus \{-1\}} (k - \alpha + \frac{1}{2})^2$
- Passing to logarithmic variables ▷ see slide +1

$$\iint_{\mathbb{R} \times \mathbb{S}} \left(|\partial_s \phi|^2 + |(\sigma \cdot L - \alpha + \frac{1}{2}) \phi|^2 \right) ds d\omega \geq C_{\alpha, p} \left(\iint_{\mathbb{R} \times \mathbb{S}} |\phi|^p ds d\omega \right)^{2/p}$$

- Monotonicity properties: for some $\alpha_* : (2, 6) \rightarrow [-1/2, 0]$

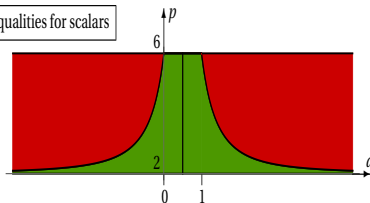
$$C_{\alpha, p} < C_{\alpha, p}^* \quad \text{if} \quad -1/2 \leq \alpha < \alpha_*(p)$$

$$C_{\alpha, p} = C_{\alpha, p}^* \quad \text{if} \quad \alpha_*(p) \leq \alpha < 1/2$$

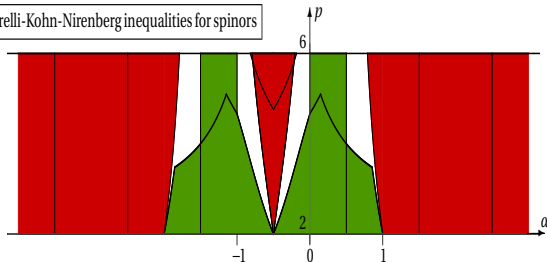
- A Gagliardo-Nirenberg interpolation inequality for spinors on the sphere based on tools of harmonic analysis ▷ see slide +2
- A Keller-Lieb-Thirring estimate
- A chain of (optimal) estimates
- Instability: study of the quadratic form obtained by linearization and representation using spherical harmonics ▷ see slide +2

Logarithmic variables

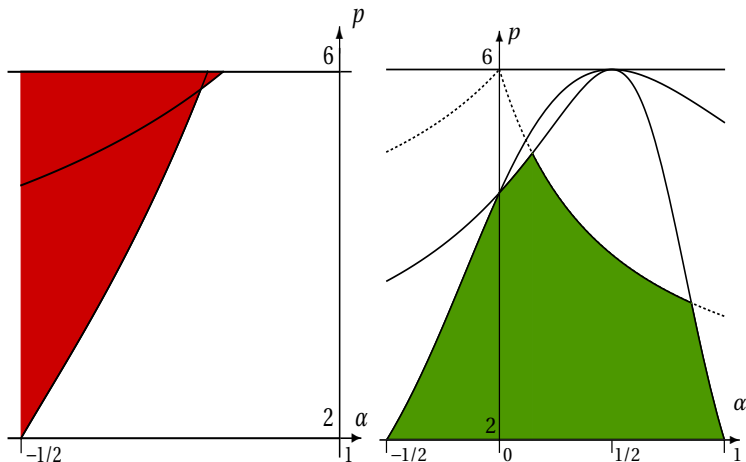
Caffarelli-Kohn-Nirenberg inequalities for scalars



Caffarelli-Kohn-Nirenberg inequalities for spinors



Symmetry *versus* symmetry breaking (details)



Symmetry results for spinors in dimension $d = 2$

the $d = 2$ spinorial Caffarelli-Kohn-Nirenberg inequality

$$\int_{\mathbb{R}^2} \frac{|\sigma \cdot \nabla \psi|^2}{|x|^{2\alpha}} dx \geq C_{\alpha,p} \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^{\beta p}} dx \right)^{2/p} \quad (\text{SCKN})$$

for spinor valued functions $\psi : \mathbb{R}^2 \rightarrow \mathbb{C}^2$

the logarithmic Caffarelli-Kohn-Nirenberg inequality

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{S}^1} \left(|\partial_s \phi(s, \theta)|^2 + |(\alpha - i\sigma_3 \partial_\theta) \phi(s, \theta)|^2 \right) ds d\theta \\ \geq C_{\alpha,p} \left(\int_{\mathbb{R}} \int_{\mathbb{S}^1} |\phi(s, \theta)|^p ds d\theta \right)^{2/p} \end{aligned}$$

Interpolation inequalities for Aharonov-Bohm magnetic fields

$$A(x) = (x_2, -x_1)/|x|^2$$

$$\int_{\mathbb{R}^2} |(-i\nabla - \alpha A)\psi|^2 dx \geq C_{\alpha,p}^{\text{AB}} \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2} dx \right)^{2/p} \quad (\text{AB})$$

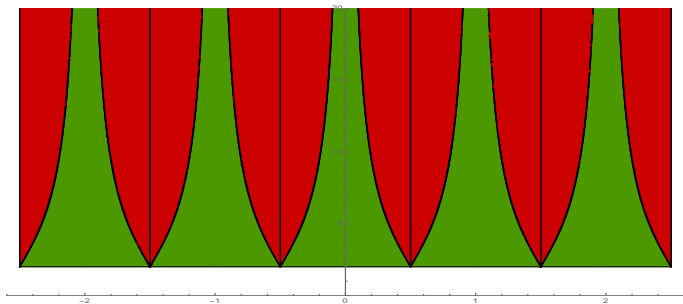
Theorem (JD, Frank, Weixler)

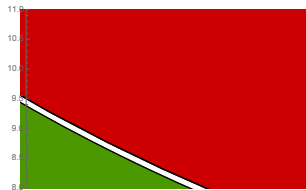
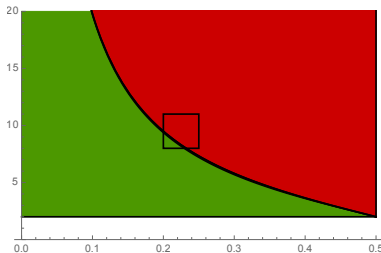
$$C_{\alpha,p} = C_{\alpha,p}^{\text{AB}} \text{ for any } (\alpha, p) \in (0, 1/2) \times (2, +\infty)$$

Symmetry *versus* symmetry breaking

Theorem (JD, Frank, Weixler)

- For every $\alpha \in (0, 1/2)$ and $p > 2$, there is an optimizer with $C_{\alpha,p} > 0$ and $\lim_{\alpha \rightarrow 0+} C_{\alpha,p} = 0$. Symmetry holds if and only if $\alpha \in (0, \alpha(p)]$ for some function $p \mapsto \alpha(p) : (2, \infty) \rightarrow (0, 1/2)$
- The symmetry and symmetry breaking regions are symmetric with respect to $\alpha = 0$ and 1-periodic





(SCKN) with $d = 2$. Horizontal axis: $\alpha \in (0, 1/2)$. Vertical axis: $p \in (2, \infty)$

● Symmetry range: green, by the equivalence with Aharonov-Bohm problem and entropy methods for flows associated to (CKN) inequalities

● Symmetry breaking range: red and blue; Undecided in the tiny white gap

● magnetic ring: an interpolation inequality on S^1

[JD, Esteban, Laptev, Loss]

● Aharonov-Bohm and Caffarelli-Kohn-Nirenberg inequalities

[Bonheure, JD, Esteban, Laptev, Loss]

● a Gegenbauer polynomial basis to study linear instability numerically

The Keller-Lieb-Thirring inequality (Schrödinger)

With $q < 2^* := 2d/(d-2)$ if $d \geq 3$, and $\vartheta = d(q-2)/(2q)$

The *Gagliardo-Nirenberg-Sobolev inequality*

$$\|\nabla u\|_{L^2(\mathbb{R}^d)}^\vartheta \|u\|_{L^2(\mathbb{R}^d)}^{1-\vartheta} \geq \mathcal{C}_q \|u\|_{L^q(\mathbb{R}^d)}$$

can be rewritten as

$$\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \lambda \|u\|_{L^2(\mathbb{R}^d)}^2 \geq C_q \lambda^{1-\vartheta} \|u\|_{L^q(\mathbb{R}^d)}^2$$

for any $(\lambda, u) \in (0, +\infty) \times H^1(\mathbb{R}^d)$, with $\mathcal{C}_q^2 = \vartheta^\vartheta (1-\vartheta)^{1-\vartheta} C_q$

Let λ be such that $C_q \lambda^{1-\vartheta} = \|V\|_{L^p(\mathbb{R}^d)}$. The Schrödinger energy is

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla u|^2 dx - \int_{\mathbb{R}^d} V |u|^2 dx &\geq \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - \|V\|_{L^p(\mathbb{R}^d)} \|u\|_{L^q(\mathbb{R}^d)}^2 \\ &\geq - \left(C_q^{-1} \|V\|_{L^p(\mathbb{R}^d)} \right)^{1/(1-\vartheta)} \|u\|_{L^2(\mathbb{R}^d)}^2 \end{aligned}$$

Keller-Lieb-Thirring inequality : with $\eta = 1/(1-\vartheta) = 2p/(2p-d)$

$$\forall V \in L^p(\mathbb{R}^d), \quad (\lambda_1 - (-\Delta - V))_- \geq -K_p \|V\|_{L^p(\mathbb{R}^d)}^\eta$$

The *free Dirac operator* in dimension d is defined by

$$\mathcal{D}_m := \sum_{j=1}^d \alpha_j (-i \partial_j) + m \beta = \alpha \cdot (-i \nabla) + m \beta$$

• $d = 1$, $\alpha = \sigma_2$ and $\beta = \sigma_3$: $\mathcal{D}_m = \sigma_2 (-i \partial_1) + m \sigma_3$

• $d = 2$, $\alpha = (\sigma_j)_{j=1,2}$ and $\beta = \sigma_3$: $\mathcal{D}_m = \sum_{j=1}^2 \sigma_j (-i \partial_j) + m \sigma_3$

• $d = 3$, $\alpha = (\alpha_k)_{k=1,2,3}$ and β such that

$$\alpha_k := \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}$$

Here $(\sigma_j)_{j=1,2,3}$ are the Pauli matrices

Ground state : $\lambda_D(V)$ is the lowest eigenvalue in $(-m, m)$ of $\mathcal{D}_m - V$

$$\Lambda_D(\alpha, p) := \inf \left\{ \lambda_D(V) : V \in L^p(\mathbb{R}^d, \mathbb{R}^+) \text{ and } \|V\|_{L^p(\mathbb{R}^d)} = \alpha \right\}$$

A Sobolev/Keller inequality for a Dirac operator

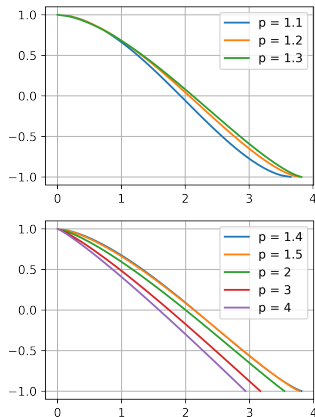
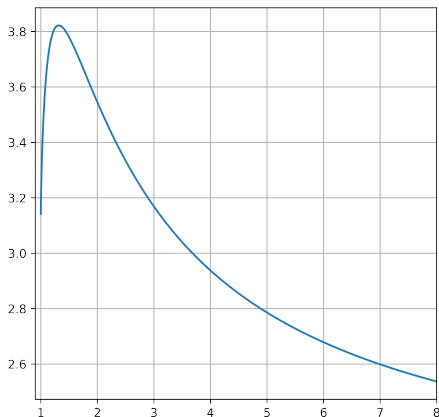
Theorem (JD, Gontier, Pizzichillo, van den Bosch)

Let $p \geq d \geq 1$. There exists $\alpha_*(p) > 0$ such that the map $\alpha \mapsto \Lambda_D(\alpha, p)$ defined on $[0, \alpha_*(p))$ is continuous, strictly decreasing, takes values in $(-m, m]$, and such that

$$\lim_{\alpha \rightarrow 0_+} \Lambda_D(\alpha, p) = m \quad \text{and} \quad \lim_{\alpha \rightarrow \alpha_*(p)} \Lambda_D(\alpha, p) = -m$$

If $(p, d) \neq (1, 1)$, then $\Lambda_D(\alpha, p)$ is attained on $(0, \alpha_*(p))$

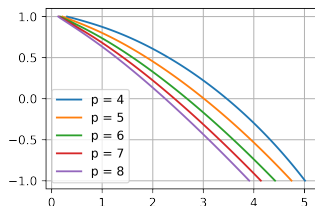
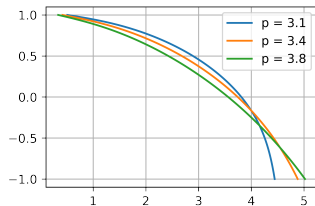
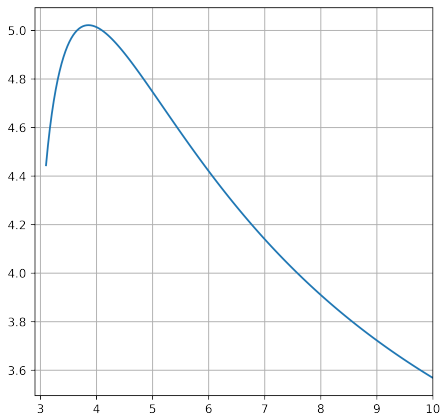
The case of dimension $d = 1$



With $m = 1$ The function $p \mapsto \alpha_*(p)$ (left), has a maximum at $p \approx 1.32$ and
 $\lim_{p \rightarrow 1+} \alpha_*(p) = \pi$ and $\lim_{p \rightarrow +\infty} \alpha_*(p) = 2$

The function (right) $\Lambda_D(\alpha_*(p), p)$ for various p is such that $\Lambda_D(\alpha_*(p), p) = -1$

The radial case in dimension $d = 3$

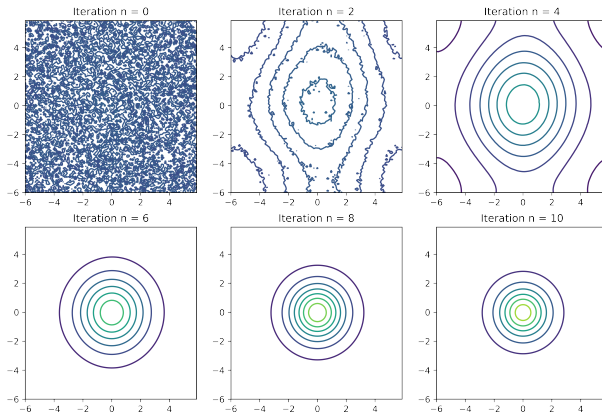


Radial case with $d = 3$ and $m = 1$

(Left) The function $p \mapsto \alpha_^{\text{rad}}(p)$ reaches its maximum at $p \approx 3.86$*

(Right) The maps $\alpha \mapsto \Lambda_D^{\text{rad}, (\kappa=1)}(\alpha, p)$

Is the optimal potential radial?



A numerical answer... Contour lines of the potential (by a fixed point method) for $p = 3$ and $\lambda = 1/2$, for some initial potential chosen at random

Conjecture. The optimal potential at $\alpha = \alpha_*(p)$ is
 an **Aubin-Talenti** profile up to an angular spinor

References

With D. Gontier, F. Pizzichillo, and H. van den Bosch
***Keller and Lieb-Thirring estimates of the eigenvalues
in the gap of Dirac operators***

Revista Matemática Iberoamericana, 40 (2): 649-692, 2023.

With M.J. Esteban, R. Frank, and M. Loss
***The CKN inequality for spinors:
symmetry and symmetry breaking***

[arXiv:2504.16909](#)

With R. Frank, and J. Weizler
***Symmetry and symmetry breaking in interpolation
inequalities for two-dimensional spinors***
Preliminary results

[arXiv:2504.16909](#)

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Thank you for your attention !